Differential Equations

Based on lectures by Prof. Adler

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January 22, 2013

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0 Review of Integration Techniques

0.1 Integration by Parts

The equation for integration by parts is

\[ \int u dv = uv - \int v du \]

The priority in which you choose what \( u \) is can be generalized by the acronym LIPET, elaborated in the list below with descending priority:

- Logarithmic Functions
- Inverse Functions
- Polynomial Functions
- Exponential Functions
- Trigonometric Functions

0.2 Partial Fractions

0.2.1 Simple Partial Fractions

Simple partial fractions can have their denominators factored to \((x - r)\)

Method 1:

1. Factor denominator
2. Decompose the fraction by creating dummy variables
3. Solve for the roots of the denominator
4. Plug the roots into the numerator and denominator of the decomposed fractions and solve
5. Integrate the partial fraction using the numbers previously solved for in place of the dummy variables

\[ \int \frac{5x - 10}{x^2 - 3x - 4} \, dx = \int \frac{5x - 10}{(x - 4)(x + 1)} \, dx = \int A \frac{1}{x - 4} + B \frac{1}{x + 1} \, dx \]

The roots are \( x = 4 \) and \( x = -1 \)

\[ A = \frac{5(4) - 10}{4 + 1} = 2 \]
\[ B = \frac{5(-1) - 10}{-1 - 4} = 3 \]

\[ \int \frac{2}{x - 4} + \frac{3}{x + 1} \, dx = 2 \ln |x - 4| + 3 \ln |x + 1| + C \]
Method 2:
1. Factor denominator
2. Decompose the fraction with dummy variables
3. Multiply both sides of the equation by the original function’s denominator
4. Group coefficients for each power of $x$
5. Set each grouped order equal to the coefficients of the powers of the original numerator
6. Solve for the constants in the numerator

$$\int \frac{5x - 10}{x^2 - 3x - 4} \, dx = \int \frac{5x - 10}{(x - 4)(x + 1)} \, dx = \int \frac{A}{x - 4} + \frac{B}{x + 1} \, dx$$

The roots are $x = 4$ and $x = -1$

$$5x - 10 = A(x + 1) + B(x - 4)$$
$$5x - 10 = Ax + A + Bx - 4B$$
$$5x - 10 = (A + B)x + A - 4B$$

$A + B = 5$ and $A - 4B = -10 : A = 2, B = 3$

$$\int \frac{2}{x - 4} + \frac{3}{x + 1} \, dx = 2 \ln |x - 4| + 3 \ln |x + 1| + C$$

0.2.2 Repeated Linear Factors

Repeated linear factors are those that, when the denominator is factored, have a higher degree than one

1. Factor the denominator
2. Check the highest ordered power for each factored term
   There must be a partial fraction for each power of $(x - r)$
3. Solve for the known quantities as in Method 2 of the Simple Partial Fractions section

$$\int \frac{2x + 4}{x^3 - 2x^2} \, dx = \int \frac{2x + 4}{x^2(x - 2)} \, dx = \int \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x - 2} \, dx$$

$$2x + 4 = A(x - 2) + Bx(x - 2) + Cx^2$$
$$2x + 4 = Ax - 2A + Bx^2 - 2Bx + Cx^2$$
$$2x + 4 = x^2(B + C)x(A - 2B) - 2A$$

$B + C = 0$ and $A - 2B = 2$ and $-2A = 4 : A = -2, B = -2, C = 2$

$$\int \frac{-2}{x^2} + \frac{-2}{x} + \frac{2}{x - 2} \, dx$$

$$\frac{2}{x} - 2 \ln |x| + 2 \ln |x - 2| + C$$
0.2.3 Irreducible Quadratic Factors

Irreducible quadratic factors have factors in the denominator that are quadratic but not reducible, such as $x^2 + 1$

1. Factor the denominator

2. If an irreducible quadratic remains in the denominator, check the highest order

The numerator for that partial fraction should contain a variable that range from $x^0$ through one less than the highest degree of the denominator

3. Set up partial fractions and solve via Method 2 of the Simple Partial Fractions section

\[ \int \frac{x^2 + x - 2}{3x^3 - x^2 + 3x - 1} \, dx = \int \frac{x^2 + x - 2}{x^2(3x - 1) + 3x - 1} \, dx = \int \frac{x^2 + x - 2}{(3x - 1)(x^2 + 1)} \, dx = \int \frac{A}{3x - 1} + \frac{Bx + C}{x^2 + 1} \, dx \]

\[ x^2 + x - 2 = A(x^2 + 1) + (Bx + C)(3x - 1) \]
\[ x^2 + x - 2 = Ax^2 + A + 3Bx^2 - Bx + 3Cx - C \]
\[ x^2 + x - 2 = x^2(A + 3B) + x(-B + 3C) + A - C \]

A + 3B = 1 and $-B + 3C = 1$ and $A - C = -2 \therefore A = -\frac{7}{5}, B = \frac{4}{5}, C = \frac{3}{5}$

\[ \int \frac{-7}{5} \frac{4x + 3}{3x - 1} + \frac{1}{5} \frac{5}{x^2 + 1} \, dx = -\frac{7\ln|3x - 1|}{5} + \frac{2\ln|x^2 + 1|}{5} + \frac{3\arctan(x)}{5} + C \]
1 Introduction

1.1 Definitions

What is a differential equation? A differential equation is an equation with a derivative in it. The order of a differential equation is the highest order of the differentiation that appears in the equation.

An ordinary differential equation\(^1\) only contains ordinary derivatives.

A general solution of an ordinary differential equation of order \(n\) is a formula that describes all solutions of the equation up to at least order \(n\).

A specific solution is determined by certain initial conditions

1.2 Separation of Variables

1.2.1 General Formula

If you can write \(\frac{dx}{dt} = f(x)g(t)\) then the O.D.E. is separable.

1. Multiply by \(dt\) to get differential form \(dx = f(x)g(t)\, dt\)

2. Separate variables \(\frac{1}{f(x)}\, dx = g(t)\, dt\)

3. Integrate both sides \(\int \frac{1}{f(x)}\, dx = \int g(t)\, dt\)

4. Solve for \(x(t)\) if possible

5. Check if \(f(r) = 0\) for some \(r\)

6. Plug back in to check solution

1.2.2 Example 1

\[(t^2 - 1) \frac{dx}{dt} + 2x = 0\]
\[\frac{dx}{dt} (t^2 - 1) = -2x\]
\[dx(t^2 - 1) = -2x\, dt\]
\[\int -\frac{1}{2x}\, dx = \int \frac{1}{t^2 - 1}\, dt\]
\[-\frac{1}{2} \ln |x| = \ln \left| \frac{t + 1}{t - 1} \right| + C\]
\[|x| = e^{\ln \left| \frac{t + 1}{t - 1} \right|} e^C\]
\[x = k \left( \frac{t + 1}{t - 1} \right)\]

\(^1\)Typically abbreviated as O.D.E.
1.3 First-Order Linear Equations

1.3.1 Definitions

A first order O.D.E. is **linear** if it can be written as \( a_1(t) \frac{dx}{dt} + a_2(t)x = E(t) \)

The O.D.E. is **normal** on an interval \( \alpha \leq t \leq \beta \) if \( a_1(t), a_2(t), E(t) \) are continuous for \( \alpha \leq t \leq \beta \) and \( a_1(t) \neq 0 \) for all \( \alpha \leq t \leq \beta \)

A first order O.D.E. is **homogeneous** if \( E(t) = 0 \)

The **standard form** of a linear first order O.D.E. is \( \frac{dx}{dt} + r(t)x = q(t) \)

1.3.2 General Formula for Variation of Parameters

1. Divide by \( a_1(t) \) to obtain the equation in standard form \( (r(t) = a_0(t)/a_1(t) \) and \( q(t) = E(t)/a_1(t) \)

\[
\frac{dx}{dt} + r(t)x = q(t)
\]

2. Separate variables in the related homogeneous equation to obtain the general homogeneous solution \( (k \) is an arbitrary constant and \( h(t) = e^{\int r(t) dt} \)

\[
\frac{dx}{dt} + r(t)x = 0 \rightarrow x = kh(t)
\]

3. We expect the solutions to be of the following form

\[ x = k(t)h(t) \]

Substituting this into the standard form equation after differentiation yields

\[ k'(t)h(t) = q(t) \]

which we can solve for \( k'(t) \)

4. Integrate the formula for \( k'(t) \) to obtain \( k(t) \)

\[ k(t) = \int k'(t) dt + C \]

Multiply \( k(t) \) by \( h(t) \) to obtain \( x(t) \)

\[ x(t) = k(t)h(t) \]

The final solution has the form

\[ x = h(t) \int \frac{q(t)}{h(t)} dt + ch(t) \]
1.3.3 Example 1

Solve the differential equation \( \frac{dx}{dt} = -\frac{1}{2}x + 15 - t \)

1. Put in standard form
   \( \frac{dx}{dt} + \frac{1}{2}x = 15 - t \)

2. Set up the homogeneous equation
   \( \frac{dx}{dt} + \frac{1}{2}x = 0 \)

3. Separate the variables
   \( \frac{1}{x} dx = -\frac{1}{2} dt \)

4. Integrate both sides and solve for \( x \)
   \( x = ke^{-\frac{1}{2}t} \)

5. Rewrite the equation to make \( k \) a function of \( t \)
   \( x = k(t)e^{-\frac{1}{2}t} \)

6. Find \( \frac{dx}{dt} \) based on this equation
   \( \frac{dx}{dt} = k'(t)e^{-\frac{1}{2}t} - \frac{1}{2}e^{-\frac{1}{2}t}k(t) \)

7. Plug this into the original equation to make an equality
   \( k'(t)e^{-\frac{1}{2}t} - \frac{1}{2}e^{-\frac{1}{2}t}k(t) = -\frac{1}{2}x + 15 - t \)

8. Replace all instances of \( x \) to allow for cancellation by substituting in the equation from Step 5
   \( k'(t)e^{-\frac{1}{2}t} - \frac{1}{2}e^{-\frac{1}{2}t}k(t) = -\frac{1}{2}k(t)e^{-\frac{1}{2}t} + 15 - t \)

9. Simplify and solve for \( k'(t) \)
   \( k'(t) = (15 - t)e^{\frac{1}{2}} \)

10. Integrate both sides
    \( k(t) = -2e^{\frac{1}{2}}(t - 17) + C \)

11. Plug \( k(t) \) into the equation from Step 5
    \( x = \left(-2e^{\frac{1}{2}}(t - 17) + C\right)e^{-\frac{1}{2}t} \)

12. Simplify
    \( x = -2(t - 17) + Ce^{-\frac{1}{2}t} \)

1.4 Graphing Differential Equations

1.4.1 Definitions

The process of graphing differential equations is useful when the differential equation is unsolvable or simply too difficult to attempt.

**Isoclines:** Where \( \frac{dx}{dt} \) is a constant value \( (k) \)
1.4.2 Procedure for graphing solutions of $\frac{dx}{dt} = f(t, x)$

1. Try to sketch isoclines for a few specific values of $k$ for $f(t, x) = k$
   (a) Replace $\frac{dx}{dt}$ with $k$ to make it $k = f(t, x)$
   (b) Solve for $x$
   (c) Plot a few values, especially $k = 0$ if possible

2. For each value of $k$, draw line segments (“tick marks”) with a slope the chosen value of $k$ along the corresponding isocline

3. Find any values of $k$ for which the isocline is a straight line of slope $k$
   (a) Any isocline of this type is itself a solution curve

4. Fit solution curves tangent to the segments drawn
   (a) Any maxima or minima occur along the isocline with $k = 0$

1.5 Euler’s Method

The step size, $h$, for Euler’s formula is the following with $n$ steps, where $T$ is the upper value of $t$ and $t_0$ is the initial value:

$$h = \frac{T - t_0}{n}$$

Euler’s formula is the following recurrence relation:

$$x_n = x_{n-1} + h \cdot f(t_{n-1}, x_{n-1})$$

2 Matrices and Determinants

2.1 Operations with Matrices

2.1.1 Addition/Subtraction

- Matrices can only be added or subtracted if they are of the same size
- To add or subtract matrices, perform the addition/subtraction horizontally across

$$\begin{bmatrix} 3 & -2 \\ -4 & 1 \end{bmatrix} - \begin{bmatrix} -5 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 8 & -3 \\ -1 & -1 \end{bmatrix}$$

2.1.2 Scalar and Matrix Multiplication

- For scalar multiplication, simply distribute outside of the matrix to each element of the matrix

$$3 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 & 21 \\ -3 & 6 \end{bmatrix}$$

- For matrix multiplication of $[A] \cdot [B]$, the number of columns in matrix $A$ must equal the number of rows in matrix $B$
  - The product matrix will have the following order: (number of rows in A) x (number of columns in B)

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 41 & 48 \end{bmatrix}$$
2.2 Determinants of 2x2 and 3x3 Matrices

If \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), then the determinant, or \( \det \), of \( A \) is equal to: \( ad - bc \)

If \( A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \), then \( \det(A) = (aei + bfg + cdh) - (bdi + afh + ceg) \)

2.3 Solving Large Systems of Equations

2.3.1 Cramer's Determinant Test

Given the system of \( n \) equations \( b_{11}u_1 + b_{12}u_2 + \cdots + b_{1n}u_n = r_1 \) all the way to \( b_{n1}u_1 + b_{n2}u_2 + \cdots + b_{nn}u_n = r_n \):

\[
\Delta = \text{Determinant of Coefficients} = \det \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}
\]

1. If \( \Delta \neq 0 \), then the system of equations has a unique solution
2. If \( \Delta = 0 \), then the system of equations either has no solutions or infinitely many solutions

2.3.2 Expansion by Minors

In the Expansion of Minors, the \((i,j)\)th minor of a matrix is obtained by omitting the \( i \)th and \( j \)th row/column to make an \((n-1)\times(n-1)\) matrix.

To Compute \( \det(B) = \Delta \):

1. Pick any row or column\(^2\) of \( B \)
2. For every entry, \( b_{ij} \), in that row or column, compute the determinant of the minor produced by removing the row and column of that entry \((i,j)\) minor
3. Multiply that determinant by \( \pm b_{ij} \)
   
   (a) + if it's even, – if it's odd
4. Add up all the terms

2.3.3 Cramer's Rule

If \( \Delta \neq 0 \), then let \( \Delta_i \) be the determinant we obtain from \( \Delta \) by replacing the entries of the \( i \)th column with the right side \( r_1, \ldots, r_n \):

\[
\Delta_i = \det \begin{bmatrix} b_{11} & \cdots & b_{1i-1} & r_1 & b_{1i+1} & \cdots & b_{1n} \\ b_{21} & \cdots & b_{2i-1} & r_2 & b_{2i+1} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{ni-1} & r_n & b_{ni+1} & \cdots & b_{nn} \end{bmatrix}
\]

\(^2\)It is typically best to choose the row or column with the most zeroes in it
The unique solution of the system is given by the formulas:

\[ u_1 = \frac{\Delta_1}{\Delta}, \quad u_2 = \frac{\Delta_2}{\Delta}, \quad \ldots, \quad u_n = \frac{\Delta_n}{\Delta} \]

### 2.3.4 Example 1

Solve the system of four equations for \( w \):

1. \( x + y - z + w = 5 \)
2. \( y + 3w = 1 \)
3. \( x + z + w = 2 \)
4. \( y + z + w = 0 \)

1. Set up the equations in the form of matrices

\[
\begin{bmatrix}
1 & 1 & -1 & 1 \\
0 & 1 & 0 & 3 \\
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
w
\end{bmatrix} =
\begin{bmatrix}
5 \\
1 \\
2 \\
0
\end{bmatrix}
\]

2. Use the expansion by minors theorem. I chose the first column here because it had the most zeroes. I mentally cover up the first column to be left with the following 4x3 matrix

\[
\begin{bmatrix}
1 & -1 & 1 \\
1 & 0 & 3 \\
0 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}
\]

3. Then take the first element of the column or row I chose to remove (first column in this case) and multiply that by the determinant of the matrix that results from removing the corresponding number row (first row)

\[ 1 \left| \begin{array}{cc}
1 & 0 \\
0 & 1 \\
1 & 1
\end{array} \right| = 1 \det
\]

(a) The sign of the term is given by the corresponding same-size sign matrix, which in this case would be

\[
\begin{bmatrix}
+ & - & + & - \\
- & + & - & + \\
+ & - & + & - \\
- & + & - & +
\end{bmatrix}
\]

4. Repeat Step 3 for the rest of the elements in the column or row chosen to make an additive algebraic equation

\[
\Delta = 1 \det
\]

\[
0 + 1 \det
\]

\[ - 0 \]

\[ - 0 \]

\[ - 0 \]
5. Repeat the expansion by minors theorem until you reach a matrix you can solve the determinant for easily (in this case I chose to remove the first row from the first matrix and the second row from the second matrix)

\[
\begin{vmatrix}
1 & 0 & 3 \\
0 & 1 & 1 \\
1 & 1 & 1
\end{vmatrix}
\rightarrow
\begin{vmatrix}
1 & 1 \\
0 & 1 \\
1 & 1
\end{vmatrix}
- 0 + 3 \begin{vmatrix}
0 & 1 \\
1 & 1
\end{vmatrix}
\]

\[
\begin{vmatrix}
1 & -1 & 1 \\
1 & 0 & 3 \\
1 & 1 & 1
\end{vmatrix}
\rightarrow
\begin{vmatrix}
-1 & 1 \\
1 & 1 \\
1 & 1
\end{vmatrix}
+ 0 - 3 \begin{vmatrix}
1 & -1 \\
1 & 1
\end{vmatrix}
\]

\[
\therefore \Delta = \left( \begin{vmatrix}
1 & 1 \\
0 & 1
\end{vmatrix}
- 0 + 3 \begin{vmatrix}
0 & 1 \\
1 & 1
\end{vmatrix}\right) + \left( -1 \begin{vmatrix}
-1 & 1 \\
1 & 1
\end{vmatrix}
+ 0 - 3 \begin{vmatrix}
1 & -1 \\
1 & 1
\end{vmatrix}\right)
\]

\[
\Delta = -7 \therefore \Delta \neq 0
\]

6. Since \( w = \frac{\Delta_w}{\Delta} \) and \( \Delta \neq 0 \), we need to solve for \( \Delta_w \) by using Cramer’s Rule to replace, in this case, the last column with the vector of the solutions from the system of linear equations

\[
\Delta_w = \begin{vmatrix}
1 & 1 & -1 & 5 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 2 \\
0 & 1 & 1 & 0
\end{vmatrix}
\]

7. Repeat Steps 3 and 4 for this determinant. The final result is

\[
\Delta_w = -2 - 1 + 5 - 2 = 0
\]

8. Plug into the equation for \( w \)

\[
w = \frac{\Delta_w}{\Delta} = \frac{0}{-7} = 0
\]


2.4 Row Reductions

2.4.1 Definition and Elementary Row Operations

There are three elementary row operations that can be done on matrices:

1. Multiplying a row by a constant
2. Switching two rows
3. Adding a constant times a row to another row

A matrix is reduced provided:

1. Any row consisting entirely of zeros is on the bottom
2. The first nonzero entry, or corner entry, of each nonzero row is 1
3. The corner entry of each nonzero row is further to the right than the corner entries of the preceding rows
4. The corner entries are the only nonzero entries in their columns

2.4.2 Steps for Row Reduction

The following are systematic steps to reduce a matrix:\(^3\):

1. Perform elementary row operations to yield a 1 in the first row, first column
2. Create zeros in all the rows of the first column except the first row by adding the first row times a constant to each other row
3. Perform elementary row operations to yield a 1 in the second row, second column
4. Create zeros in all the rows of the second column except the second row by adding the second row times a constant to each other row
5. Perform elementary row operations to yield a 1 in the third row, third column
6. Create zeros in all the rows of the third column except the third row by adding the third row times a constant to each other row
7. Continue this process until the first m x m entries form the identity matrix

2.4.3 Solving Equations Via Row Reduction

- Set up the system of linear equations in the format of matrices
- Include the vector for the solutions to the linear equations inside the matrix of coefficients that is getting row reduced
  - The vector is usually separated by a large vertical bar and put on the right side of the matrix
  - The properties of a reduced row do not need to rely on the vector on the right side of the bar
  - All row operations done to the main matrix must also be done to the included vector

\(^3\)Steps taken from Sparknotes
When the matrix is reduced, solve the system of linear equations

- Sometimes there will be infinitely many solutions, in which case a variable is made to symbolize the solution set
- Sometimes there will be no solutions because of an untrue statement (e.g. $0 = 1$)

If asked to solve $Bu = k$ for $u$, it is this exact process. The solution set is given as the matrix $u$

### 2.4.4 Example 1

Solve the system of linear equations:

1. $x - y + z = 1$
2. $2x + y + 2z = 5$
3. $x + 2y + z = 4$

1. Write the system of equations in matrix form

$$
\begin{bmatrix}
1 & -1 & 1 \\
2 & 1 & 2 \\
1 & 2 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} =
\begin{bmatrix}
1 \\
5 \\
4
\end{bmatrix}
$$

2. Write out the matrix to be row reduced

$$
\begin{bmatrix}
1 & -1 & 1 & | & 1 \\
2 & 1 & 2 & | & 5 \\
1 & 2 & 1 & | & 4
\end{bmatrix}
$$

3. Row reduce the matrix

$$
\begin{bmatrix}
1 & 0 & 1 & | & 2 \\
0 & 1 & 0 & | & 1 \\
0 & 0 & 0 & | & 0
\end{bmatrix}
$$

4. Set up and solve the system of linear equations

$$
x = 2 - z, \quad y = 1, \quad 0 = 0
$$

(a) Make sure there are no untrue statements

5. If needed, create a new variable to represent the infinite solution set ($z = a$ for this example)

$$
x = 2 - a, \quad y = 1, \quad z = a
$$

6. If needed, put the solution set in matrix form

$$
\begin{bmatrix}
2 \\
1 \\
0
\end{bmatrix} + a
\begin{bmatrix}
-1 \\
0 \\
1
\end{bmatrix}
$$
3 Linear Differential Equations

3.1 \( \frac{dx^n}{dt^n} = E(t) \)

- An \( n \)-th order O.D.E. is **linear** if it can be written in the form \( a_n(t) \frac{d^n x}{dt^n} + \cdots + a_1(t) \frac{dx}{dt} + a_0(t) x = E(t) \)

- An easy family to solve differential equations for is \( \frac{dx^n}{dt^n} = E(t) \) since it can be solved via simple integration.
  
  - For instance, integrating thrice on \( \frac{d^3 x}{dt^3} = e^t + 3 \) will yield the solution \( x(t) = e^t + \frac{1}{2} t^3 + \frac{1}{2} c_1 t^2 + c_2 t + c_3 \)

- All solutions can be described as \( x(t) = h(t)p(t) \), where \( h(t) \) is the function with all constants and \( p(t) \) is the specific solution.
  
  - With the previous problem, \( h(t) = \frac{1}{2} c_1 t^2 + c_2 t + c_3 \) and \( p(t) = e^t + \frac{1}{2} t^3 \)

- \( h(t) \) is a general solution to the homogeneous equation, and \( p(t) \) is a particular solution to the original O.D.E.

- If \( E(t) = 0 \), the O.D.E. is homogeneous. It’s normal if \( a_n(t) \neq 0 \) on the interval, \( I \), and \( a_n(t), \ldots, a_1(t), a_0(t), x(t) \) are continuous on the \( I \)

3.2 Existence and Uniqueness Theorem

- In essence, for some \( f(t, x) \) at some point \( (a, b) \): if \( f(t, x) \) is continuous near this point and if \( \frac{\partial f}{\partial x} \) is continuous at this point, then the existence and uniqueness theorem applies.

3.3 Operator Notation

The derivative operator is: \( D^n x = \frac{d^n x}{dt^n} \)

The operator \( L \) is defined as

\[
a_n(t) \frac{d^n x}{dt^n} + \cdots + a_1(t) \frac{dx}{dt} + a_0(t) x = E(t) \rightarrow [a_n(t)D^n + \cdots + a_1(t)D + a_0(t)] x = E(t) \rightarrow Lx = E(t)
\]

Example: If \( L = D^2 + t \), find \( L(\sin(2t)) \)

Solution: \( (D^2 + t)(\sin(2t)) = -4\sin(2t) + t\sin(2t) \)

The **principles of superposition** states that

\[
L(x_1(t) + x_2(t)) = L(x_1(t)) + L(x_2(t))
\]

The **principle of proportionality** states that

\[
L(cx_1) = cL(x_1)
\]

A differential equation is **linear** if it can be written as \( Lx = E(t) \)
3.4 How to Solve Linear O.D.E.s

1. Find the general solution to the homogeneous equation, \( h(t) \)
2. Find any particular solution, \( p(t) \)
3. The general solution of \( Lx = E(t) \) will be \( x(t) = h(t) + p(t) \)

3.4.1 Example 1

Given \((D^2 - D - 2)x = -t + 4\), \( h(t) = c_1e^{2t} + c_2e^{-t} \), and \( p(t) = At + B \), find the values of \( A \) and \( B \). Then find a general equation to the linear O.D.E.

1. Substitute in for the derivative operator. Be careful to distribute the \( x \) to make \( 2x \)
   \[
   0 - A - 2At - 2B = -t + 4
   \]
2. Solve for \( A \) and \( B \) by “grouping”
   \[
   -2A = -1 \quad \therefore A = \frac{1}{2} \quad -\frac{1}{2} - 2B = 4 \quad \therefore B = -\frac{9}{4}
   \]
3. Create the equation \( x(t) = h(t) + p(t) \)
   \[
   x = c_1e^{2t} + c_2e^{-t} + \frac{1}{2}t - \frac{9}{4}
   \]

3.5 Homogeneous Linear Equations

3.5.1 Introduction

- In general, suppose \( h_1(t), h_2(t), \ldots, h_n(t) \) are solutions to an \( n^{th} \) order linear, homogeneous O.D.E. \((Lx = 0)\) and \( c_1, c_2, \ldots, c_n \) are constants. Then, \( h(t) = c_1h_1(t) + c_2h_2(t) + \ldots + c_nh_n(t) \) is also a solution because \( h(t) \) is a linear combination of \( h_1(t), h_2(t), \ldots, h_n(t) \).
- Any linear combination of solutions of a homogeneous linear O.D.E. is also a solution.
  - If \( h(t) \) is the general solution, we say that \( h_1(t), h_2(t), \ldots, h_n(t) \) generate the general solution.

3.5.2 Example 1

Is \( h(t) \) the general solution to \((D^2 - 3D)x = 0\) if \( h_1(t) = 1 \) and \( h_2(t) = e^{3t} \)

1. Write out \( h(t) \)
   
   (a) \( h(t) = c_1 + c_2e^{3t} \)
2. Consider initial conditions (use \( \alpha \) as a place-holder) up to one less than the highest derivative degree
   
   (a) \( x(0) = \alpha_0 \rightarrow h(0) = c_1 + c_2 = \alpha_0 \)
   (b) \( x'(0) = \alpha_1 \rightarrow h'(0) = 3c_2 = \alpha_1 \)
3. Solve the initial value problem
   
   (a) \( c_2 = \frac{\alpha_1}{3} \)
   (b) \( c_1 = \alpha_0 - \frac{\alpha_1}{3} \)
4. Rewrite \( h(t) \)
   
   (a) \( h(t) = \left( \alpha_0 - \frac{\alpha_1}{3} \right) + \frac{\alpha_1}{3}e^{3t} \)
5. Since we can write the equation with initial conditions, it is general.
3.5.3 Example 2

Is \( h(t) \) the general solution to \( D^3x = 0 \) if \( h_1(t) = t^2 \), \( h_2(t) = 1 \), and \( h_3(t) = 3t^2 + 7 \)

1. Write out \( h(t) \)
   
   \( h(t) = c_1t^2 + c_2 + c_3(3t^2 + 7) \)

2. Consider initial conditions
   
   (a) \( x(0) = \alpha_0 \rightarrow h(0) = c_2 + 7c_3 = \alpha_0 \)
   
   (b) \( x'(0) = \alpha_1 \rightarrow h'(0) = 0 = \alpha_1 \)
   
   (c) \( x''(0) = \alpha_2 \rightarrow h''(0) = 2c_1 + 6c_3 = \alpha_2 \)

3. \( h(t) \) is not a general solution because it only works if \( \alpha_1 = 0 \)

3.5.4 Method

- For an \( n \)-th order homogeneous linear O.D.E., we need \( n \) initial conditions to guarantee a solution
- Pick a \( t_0 \) in an interval where \( L \) is normal
  
  \( x(t_0) = \alpha_0, x'(t_0) = \alpha_1, \ldots, x^{n-1}(t_0) = \alpha_{n-1} \)
- Need \( n \) parameters (constants) in the general solution. So, the general solution must satisfy \( h(t) = c_1t + \ldots + c_n h_n(t) \)
  
  \( h(t_0) = c_1 h_1(t_0) + \ldots + c_n h_n(t_0) = \alpha_0 \) through \( h^{n-1}(t_0) = c_1 h_1^{n-1}(t_0) + \ldots + c_n h_n^{n-1}(t_0) = \alpha_{n-1} \)
- If you can find \( c_1 \) through \( c_n \) that satisfy this uniquely for any \( \alpha_0 \) to \( \alpha_{n-1} \), the linear combination of \( h(t) \) must generate the general solution

3.5.5 Wronskian Test

- The Wronskian of \( h_1(t), \ldots, h_n(t) \) is defined as \( W[h_1, h_2, \ldots, h_n](t_0) = \det \begin{bmatrix} h_1(t_0) & h_2(t_0) & \ldots & h_n(t_0) \\ h'_1(t_0) & h'_2(t_0) & \ldots & h'_n(t_0) \\ \vdots & \vdots & \ddots & \vdots \\ h^{n-1}_1(t_0) & h^{n-1}_2(t_0) & \ldots & h^{n-1}_n(t_0) \end{bmatrix} \)
- The Wronskian test is as follows: suppose \( Lx = 0 \) is an \( n \)-th order linear homogeneous O.D.E. that is normal on an interval \( I \). Let \( t_0 \) be in \( I \). The functions \( h_1(t), \ldots, h_n(t) \) generate the general solution if and only if they satisfy \( L_{hi} = 0 \) for \( i = 1, \ldots, n \) and \( W[h_1, h_2, \ldots, h_n](t_0) \neq 0 \)

3.5.6 Example 3

Question: Is \( h(t) \) a general solution to \( D^3x = 0 \) if \( h_1(t) = t^2 \), \( h_2(t) = 1 \), and \( h_3(t) = 3t^2 + 7 \) at \( t_0 = 0 \)

Solution: \( W[t^2, 1, 3t^2 + 7](0) = \det \begin{bmatrix} 0 & 1 & 7 \\ 0 & 0 & 0 \\ 2 & 0 & 6 \end{bmatrix} = 0 \); \( h(t) \) is not the general solution
3.5.7 Example 4

Question: Do solutions of the form \( e^{\lambda t} \) generate a solution of \( (D^2 - 4D + 4)x = 0 \)

Solution:

1. \( (D^2 - 4D + 4)e^{\lambda t} = 0 \rightarrow \lambda^2 e^{\lambda t} - 4\lambda e^{\lambda t} + 4e^{\lambda t} = 0 \)

2. \( (\lambda - 2)^2 e^{\lambda t} = 0 \)

\[ \therefore \quad \lambda = 2 \therefore \quad e^{2t} \text{ is a solution} \]

Is \( c_1 e^{2t} \) the general solution?

1. \( W[e^{2t}, 0](0) = \det \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = 0 \therefore \quad c_1 e^{2t} \text{ is not a general solution} \)

(a) This is to be expected, as it is is a second order O.D.E. and needs 2 solutions

3.6 Test for Linear Independence

- The functions \( h_1(t), h_2(t), \ldots, h_n(t) \) are linearly dependent on an interval \( I \) if there exists constants \( c_1, c_2, \ldots, c_n \) such that with at least one \( c_i \neq 0 \)
- The functions are linearly independent if the constants are all equal to zero

3.6.1 Substituting Values for Testing Linear Independence

Assume I have \( h_1(t) = t \), and \( h_2(t) = t^2 \) on \( -\infty < t < \infty \)

1. Create the possible general solution

   (a) \( c_1 t + c_2 t^2 = 0 \)

2. Check enough values of \( t \) to solve for all the constants. If they all equal zero, it’s independent. Otherwise, it’s dependent.

   (a) Check \( t = 1 \)
       i. \( c_1 + c_2 = 0 \therefore c_1 = -c_2 \)
   (b) Check \( t = 2 \)
       i. \( 2c_1 + 4c_2 = 0 \therefore c_2 = 0 \text{ and } c_1 = 0 \)

Therefore, \( t \) and \( t^2 \) are linearly independent

3.6.2 Taking Derivatives for Testing Linear Independence

Assume I have \( h_1(t) = 1 \), \( h_2(t) = t \), and \( h_3(t) = \cos t \) on \( -\infty < t < \infty \)

1. Write the possible general solution

   (a) \( c_1 + c_2 t + c_3 \cos t = 0 \)

2. Take enough derivatives of the equation as necessary to solve for all constants

   (a) \( \frac{d}{dt} = c_2 - c_3 \sin t = 0 \)
   (b) \( \frac{d^2}{dt^2} = -c_3 \cos t = 0 \)
       i. \( c_1 = 0, c_2 = 0, c_3 = 0 \)

Therefore, \( 1, t, \) and \( \cos t \) are linearly independent
3.6.3 Wronskian Test for Linear Independence

- If \( W \neq 0 \), there is a unique solution
  - Recall that \( c_1 = c_2 = \ldots = c_n = 0 \) is always a solution. As a result, there cannot be “no solution.”
    If \( W = 0 \) it must mean that there are infinitely many solutions.

- If \( W \neq 0 \), there is linear independence
- If \( W = 0 \), there is linear dependence

3.7 Homogeneous Equations: Real Roots

3.7.1 Definitions

- We define the characteristic polynomial of a homogeneous linear O.D.E. as \( P(r) = anr^n + \ldots + a_1r + a_0 \)
  - \( Lx = P(D)x \)
- If the real number \( \lambda \) is a root of \( P(r) \), then \( e^{\lambda t} \) is a solution of \( P(D)x = 0 \)
- The exponential shift is defined as \( P(D)[e^{\lambda t}y] = e^{\lambda t}P(D + \lambda)y \)
- If the real number \( \lambda \) is a root of \( P(r) \) with multiplicity \( k \), then the \( k \) functions \( e^{\lambda t}, te^{\lambda t}, \ldots, t^{k-1}e^{\lambda t} \) are linearly independent solutions of \( P(D)x = 0 \)

3.7.2 Example 1

Solve \((D^3 - D)x = 0\)

1. Substitute \( r \) for \( D \) and factor algebraically
   (a) \( r^3 - r = r(r - 1)(r + 1) \)
2. Check the root values and apply them to \( e^{\lambda t} \) to make a general solution
   (a) \( \lambda = -1, 0, 1 \)
   (b) \( x = c_1 + c_2e^t + c_3e^{-t} \)

3.7.3 Example 2

Solve \((D - 2)^2(D - 1)^3x = 0\)

1. Substitute \( r \) for \( D \) and factor
   (a) \( (r - 2)^2(r - 1)^3 = (r - 2)^2(r - 1)^3 \)
2. Solve for \( r \) which will be \( \lambda \) and check the multiplicity
   (a) \( \lambda = 2 \) with multiplicity 2
   (b) \( \lambda = 1 \) with multiplicity 3
3. Write out the general solution
   (a) \( x = c_1e^{2t} + c_2te^{2t} + c_3e^t + c_4te^t + c_5t^2e^t \)
3.8 Homogeneous Equations: Complex Roots

- Suppose $\alpha \pm \beta i$ are roots of $P(r)$, where $\alpha$ and $\beta$ are real and $\beta \neq 0$. Then $e^{\alpha t} \cos(\beta t)$ and $e^{\alpha t} \sin(\beta t)$ are linearly independent real-value solutions of $P(D)x = 0$

  - Since a $\pm$ indicates two roots, there should be a corresponding cosine and sine function for each multiplicity of $\alpha \pm \beta i$

- An annihilator is an operator, $A(D)$, that satisfies $A(D)E(T) = 0$

3.8.1 Example 1

Solve $(D^2 + 4D + 5)^2 x = 0$

1. Factor to find the roots of the characteristic polynomial
   
   (a) Roots are $-2 \pm i$

2. Write out the general solution
   
   (a) $x = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t + c_3 t e^{-2t} \cos t + c_4 t e^{-2t} \sin t$

3.8.2 Example 2

Find an annihilator with constant coefficients that annihilates $1 + 65e^t \cos(2t)$

1. Recognize that $r = 0$ and $r = 1 \pm 2i$ are the roots

2. To find what will get $1 \pm 2i$, we can solve algebraically
   
   (a) $r [r - (1 + 2i)] [r - (1 - 2i)] = r (r^2 - 2r + 5)$

3. Therefore, the annihilator is $D (D^2 - 2D + 5)$

3.9 Nonhomogeneous Linear Equations: Undetermined Coefficients

To solve a constant-coefficient equation of the form $P(D)x = E(t)$:

1. Solve the roots related homogeneous equation of $P(D)x = 0$ once the original equations is in standard form
   
   (a) Solve for the homogeneous general equation, $h(t)$

2. Find an annihilator$^4$, $A(D)$, for the right side of the nonhomogeneous equation: $A(D)E(t) = 0$
   
   (a) Solve for the roots and find the characteristic polynomial

3. Remove any redundant terms of the particular solution, but be careful

4. Plug in the particular solution, $p(t)$, in for $x$ in the original nonhomogeneous equation
   
   (a) Solve for the constants of $p(t)$

5. Write out the general solution by combining the particular and homogeneous solutions

$^4$An annihilator for $t^k$ is $D^{k+1}$
3.9.1 Example 1

Solve \((D^2 - 2D - 3) x = 6 - 8e^t\)

1. The homogeneous equation is \((D^2 - 2D - 3) x = 0\) with roots \((r - 3)(r + 1)\)
2. Thus the homogeneous general equation can be written as \(h(t) = c_1e^{3t} + c_2e^{-t}\)
3. An annihilator for the right side of the nonhomogeneous equation has roots \(r = 0\) and \(r = 1\) of an annihilator \(A(D) = D(D - 1)\)
4. The particular solution’s annihilator is \((r - 1)(r - 3)(r + 1)\)
5. The particular solution, \(p(t)\), is thus \(p(t) = k_1 + k_2e^t + k_3e^{3t} + k_4e^{-t}\)
   
   (a) This can be simplified to \(p(t) = k_1 + k_2e^t\)
6. Substitute \(x = p(t)\) into the original nonhomogeneous equation
   
   (a) \((D^2 - 2D - 3)(k_1 + k_2e^t) = 6 - 8e^t \rightarrow -3k_1 - 4k_2e^t = 6 - 8e^t\)
   
   (b) Therefore, \(k_1 = -2\) and \(k_2 = 2\)
7. The particular solution is \(p(t) = -2 + 2e^t\)
8. The general solution of the given nonhomogeneous equation is \(x = c_1e^{3t} + c_2e^{-t} - 2 + 2e^t\)

3.9.2 Example 2

Solve \((D^2 - 4) x = e^t + 2e^{2t}\)

1. The homogeneous equation is \((D^2 - 4) x = 0\) with roots \(r = \pm 2\) from \((r - 2)(r + 2)\)
2. The homogeneous general equation is \(h(t) = c_1e^{-2t} + c_2e^{2t}\)
3. An annihilator for the right side of the nonhomogeneous equation has roots \(r = 1, r = 2\) of an annihilator \(A(D) = (D - 1)(D - 2)\)
4. The particular solution’s annihilator is \((D - 1)(D - 2)(D - 2)(D + 2)\), which is identical to \((D - 1)(D - 2)^2(D + 2)\)
   
   (a) Notice here that it’s not as simple as crossing out the roots already used in the homogeneous characteristic polynomial due to the multiplicity of \((D - 2)\)
5. The particular solution, \(p(t)\), is thus \(p(t) = k_1e^t + k_2e^{2t} + k_3te^{2t} + k_4e^{-2t}\)
   
   (a) This can be simplified to \(p(t) = k_1e^t + k_3te^{2t}\)
6. Substitute \(x = p(t)\) into the original nonhomogeneous equation
   
   (a) \((D^2 - 4)(k_1e^t + k_3te^{2t}) = e^t + 2e^{2t} \rightarrow -3k_1e^t + 4k_3e^{2t} = e^t + 2e^{2t}\)
   
   (b) Therefore, \(k_1 = -\frac{1}{3}\), and \(k_3 = \frac{1}{2}\)
7. The particular solution is \(p(t) = -\frac{1}{3}e^t + \frac{1}{2}te^{2t}\)
8. The general solution of the given nonhomogeneous equation is \(x = c_1e^{-2t} + c_2e^{2t} - \frac{1}{3}e^t + \frac{1}{2}te^{2t}\)
3.10 Nonhomogeneous Equations: Variation of Parameters

To solve an \( n \)th-order nonhomogeneous linear equation,

1. Put the equation in standard form
2. Find the general solution of the related homogeneous equation
3. Solve the following system of equations for the unknown constants, where \( q(t) \) is the right side of the equation in standard form:

\[
\begin{align*}
    c_1'(t)h_1(t) + \ldots + c_n'(t)h_n(t) &= 0 \\
    c_1'(t)h_1'(t) + \ldots + c_n'(t)h_n'(t) &= 0 \\
    \ldots & \quad \ldots \quad \ldots & = 0 \\
    c_1'(t)h_1^{(n-2)}(t) + \ldots + c_n'(t)h_n^{(n-2)}(t) &= 0 \\
    c_1'(t)h_1^{(n-1)}(t) + \ldots + c_n'(t)h_n^{(n-1)}(t) &= q(t)
\end{align*}
\]

4. Integrate to find specific values of the constants
5. Find the particular solution of the nonhomogeneous equation
6. The general solution is obtained by adding the particular with the general homogeneous solution

3.10.1 Example 1

Solve \((t^2D^3 + 2tD^2 - 2D)x = t^3, \quad 0 < t < +\infty\)

1. The equation in standard form is \((D^3 + 2t^{-1}D^2 - 2t^{-2}D)x = t\)
2. The related homogeneous equation is \((D^3 + 2t^{-1}D^2 - 2t^{-2}D)x = 0\)
   
   (a) The general solution is \(h(t) = c_1t^2 + c_2\frac{1}{t} + c_3\)

   (b) Therefore, we look for a particular solution in the form \(x = p(t)\) with \(p(t) = c_1(t)t^2 + c_2(t)\frac{1}{t} + c_3(t)\)
3. We must solve the following system:

\[
\begin{align*}
    t^2c_1'(t) + \frac{1}{t}c_2'(t) + c_3'(t) &= 0 \\
    2tc_1'(t) - \frac{1}{t^2}c_2'(t) + 0 &= 0 \\
    2c_1'(t) + \frac{2}{t^3}c_2'(t) + 0 &= t
\end{align*}
\]

4. As a result, \(c_1'(t) = \frac{2}{t}, \ c_2'(t) = \frac{t^4}{3}, \) and \(c_3'(t) = -\frac{t^3}{2}\)
5. We integrate these constants to get \(c_1(t) = \frac{t^2}{12}, \ c_2(t) = \frac{t^5}{15}, \) and \(c_3(t) = -\frac{t^4}{8}\)
6. The particular solution is thus \(p(t) = \frac{t^4}{12} + \frac{t^4}{15} - \frac{t^4}{8} - \frac{t^4}{40}\)
7. The general solution is \(x = c_1t^2 + c_2\frac{1}{t} + c_3 + \frac{t^4}{40}\)
4 The Laplace Transform

4.1 Definitions and Basic Calculations

- The Laplace Transform, a linear operator, is defined as
  \[ F(s) = \int_0^\infty e^{-st} f(t) \, dt \]
- We write the Laplace Transform as
  \[ F(s) = \mathcal{L}[f(t)] \]
- If \( F(s) = \mathcal{L}[f(t)] \), then we say that \( f(t) \) is the inverse Laplace transform, written as
  \[ f(t) = \mathcal{L}^{-1}[F(s)] \]
- Linearity:
  \[ \mathcal{L}[c_1f_1(t) + c_2f_2(t)] = c_1\mathcal{L}[f_1(t)] + c_2\mathcal{L}[f_2(t)] \]
  \[ \mathcal{L}^{-1}[c_1F_1(s) + c_2F_2(s)] = c_1\mathcal{L}^{-1}[F_1(s)] + c_2\mathcal{L}^{-1}[F_2(s)] \]
- First Shift Formula:
  \[ \mathcal{L}[e^{\alpha t}f(t)] = F(s - \alpha), \text{ where } F(s) = \mathcal{L}[f(t)] \]
  \[ \mathcal{L}^{-1}[F(s)] = e^{\alpha t}\mathcal{L}^{-1}[F(s + \alpha)] \]
- Second Differentiation Formula:
  \[ \mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}\mathcal{L}[f(t)] \]
4.2 Basic Transforms and Inverse Transforms

\[ \mathcal{L} [e^{\lambda t}] = \frac{1}{s - \lambda} \quad \text{and} \quad \mathcal{L}^{-1} \left[ \frac{1}{s - \lambda} \right] = e^{\lambda t} \]

\[ \mathcal{L} [1] = \frac{1}{s} \quad \text{and} \quad \mathcal{L}^{-1} \left[ \frac{1}{s} \right] = 1 \]

\[ \mathcal{L} [t^n] = \frac{n!}{s^{n+1}} \quad \text{and} \quad \mathcal{L}^{-1} \left[ \frac{1}{s^n} \right] = \frac{t^{n-1}}{(n-1)!} \]

\[ \mathcal{L} [\cos (\beta t)] = \frac{s}{s^2 + \beta^2} \quad \text{and} \quad \mathcal{L}^{-1} \left[ \frac{s}{s^2 + \beta^2} \right] = \cos (\beta t) \]

\[ \mathcal{L} [\sin (\beta t)] = \frac{\beta}{s^2 + \beta^2} \quad \text{and} \quad \mathcal{L}^{-1} \left[ \frac{1}{s^2 + \beta^2} \right] = \frac{1}{\beta} \sin (\beta t) \]

4.3 Using the Laplace Transform to Solve Initial-Value Problems

1. Transform both sides of the O.D.E., incorporating the initial data by means of the first differentiation formula,

\[ \mathcal{L} \left[ D^k x \right] = s^k \mathcal{L} [x] - s^{k-1} x(0) - s^{k-2} x'(0) - \ldots - x^{(k-1)}(0) \]

2. Solve algebraically for \( \mathcal{L} [x] \) in terms of \( s \)

3. Obtain \( x \) as the inverse Laplace transform of \( \mathcal{L} [x] \)

(a) The linearity of the Laplace (and inverse Laplace) transform as well as partial fractions are typically used in this step

4.4 Functions Defined in Pieces

Any function \( g(t) \) that is defined in pieces can be written as a sum of terms of the form \( u_a(t) f(t) \), where \( u_a(t) \) is the following unit step function,

\[ u_a(t) = \begin{cases} 
0 & \text{if } t < a \\
1 & \text{if } t \geq a 
\end{cases} \]

To obtain this expression for \( g(t) \), we note that:

1. \( u_a(t) \) switches on at \( t = a \)

2. At every interface of \( g(t) \), we want to switch off the immediately preceding formula and switch on the next one

Second Shift Formula\(^5\):

\[ \mathcal{L} [u_a(t) f(t)] = e^{-as} \mathcal{L} [f(t + a)] \]

\[ \mathcal{L}^{-1} \left[ e^{-as} F(s) \right] = u_a(t) f(t - a) \text{, where } f(t) = \mathcal{L}^{-1} [F(s)] \]

Also, a helpful Laplace Transform to remember is the following, where \( n \) is a constant:

\[ \mathcal{L} [u_n(t)] = \frac{e^{-ns}}{s} \]

\( ^5 \) For a problem like \( \mathcal{L} [u_a(t) \sin(t)] \), the second shift formula yields \( e^{-as} \mathcal{L} [\sin(t + \pi)] \). To solve this, we can use the trigonometric properties to note that \( \sin(t + \pi) \) is simply \( -\sin(t) \), and, therefore, the transform is \( \frac{e^{-\pi s}}{s+1} \).
4.4.1 Example 1

Find the Laplace transform of \( f(t) = |t - 3| \)

1. Define the function in a step-wise fashion

\[
f(t) = \begin{cases} 
-t + 3 & \text{if } t < 3 \\
 t - 3 & \text{if } t \geq 3
\end{cases}
\]

2. Write out \( f(t) \) with the step function

(a) \( f(t) = -t + 3 + u_3(t) \left| t - 3 + t - 3 \right| \)

i. This simplifies to \( f(t) = -t + 3 + 2u_3(t) \left| t - 3 \right| \)

3. Take the Laplace transform

(a) \( \mathcal{L} \left[ f(t) \right] = \mathcal{L} \left[ -t + 3 \right] + 2 \mathcal{L} \left[ u_3(t) \left| t - 3 \right| \right] \)

4. Solve the Laplace transform

(a) \( -\frac{1}{s^2} + \frac{3}{s} + \frac{2e^{-3s}}{s^2} \)

4.4.2 Example 2

Find the inverse Laplace transform of \( \frac{e^{-s}}{s(s+1)} \)

1. Find the inverse Laplace transform of the function without \( e^{-s} \) by breaking it into partial fractions

(a) \( \frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \)

i. This becomes \( \frac{1}{s} - \frac{1}{s+1} \)

2. Solve the inverse Laplace transform

(a) \( \mathcal{L}^{-1} \left[ \frac{1}{s} \right] - \mathcal{L}^{-1} \left[ \frac{1}{s+1} \right] = 1 - e^{-t} \)

3. Use the second shift formula to account for \( e^{-s} \)

(a) \( u_1(t) \left[ 1 - e^{-(t-1)} \right] \)

4.4.3 Cautionary Notes

1. When solving Laplace transforms, it is important to note the following \textit{false} property. Laplace transforms can be thought to have analogous properties with integrals, so: \( \mathcal{L} \left[ A \cdot B \right] \neq \mathcal{L} \left[ A \right] \cdot \mathcal{L} \left[ B \right] \)

(a) The same applies for inverse Laplace transforms

2. When breaking up an absolute value function into a step-wise function, the negated version is valid before the first break-point, and the function with the absolute value bars removed is valid at and after the break-point

3. If the following type of Laplace transform is seen, recognize you can factor out constants to solve it: \( \mathcal{L} \left[ e^{-(t+3)} \right] \)

(a) To solve this, note that the inside is \( e^{-3} \cdot e^{-t} \), so it is the same as \( e^{-3} \mathcal{L} \left[ e^{-t} \right] = \frac{1}{e^3(s+1)} \)
4.5 Convolution

- Convolution shares the distributive, associative, and commutative properties of multiplication but does not share all properties.
  - For instance, a convolution with 1 will produce a different product
- The convolution is defined as
  \[
  (f * g)(t) = \int_0^t f(t-u)g(u)\,du
  \]
- The convolution formula for Laplace transforms is as follows but is frequently used in its inverse form
  \[
  \mathcal{L}[(f * g)(t)] = \mathcal{L}[f(t)]\mathcal{L}[g(t)]
  \]
  \[
  \mathcal{L}^{-1}[F(s)G(s)] = \mathcal{L}^{-1}[F(s)] \ast \mathcal{L}^{-1}[G(s)]
  \]

5 Spring Models

- The general equation for a spring is the following where \(m\) is mass, \(b\) is the damping coefficient, \(k\) is the spring constant, and \(E(t)\) is an applied external force:
  \[
  [mD^2 + bD + k]x = E(t)
  \]
- A spring is underdamped if \(b^2 - 4mk < 0\)
- A spring is overdamped if \(b^2 - 4mk > 0\)
- A spring is critically damped if \(b^2 - 4mk = 0\)

6 Linear Equations: Power Series

6.1 Properties of Series

- The following is the additive property of series
  \[
  \sum_{k=0}^{\infty} b_k (t-t_0)^k + \sum_{k=0}^{\infty} c_k (t-t_0)^k = \sum_{k=0}^{\infty} (b_k + c_k)(t-t_0)^k
  \]
- Series with constants in the summands can be factored out of the series
- The following is the property of taking a derivative of a series
  \[
  \frac{d}{dt} \left( \sum_{k=0}^{\infty} b_k (t-t_0)^k \right) = \sum_{k=1}^{\infty} kb_k (t-t_0)^{k-1}
  \]
- To change the index, substitutions can be made
- If \(a_n(t_0) \neq 0\), it is said to be an ordinary point. If \(a_n(t_0) = 0\), it is said to be a singular point
6.2 Solutions about Ordinary Points

- When solving linear differential equations, the following substitution can be made the following series, where \( t_0 \) is where the series is centered around:

\[
x(t) = \sum_{k=0}^{\infty} b_k (t - t_0)^k
\]

Procedure:

1. Substitute the series equivalent for \( x(t) \) into the given differential equation
2. Perform index substitutions to match up the powers of \( t \), separating out extra terms to match up the limits of summation if necessary
3. Set each combined coefficient of the resulting series equal to zero to obtain a recurrence relation
   (a) Make sure to write the range of index values for the recurrence relation (see 6.4.1)
4. If a pattern is visible, write down an explicit formula
5. Substitute the coefficients back into the the series definition of \( x(t) \) to obtain a power series expression for the solution

6.2.1 Example 1

Write out the terms up to \( t^5 \) of the general solution for \([D^2 + D + t] x = 0\) about \( t_0 = 0 \)

1. Substitute in \( x(t) = \sum_{k=0}^{\infty} b_k t^k \)
   (a) \( x'(t) = \sum_{k=1}^{\infty} k b_k t^{k-1} \) and \( x''(t) = \sum_{k=2}^{\infty} k(k-1) b_k t^{k-2} \)
   (b) The new equation is thus \( \sum_{k=2}^{\infty} k(k-1) b_k t^{k-2} + \sum_{k=1}^{\infty} k b_k t^{k-1} + \sum_{k=0}^{\infty} b_k t^{k+1} = 0 \)

2. Set all the powers of \( t \) to be the same by swapping the index values
   (a) The first index will change to \( j = k - 2 \), the second to \( j = k - 1 \), and the third to \( j = k + 1 \)
   (b) The new equation is thus \( \sum_{j=0}^{\infty} (j+2)(j+1)b_{j+2}t^j + \sum_{j=0}^{\infty} (j+1)b_{j+1}t^j + \sum_{j=1}^{\infty} b_{j-1}t^j = 0 \)

3. In order to easily do arithmetic with series, the index values must be the same. Take out as many terms as necessary to match the index values
   (a) The new equation is thus \( (2b_2 + b_1) + \sum_{j=1}^{\infty} [(j+2)(j+1)b_{j+2} + (j+1)b_{j+1} + b_{j-1})t^j] = 0 \)

4. Solve for zero by setting each set to zero
   (a) \( 2b_2 + b_1 = 0 \) and \( b_{j+2} = \frac{-b_{j-1} - (j+1)b_{j+1}}{(j+2)(j+1)}, j = 1, 2, 3, ... \)
      i. The index for \( j \) starts at a value that does not create a negative index and, in this case, a value of \( b_3 \) since \( b_1 \) and \( b_2 \) were already defined by a separate equation
   (b) The first equation can be rewritten as \( b_2 = -\frac{b_1}{2} \)

5. Since the goal is to obtain up to the \( t^5 \) term, we will need up to calculate \( b_0, b_1, b_2, b_3, b_4, \) and \( b_5 \)
   (a) \( b_0 = b_0 \) and \( b_1 = b_1 \)
      i. \( b_2 = -\frac{b_1}{2} \), as already stated
(b) \( b_3 = \frac{-b_0 - 2b_2}{6} = \frac{-b_0 + b_1}{6} \)
(c) \( b_4 = \frac{-b_1 - 3b_3}{12} = -\frac{3b_1 + b_0}{24} \)
(d) \( b_5 = \frac{-b_2 - 4b_4}{20} = \frac{b_1}{20} - \frac{b_0}{120} \)

6. Plug this into the series defined in step 1

(a) \( x = b_0 t^0 + b_1 t^1 - \frac{b_1}{2} t^2 + \left( -\frac{b_0}{6} + \frac{b_1}{6} \right) t^3 + \left( -\frac{1}{8} b_1 + \frac{b_0}{24} \right) t^4 + \left( \frac{b_1}{20} - \frac{b_0}{120} \right) t^5 \)

7. Collect the terms into groups of \( b_0 \) and \( b_1 \)

(a) \( x = b_0 \left( 1 - \frac{1}{6} t^3 + \frac{1}{24} t^4 - \frac{1}{120} t^5 + \ldots \right) + b_1 \left( t - \frac{1}{2} t^2 + \frac{1}{6} t^3 - \frac{1}{8} t^4 + \frac{1}{20} t^5 - \ldots \right) \)

6.2.2 Example 2

Find the recurrence relation of \( [D^2 - (t - 2)] x = 0 \) subject to the condition at \( t = 2 \), where \( x(2) = 1 \) and \( x'(2) = 0 \)

1. Since \( t_0 = 2 \), the power series representation of \( x(t) \) is \( x(t) = \sum_{k=0}^{\infty} b_k (t - 2)^k \)
2. The given conditions say that \( b_0 = 1 \) and \( b_1 = 0 \)
3. Since \( t_0 = 2 \) is a singular point, we make the substitution of \( T = t - 2 \)

(a) This makes the series representation of \( x(t) \) as \( x(T + 2) = \sum_{k=0}^{\infty} b_k T^k \)
4. Substituting the equation for \( x(t) \) into the O.D.E. is \( \sum_{k=2}^{\infty} k(k-1) b_k T^{k-2} - \sum_{k=0}^{\infty} b_k T^{k+1} = 0 \)
5. To make the powers of \( T \) the same, the substitution of \( j = k - 2 \) is used for the first summation and \( j = k + 1 \) for the second summation

(a) This yields \( 2b_2 + \sum_{j=1}^{\infty} \left[ (j + 2)(j + 1) b_{j+2} - b_{j-1} \right] T^j = 0 \)
6. The previous equation implies that \( b_2 = 0 \) and \( b_{j+2} = \frac{b_{j-1}}{(j + 2)(j + 1)} \), \( j = 1, 2, 3, \ldots \)

7 Linear Systems of Differential Equations

7.1 Linear Systems of O.D.E.’s

- To multiply matrices, the following general equation applies,

\[
\begin{bmatrix}
  a_{11} & \ldots & a_{1m} \\
  \ldots & \ldots & \ldots \\
  a_{n1} & \ldots & a_{nm}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  \ldots \\
  x_m
\end{bmatrix} =
\begin{bmatrix}
  a_{11}x_1 + \ldots + a_{1m}x_m \\
  \ldots \\
  a_{n1}x_1 + \ldots + a_{nm}x_m
\end{bmatrix}
\]
A system of O.D.E.’s is **linear** if it can be written in the form
\[ x_1' = a_{11}x_1 + \ldots + a_{1n}x_n + E_1(t) \]
\[ \quad \vdots \]
\[ x_n' = a_{n1}x_1 + \ldots + a_{nn}x_n + E_n(t) \]
- The coefficients of \( a_{ij} \) may be constants or functions of \( t \)

- The order of a system of linear O.D.E.s is the number of unknowns it has

- The matrix form of the above system is
\[
-D\vec{x} = A\vec{x} + E(t) \\
\text{For this, } \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix}, \quad \text{and } E(t) = \begin{bmatrix} E_1(t) \\ \vdots \\ E_n(t) \end{bmatrix}
\]

- The order of the system is \( n \), and the system is **homogeneous** if \( E(t) = 0 \)

7.1.1 Example 1

**Question**: Write \( x' = -ty - z + t, \ y' = -\frac{x}{t} - \frac{z}{t} + 1, \) and \( z' = x-ty \) in matrix form and label if it’s homogeneous as well as the order

**Answer**: The order is 3 for this nonhomogeneous system. The matrix representation is,
\[
D\vec{x} = A\vec{x} + E(t) \\
\text{where } \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad A = \begin{bmatrix} 0 & -t & -1 \\ -t & 0 & -t^{-1} \\ 1 & -t & 0 \end{bmatrix}, \quad \text{and } E(t) = \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix}
\]

7.1.2 Example 2

Write the general solution of the equivalent system for \((D^2 - 1)x = t\)

1. Set \( x_1 = x \) and \( x_2 = x' \)
   - (a) By extension, \( x' = x_1' = x_2 \)
2. For the given O.D.E., the equivalent system is \( x_1' = x_2 \) and \( x_2' = t + x_1 \)
3. Solve the O.D.E.
   - (a) This will yield \( x(t) = c_1e^t + c_2e^{-t} - t \)
4. Rewrite the solution as a solution set for \( x_1 \) and \( x_2 \) to yield the general solution
   - (a) \( x_1 = x = c_1e^t + c_2e^{-t} - t \)
   - (b) \( x_2 = x' = c_1e^t - c_2e^{-t} - 1 \)
5. Set up the equivalent system in the form \( Dx = Ax + E(t) \)
   - (a) \( D\vec{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\vec{x} + \begin{bmatrix} 0 \\ t \end{bmatrix}, \quad \text{where } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \)
6. Write the general solution of the equivalent system
   - (a) \( \vec{x} = c_1\begin{bmatrix} e^t \\ e^t \end{bmatrix} + c_2\begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} + \begin{bmatrix} -t \\ -1 \end{bmatrix} \)
7.2 Linear Independence of Vectors

- Vectors are considered linearly dependent if there exists at least one constant that does not equal zero so that, \( c_1 \vec{v}_1 + \ldots + c_k \vec{v}_k = \vec{0} \)
- Vectors are linearly independent if all constants equal zero

7.3 Homogeneous Systems, Eigenvalues, and Eigenvectors

- To find the eigenvalues of a matrix, find when
  \[ \det(A - \lambda I) = 0 \]
- The corresponding eigenvectors of \( A \) are the nonzero solutions of
  \[ (A - \lambda I) \vec{v} = \vec{0} \]
- If \( \lambda \) is an eigenvalue of \( A \) and \( \vec{v} \) is an eigenvector of \( A \) corresponding to \( \lambda \), then there is a solution,
  \[ \vec{x} = e^{\lambda t} \vec{v} \]

7.4 Homogeneous Systems: Real Roots

1. Find the roots of the characteristic polynomial \( \det(A - \lambda I) \)
2. For each real root of \( \lambda \), find as many linearly independent eigenvectors corresponding to \( \lambda \) as possible
   (a) If it is consistent and safe to set each noncorner variable equal to 1 while taking all other noncorner variables equal to 0
3. Associate each eigenvector to the solution \( e^{\lambda t} \vec{v} \)
4. Combine the solutions of \( D \vec{x} = A \vec{x} \)

7.5 Homogeneous Systems: Complex Roots

1. Do the same procedure as when solving a homogeneous system with real roots
2. When a complex root pair is obtained, work with one of the roots as if it were real to obtain complex solutions
   (a) Associate to the pair of eigenvalues the real and imaginary parts of these complex solutions
3. The real and imaginary parts of the complex solution are independent initial vectors (without the factor of \( i \))

Typically, Euler’s Formula must be used:

\[ e^{it} = \cos t + i \sin t \]
7.6 Double Roots and Matrix Products

- If \( B \) is an \( n \times n \) matrix, \( C \) is an \( m \times k \) matrix, and \( \vec{v} \) is a \( k \)-vector, then
  \[ (BC) \vec{v} = B (C \vec{v}) \]

- If \( \lambda \) is a double root of the characteristic polynomial \( \det (A - \lambda I) \), then a generalized eigenvector corresponding to the eigenvalue \( \lambda \) is a vector \( \vec{v} \neq \vec{0} \) satisfying,
  \[ (A - \lambda I)^2 \vec{v} = \vec{0} \]

- Associated to each generalized eigenvector is a solution,
  \[ \vec{h}(t) = e^{\lambda t} \left( \vec{v} + t [A - \lambda I] \vec{v} \right) \]

7.7 Homogeneous Systems: Multiple Roots

To solve \( D \vec{x} = A \vec{x} \), where \( A \) is an \( n \times n \) matrix:

1. Find the roots of the characteristic polynomial \( \det (A - \lambda I) \) and their multiplicities
2. For each real root \( \lambda \) of multiplicity \( m \), find \( m \) linearly independent solutions of \( (A - \lambda I)^m \vec{v} = \vec{0} \)
   
   - (a) Associate to each of these generate eigenvectors the solution of \( D \vec{x} = A \vec{x} \)
     \[ \vec{h}(t) = e^{\lambda t} \left( \vec{v} + t [A - \lambda I] \vec{v} + \frac{1}{2} t^2 [A - \lambda I]^2 \vec{v} + ... + \frac{1}{(m-1)!} t^{m-1} [A - \lambda I]^{m-1} \vec{v} \right) \]

3. For each pair of complex roots \( \alpha \pm \beta i \), work with one of the roots as if it were real as before
4. Combine the solutions to form \( \vec{x} \)

7.8 Nonhomogeneous Systems

To solve \( D \vec{x} = A \vec{x} + \vec{E}(t) \):

1. Find the general solution of the related homogeneous system, \( \vec{H}(t) \)
2. Look for a particular solution by solving the following equation
   \[ c_1'(t) \vec{h}_1(t) + ... + c_n'(t) \vec{h}_n(t) = \vec{E}(t) \]
3. Integrate \( c_1', c_2', ..., c_n' \) to obtain the particular solution
4. Write the general solution by adding the homogeneous and particular solutions

8 Qualitative Theory of Systems of O.D.E.’s

8.1 Stability in First-Order Equations

- For a given phase portrait, if the arrowheads point towards \( x = c \) on both sides, \( x = c \) is an attractor
- If the arrowheads point in opposite directions, \( x = c \) is a repeller
- An equation is autonomous if it’s in the form \( \frac{dx}{dt} = f(x) \)
8.2 Phase Portraits of Linear Systems

8.2.1 Two Distinct Real and Nonzero Eigenvalues

1. Plot two lines by plotting a point from each eigenvector and extrapolating a line from each through the origin

2. Add arrowheads to each line by looking at the corresponding eigenvalue
   
   (a) If the eigenvalue is negative, it is an attractor
   
   (b) If the eigenvalue is positive, it is a repeller

3. For the case of $\lambda_1 < 0 < \lambda_2$,

   ![Figure 4.9](image)

   1. For the case of $0 < \lambda_1 < \lambda_2$ and $\lambda_1 < \lambda_2 < 0$,
      
      (a) If the eigenvalues are positive, they’ll be pointing away from the origin
      
      (b) If the eigenvalues are negative, they’ll be pointing toward the origin

   ![Figure 4.11](image)

8.2.2 Two Equal Real and Nonzero Eigenvalues

1. If $A$ has two independent eigenvectors, all lines go through the origin. This should look somewhat like an asterisk (*)
   
   (a) If the eigenvalues are negative, the arrows are inward
   
   (b) If the eigenvalues are positive, the arrows are outward

---

6 All graphs taken from “Differential Equations: A First Course,” 3rd Edition by Guterman and Nitecki
1. If $A$ has one independent eigenvector, you get one of the following pictures

(a) To test if it is clockwise or counterclockwise, check $\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ by plugging it into $A\vec{x}$

i. If $A\vec{x}$ has the first element positive, this means $\frac{dx}{dt} > 0$, so there is clockwise motion

ii. The converse statement indicates counterclockwise motion

8.2.3 Two Eigenvalues with One Zero-Eigenvalue

1. The phase portrait will have a line through the origin based on the eigenvector that corresponds to the zero-eigenvalue

2. The integral curves will be perpendicular to the first line based on the nonzero eigenvalue’s corresponding eigenvector

(a) The arrowheads will point away from the origin if the eigenvalue is positive

(b) The arrowheads will point toward the origin if the eigenvalue is negative

Phase portrait for $dx - Ax, \lambda_1 = 0, \lambda_2 > 0$. 

8.2.4 Two Zero-Eigenvalues

1. Plot a line from the eigenvector corresponding to the zero-eigenvalue
2. All integral curves are parallel to this line through the origin

8.2.5 Eigenvalues of the Form $\alpha \pm \beta i$

1. If the real part of the eigenvalues is negative, the integral curve spirals inward
2. If the real part of the eigenvalues is positive, the integrals spirals outward

(a) To determine the counterclockwise or clockwise orientation, check $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as before

8.2.6 Imaginary Eigenvalues with $\alpha = 0$

1. The plots will be ovals or circles around the origin
2. It is necessary to check if it’s counterclockwise or clockwise, as before

8.3 Linearization and Stability of Equilibria

- If every eigenvalue of $A$ has negative real parts, then $\begin{bmatrix} x \\ y \end{bmatrix} = 0$ is an attractor
- If every eigenvalue of $A$ has positive real parts, then $\begin{bmatrix} x \\ y \end{bmatrix} = 0$ is a repeller
- If an eigenvalue of $A$ has real parts of opposite signs or the presence of zero or pure imaginary eigenvalues, $\begin{bmatrix} x \\ y \end{bmatrix} = 0$ is neither an attractor or repeller
- The Hartman-Grobman Theorem states that if the linearization matrix $A$ has no zero or pure imaginary eigenvalues, then the phase portrait near the equilibrium can be obtained from the phase portrait of the linear system via a continuous change of coordinates
- The linearization matrix is

$$A = \begin{bmatrix} f_x(a, b) & f_y(a, b) \\ g_x(a, b) & g_y(a, b) \end{bmatrix}$$
8.3.1 Example 1

8.4 Constants of Motion

- To test if $E(x, y)$ is a constant of motion,
  \[ f(x, y) \frac{\partial E}{\partial x} + g(x, y) \frac{\partial E}{\partial y} = 0 \]

- The discriminant of $E(x, y)$ is
  \[ \Delta(x, y) = \frac{\partial^2 E}{\partial^2 x} \cdot \frac{\partial^2 E}{\partial^2 y} - \left( \frac{\partial^2 E}{\partial x \partial y} \right)^2 \]

- If $\Delta > 0$, then there is a strict local extremum
  - If $\frac{\partial^2 E}{\partial^2 x} < 0$, then the point is a maximum
  - If $\frac{\partial^2 E}{\partial^2 x} > 0$, then the point is a minimum

- If $\Delta < 0$, then $E(x, y)$ has a saddle point

- If $\Delta = 0$, the point is a degenerate critical point where no conclusion can be made

Note: The second derivative test is used as follows. Firstly, find the equilibrium points of the original system. Then, do the second derivative test on all the critical points of $E$. This will tell you the quality (stable/unstable) of any equilibrium points of the original system that are also critical points of $E$.

8.5 Lyapunov Functions

- The function $E(x, y)$ is a lyapunov function if
  \[ f(x, y) \frac{\partial E}{\partial x} + g(x, y) \frac{\partial E}{\partial y} \leq 0 \]

- If $(x_0, y_0)$ is a relative minimum for $E$, then this point is an attractor
- If $(x_0, y_0)$ is a relative maximum for $E$, then this point is a repeller
- If $(x_0, y_0)$ is a saddle point for $E$, then it is unstable but neither an attractor or repeller

8.6 Limit Cycles and Chaos

- There are no closed integral curves if $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ is always positive or always negative in a simply connected domain (e.g. first quadrant)
- There are no closed integral curves if there are real eigenvalues of opposite signs
- If all equilibria lie on the axes, there is no integral curve
- The Poincaré-Bendixson Theorem states that there is a closed integral curve if two circles are drawn of different radii that have $\frac{dr}{dt}$ of opposite signs. Also,
  \[ \frac{dx}{dt} = \frac{1}{r} \left( x \frac{dx}{dt} + y \frac{dy}{dt} \right) \] (Circle centered around origin)