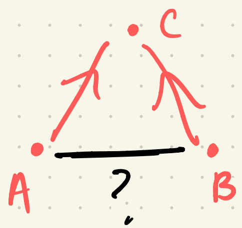
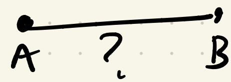
Practice Midterm 2021 Solutions

1. True/False: if a 3-candidate election has two Smith candidates out of three, then there must be a tie in the pairwise comparison graph.

True. Suppose the three candidates are A, B, C and the Smith Set $S = \{A, B\}$. We know then that there must be an arrow from A to C and an arrow from B to C, since the Smith set is a dominating set. So we have

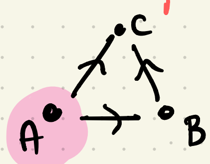


There are 3 possibilities for

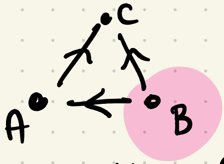


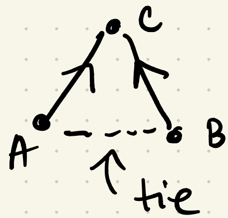
to check

Case 1:



but then $\{A\}$ is a dominating set and therefore $\{A, B\}$ cannot be the Smith set since the Smith set is defined to be the smallest dominating set.

Case 2:  but then $\{B\}$ is a dominating set and therefore $\{A, B\}$ cannot be the Smith set since the Smith set is defined to be the smallest dominating set.

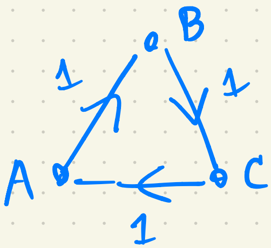
Case 3: 

This is the only possibility left, and indeed this gives the Smith set $S = \{A, B\}$.

2. Find an example (system and preference schedule) for which there is a spoiler who is a Smith candidate. In your example, are they a winning spoiler? Losing spoiler? Weak spoiler?

One approach is to try out a preference schedule in which <u>all</u> of the candidates are in the Smith set. Automatically, then, any spoiler will be a Smith candidate. So we need a system and a preference schedule in which (1) all the candidates are Smith candidates and (2) some candidate is a spoiler. Let's try out a simple example satisfying (1): a preference schedule giving rise to a Condorcet cycle. Let's see what happens when we use plurality.

×1	×1	×1
A	B	C
B	C	A
C	A	B



Smith set $S = \{A, B, C\}$

$W_{plur} = \{A, B, C\}$

What happens when C is disqualified?

×2	×1
A	B
B	A

$\Rightarrow W'_{plur} = \{A\}$

⟶

Candidate C is a spoiler because

$$\underbrace{W'_{plur}}_{\{A\}} \neq \underbrace{W - \{C\}}_{\{A, B\}}.$$

In particular:

- C is a winning spoiler because $C \notin W_{plur}$.
- C is not a losing spoiler because $C \notin W_{plur}$.
- C is not a weak spoiler because $C \in S$, where S is the Smith set, which is the set of strong (i.e., not weak) candidates,

3. Suppose there are $n = 10$ candidates and $N = 100$ voters in a particular perference schedule. How many consolidations do you have to consider to run each of these methods?: plurality, runoff, elimination, Coombs, Borda, Smith, Smithified plurality, pairwise comparison, sequential, and dictatorship. Example: Runoff requires you to identify the top two vote-getters and compare them head-to-head, so you do **one** consolidation (down to those two candidates).

In case it is helpful: a pairwise comparison graph with ten vertices has 45 edges. (★)

① Plurality: 0 consolidations

To compute the winner in plurality, we simply read off the candidate(s) with the most first-place votes from the given preference schedule.

② Runoff: 1 consolidation (see (★) above)

③ Elimination: Each time I eliminate a candidate, I have to consolidate the preference schedule. I can stop when I have 2 candidates remaining, and simply read off who has more votes at that stage. To arrive at 2 candidates remaining in the preference schedule, we successively eliminate the 8 other candidates, and each time we eliminate a candidate we consolidate the preference schedule, so we need 8 consolidations.

(4) Coombs: 8 consolidations.

The reasoning is the same as above. Coombs and elimination differ only in how a candidate is eliminated in each round; 8 eliminations (and thus consolidations) are still required to arrive at 2 candidates remaining, at which stage we just read off which candidate has more votes.

(5) Borda: 0 consolidations

We compute Borda scores for each candidate directly from the given schedule.

(6) Pairwise Comparison: 45 consolidations

To construct the PWC graph, we need to conduct a head-to-head for each pair of candidates. There are 45 such pairs (corresponding to the 45 edges in the PWC graph). To find the winner of a head-

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-to-head, we consolidate the preference schedule to the two candidates of interest in the head-to-head. Thus 45 consolidations are needed to construct the PWC graph. Once we have the PWC graph, we simply look at which candidate(s) have the greatest number of outgoing arrows to determine the winner(s).

(7) Smith: 45 consolidations

We need to construct the PWC graph and once we have that we can use it to find the Smith set, so no additional consolidations are required. So the answer is the same as for pairwise comparison above.

(8) Smithified plurality: 46 consolidations

Again, we have to find the Smith set by constructing the PWC graph, which by the above requires 45 consolidations. Then, once we find the Smith candidates we have to consolidate the preference schedule

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to those candidates and then read off the winner based on plurality from the consolidated preference schedule. So in total we need $46 = 45 + 1$ consolidations.

(9) <u>Sequential</u>: 9 consolidations

Sequential comparison requires 9 head-to-head competitions. For, say the sequential ordering is:

$$A, B, C, D, E, F, G, H, I, J$$

Then:

A vs. B → winner W_1

then W_1 vs. C → winner W_2

then W_2 vs. D → winner W_3

then W_3 vs. E → winner W_4

then W_4 vs. F → winner W_5

then W_5 vs. G → winner W_6

then W_6 vs. H → winner W_7

then W_7 vs. I → winner W_8

then W_8 vs. J → winner W_9

where W_i is just the winner in the i^{th} round.

Since determining the winner of each of the 9 head-to-heads requires a consolidation, the answer is 9.

(10) Dictatorship : 0 consolidations

Just look at the ballot of the dictator (namely their first-place choice) to determine the winner.

4. Suppose again there are $n = 10$ candidates and $N = 100$ voters in an election. Can you tell from the pairwise comparison graph alone whether an election has a Condorcet candidate? Pareto candidate? majority candidate? unanimous preference of X over Y?

Condorcet: Yes

By definition, a Condorcet candidate has an outgoing arrow to each other candidate in PWC graph

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Pareto: Yes. If a candidate X has an arrow to every other candidate with a margin of 100, then X is preferred by every voter to every other candidate (and conversely, if X is Pareto, this is what the PWC graph will have).

Majority: No. You can have a majority candidate who beats everyone by small margins, so the arrows and margins alone are not enough. If you're curious to see a concrete example for why the PWC graph does not provide enough information, see next page.

Unanimous: Yes. If X beats Y with a margin of 100, then all voters prefer X to Y (and conversely, if all voters prefer X to Y, this is what the PWC graph will have).

Schedule 1

x33	x33	x33	1
A	C	B	A
B	B	A	B
C	A	C	C
D	D	D	D
E	E	E	E
F	F	F	F
G	G	G	G
H	H	H	H
I	I	I	I
J	J	J	J

(★)

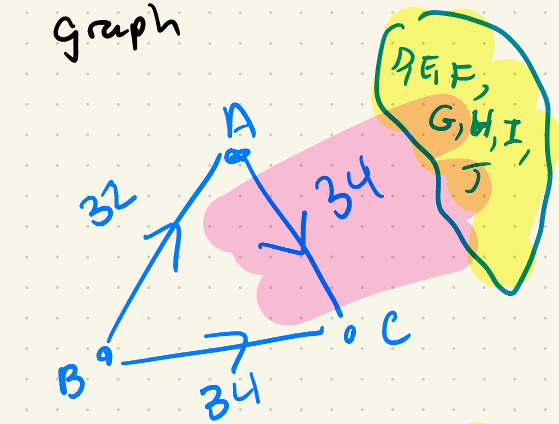
Schedule 2

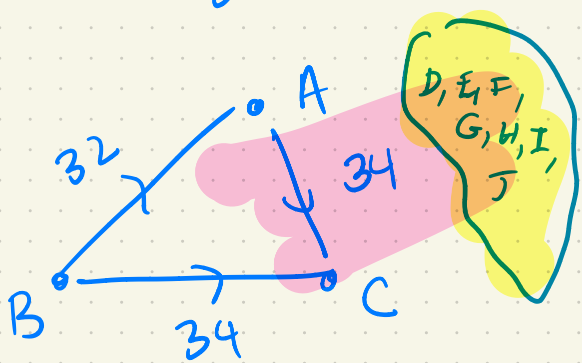
majority candidate

x33	x33	x33	x1
B	B	C	A
A	A	A	B
C	C	B	C

{ Insert (★) }

Sched 1 and Sched 2 have same pairwise comparison graphs but B is a only a majority candidate in Sched 2.
⟹ cannot tell whether there is a majority candidate from PWC graph





* Arrows and margins from {A, B, C} to {D, E, ..., I, J} and among {D, E, ..., I, J} are the same in both PWC graphs.

5. Build a preference schedule each candidate has under 40% of the first-place votes, but there is some consolidation which produces a majority candidate.

Make a divided vote in a 3 way race. Then any consolidation to two candidates will produce someone with a considerable majority.

×35	×35	×30
A	B	C
B	C	A
C	A	B

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×65	×35
A	B
B	A

6. True/False: the runoff method is unanimity-fair.

True. If there is a unanimous preference (say $X > Y$) then X appears above Y in every single column in the preference schedule. Now the runoff method takes the top two first-place vote-getters. Candidate Y has no first-place votes at all, because everyone prefers X to Y. So Y can't be one of the candidates in the runoff, so $Y \notin W$.

7. Is it possible for a move to be favorable to one candidate and neutral to another candidate? If so, give an example.

Sure. Recall a neutral move to a candidate keeps them in place (no candidates may "hop" over) while a favorable move raises a candidate while keeping the relative order of the other candidates unchanged. So take

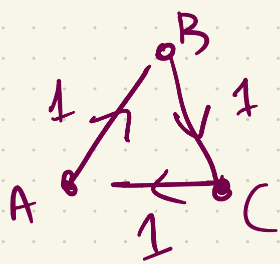
$$\begin{matrix} A \\ B \\ C \end{matrix} \quad \rightarrow \quad \begin{matrix} B \\ A \\ C \end{matrix}$$

Move is favorable to B, neutral to C.

8. Give an example of a preference schedule with five candidates so that all of them are involved in a Condorcet cycle.

Recall to create a Condorcet cycle with 3 candidates, we took

x1	x2	x3
A	B	C
B	C	A
C	A	B

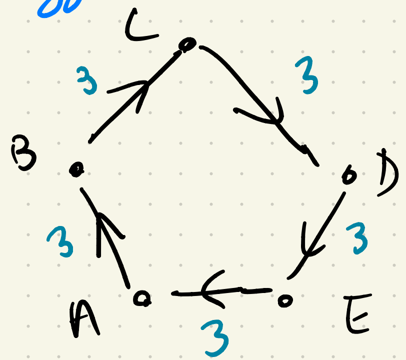


see next page

Notice to generate the ballots from left-to-right, "shift" the previous ballot up and let the ballot "wrap around".

Now we generalize the preference schedule from before to 5 candidates by using the same "shifting" strategy.

x1	x1	x1	x1	x1
A	B	C	D	E
B	C	D	E	A
C	D	E	A	B
D	E	A	B	C
E	A	B	C	D



9. Make up an example of a voting system that is unanimity-fair but not Pareto efficient, or explain why this is impossible.

Impossible! Suppose the system is unanimity-fair, and suppose there is a Pareto candidate. Then that candidate is ranked first on every ballot, but that means they're unanimously preferred to every other candidate. UF says that if there's a unanimous candidate, then the dispreferred candidate shouldn't win. For this to be true, everybody but the Pareto candidate would be eliminated from contention. But the Pareto candidate must win, so the system is automatically Pareto efficient.