Chapter 11 - Vectors and Vector-Valued Functions (using L\TeX)

**Scalar Multiplication:**
Given a scalar, \( c \), and a vector, \( \vec{v} \), the scalar multiple \( c \vec{v} \) is a vector whose magnitude is \(|c|\) multiplied by the magnitude of \( \vec{v} \).

If \( c < 0 \), then \( c \vec{v} \) and \( \vec{v} \) point in opposite directions.

Two vectors are parallel if they are scalar multiples of one another.

Because \( 0 \vec{v} = 0 \) for all vectors, the zero vector is parallel to all vectors.

**Vector Components:**
Round brackets, \( (\ ) \), are for coordinates and angled brackets, \( \langle \rangle \), are for components of a vector.

If \( \vec{v} = \langle a, b, c \rangle \), then \(|\vec{v}| = \sqrt{a^2 + b^2 + c^2}\)

**Unit Vectors:**
A unit vector is a vector of magnitude (length) 1.

If \( \vec{v} \) is not the zero vector, then the unit vector in the direction of \( \vec{v} \) is defined as: \( \hat{v} = \frac{\vec{v}}{|\vec{v}|} \)

Also, \( \pm \frac{\vec{v}}{|\vec{v}|} \) are unit vectors parallel to \( \vec{v} \).

Unit Vectors: \( \hat{i} = \langle 1, 0, 0 \rangle \), \( \hat{j} = \langle 0, 1, 0 \rangle \), \( \hat{k} = \langle 0, 0, 1 \rangle \)

**Vectors in Three Dimensions:**
If the curled fingers of the right hand are rotated from the positive x axis to the positive y axis, the thumb is in the direction of the positive z axis.

To find \( \overrightarrow{PQ} \) from \( P(2, 1, 2) \) and \( Q(4, 3, 6) \), the values must be subtracted since it’s the vector between these points.

**Distance Formula in xyz-Space:** \( \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \)

A sphere is defined as: \( (x - a)^2 + (y - b)^2 + (z - c)^2 = r^2 \)

Example: (Note: Uses “completing the square,” which entails dividing the middle component by 2, squaring it, and adding and subtracting it)
\[
\begin{align*}
    x^2 + y^2 + z^2 + 4x - 6y + 2z &= -6 \\
    (x^2 + 4x + 4) + (y^2 - 6y + 9) + (z^2 + 2z + 1) &= -6 \\
    (x + 2)^2 + (y - 3)^2 + (z + 1)^2 &= 8
\end{align*}
\]

**Dot Products:**
If \( \vec{u} = \langle u_1, u_2, u_3 \rangle \) and \( \vec{v} = \langle v_1, v_2, v_3 \rangle \), then the dot product is: \( \vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3 \)

It can also be defined as \( \vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos(\theta) \) if \( \theta \) is the angle between the 2 vectors.

Vectors are said to be orthogonal if the dot product of the vectors is zero.

**Projections:**
To define the projection of \( \vec{u} \) onto \( \vec{v} \), the following notation is used: \( \overrightarrow{\text{Proj}_\vec{v}} \vec{u} \)

The scalar projection is \( \text{Scal}_\vec{v} \vec{u} = |\overrightarrow{\text{Proj}_\vec{v}} \vec{u}| \)

\[
\begin{align*}
    \text{Scal}_\vec{v} \vec{u} &= |\overrightarrow{\text{Proj}_\vec{v}} \vec{u}| \\
    &= |\vec{u}| \cos(\theta) \\
    &= \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} \\
    \overrightarrow{\text{Proj}_\vec{v}} \vec{u} &= \left( \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) |\vec{v}| \\
    &= \overrightarrow{\text{Proj}_\vec{v}} \vec{u}
\end{align*}
\]

To find the projection of \( \vec{u} \) orthogonal to \( \vec{v} \), this equation would be used: \( \vec{u} - \overrightarrow{\text{Proj}_\vec{v}} \vec{u} \)
Note: The projection of a vector on another vector is merely a scalar multiple of the vector its being projected onto. It doesn’t even have to be in the same direction (can be antiparallel)

**Cross-Product:**

\[ \overrightarrow{u} \times \overrightarrow{v} = |\overrightarrow{u}| |\overrightarrow{v}| \sin(\theta) \]

where \( \theta \) is between 0 and \( \pi \) radians (angle between \( \overrightarrow{u} \) and \( \overrightarrow{v} \)) and the direction of the cross product is indicated by the right hand rule

- Cross product produces vector orthogonal to both vectors being crossed

The magnitude of the cross product is also the area of a parallelogram with \( \overrightarrow{u} \) and \( \overrightarrow{v} \) as the sides

- Anti-Commutative: \( \overrightarrow{u} \times \overrightarrow{v} = - (\overrightarrow{v} \times \overrightarrow{u}) \)

- \( \overrightarrow{i} \rightarrow \overrightarrow{j} \rightarrow \overrightarrow{k} \) (so, \( \overrightarrow{i} \times \overrightarrow{j} = \overrightarrow{k} \), and \( \overrightarrow{j} \times \overrightarrow{k} = \overrightarrow{i} \), and \( \overrightarrow{k} \times \overrightarrow{i} = \overrightarrow{j} \))

\[ \overrightarrow{u} \times \overrightarrow{v} = (u_2v_3 - u_3v_2)i + (u_3v_1 - u_1v_3)j + (u_1v_2 - u_2v_1)k \]

To get this, set up a 3x3 matrix:

\[ \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \overrightarrow{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \overrightarrow{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \overrightarrow{k} \]

Then do the simple diagonal rule determinant for the 2x2 matrices (SE arrow multiplication minus SW arrow)

Area of a Parallelogram with sides \( P, Q, R, S \): \( |\overrightarrow{PQ} \times \overrightarrow{PR}| \)

Area of a Triangle with sides \( P, Q, R \): \( \frac{|\overrightarrow{PQ} \times \overrightarrow{PR}|}{2} \)

**Vector-Valued Functions:**

Equation of a Line passing through \( \langle x_0, y_0, z_0 \rangle \) the direction \( \parallel \overrightarrow{v} = \langle v_1, v_2, v_3 \rangle \) is:

\[ \overrightarrow{r}(t) = \langle x_0 + tv_1, y_0 + tv_2, z_0 + tv_3 \rangle \]

Also can be written parametrically as: \( x = x_0 + tv_1, \ y = y_0 + tv_2, \ z = z_0 + tv_3 \)

What is needed is a point on the line and a vector that’s parallel

There are infinite vector-valued functions for a curve/line because the parameter can be changed using another point.

To define a line segment, a specific domain for the parameter must be written along with the vector-valued function for the whole line, which can be determined by doing the final coordinate minus the initial coordinate and making it a vector that will become \( \overrightarrow{v} \) or \( \langle v_1, v_2, v_3 \rangle \)

To find the domain, look for all values of \( t \) that will give a value at the beginning and end of the line via the given coordinates.

Equation of a curve: \( \overrightarrow{r}(t) = \langle f(t), g(t), h(t) \rangle, \ a \leq t \leq b \)

The domain of \( t \) is all \( t \) values that make \( f(t), g(t), h(t) \) defined

To graph a function, make \( z \) zero to plot in the \( xy \)-plane and then make a new plot adding \( z \) in

Finding intersections of planes and curves is straightforward, so keep it that way:

Ex: \( y = x = 0 \) and \( \overrightarrow{r}(t) = \langle 15 \cos(t), 15 \sin(t), t \rangle \)

\( 15 \sin(t) - 15 \cos(t) = 0 \) and solve for \( t \) then plug into the original

**Calculus with Vector-Valued Functions:**

As long as \( r'(t) \neq 0 \), then \( r'(t) \) is the tangent vector assuming \( r(t) \) is fully differentiable on the interval

The Unit Tangent Vector: \( \overrightarrow{T} = \frac{r'(t)}{|r'(t)|} \)

All derivative rules still apply, but be careful:

- Dot Product Rule: \( \frac{d}{dt}(\overrightarrow{u} \cdot \overrightarrow{v}) = \overrightarrow{u}' \cdot \overrightarrow{v} + \overrightarrow{u} \cdot \overrightarrow{v}' \)

- Cross Product Rule: \( \frac{d}{dt}(\overrightarrow{u} \times \overrightarrow{v}) = \overrightarrow{u}' \times \overrightarrow{v} + \overrightarrow{u} \times \overrightarrow{v}' \)

Integration is as usual, but the constant is a vector \( \langle C \rangle \) and can be added to the whole function or its components. Also, integration can be distributed in to the components:

\( \int \overrightarrow{r}(t)dt = \overrightarrow{R}(t) + \overrightarrow{C} \)
Motion in Space:
\[ \overrightarrow{r}''(t) = \overrightarrow{v}'(t) = \overrightarrow{a}(t) \]
When integrating and finding the constants using initial values \((t = 0)\) problems, remember that the constant \(C\) is in more than one dimension.

Speed \(= |\overrightarrow{v}(t)|\)

Arc Length:
\[ S = \int_{a}^{b} \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2} \, dt = \int_{a}^{b} |\overrightarrow{r}'(t)| \, dt \]
If a vector is multiplied by a scalar multiple, it is the same equation even though it is different parametrization. Even so, its arc length is identical.

Circular Motion with constant \(|\overrightarrow{r}|\):
\[ \overrightarrow{r} \cdot \overrightarrow{v} = 0 \]

Reparameterization:
If a curve or function has the dummy variable \(t\) replaced with one that is respect to arc length, it is unique and has meaning (distance on curve)
\[ S(t) = \int_{a}^{t} |\overrightarrow{r}'(u)| \, du \]
Find \(S(t)\) and then solve for \(t\) to substitute into the vector-valued function
\(a\) is simply the lowest point on the domain
Change the domain for \(S(t)\) by replacing \(t\) for the function in terms of \(s\) in the inequality for the domain and then isolate \(s\)
If \(|\overrightarrow{r}'(t)| = 1\), then arc length was already used for parameterization