Transport Phenomena I

Andrew Rosen

October 27, 2013

Contents

1 Dimensional Analysis and Scale-Up .................................................. 3
  1.1 Procedure ......................................................................................... 3
  1.2 Example .......................................................................................... 3

2 Introduction to Fluid Mechanics ............................................................. 4
  2.1 Definitions and Fundamental Equations ............................................. 4
  2.2 Hydrostatics .................................................................................... 4
    2.2.1 Pressure Changes with Elevation ................................................ 4
    2.2.2 U-Tube Example ....................................................................... 5
    2.2.3 Force on a Dam .......................................................................... 5
    2.2.4 Archimedes’ Law ...................................................................... 6
    2.2.5 Buoyancy Example .................................................................... 6
    2.2.6 Pressure Caused by Rotation ..................................................... 6

3 Shear Stress and the Shell Momentum Balance ......................................... 6
  3.1 Types of Stress .................................................................................. 6
  3.2 Shell Momentum Balance ................................................................... 7
    3.2.1 Procedure .................................................................................. 7
    3.2.2 Boundary Conditions .................................................................. 7
  3.3 Flow of a Falling Film ....................................................................... 8
  3.4 Flow Through a Circular Tube ........................................................... 9
  3.5 Flow Through an Annulus .................................................................. 10

4 Mass, Energy, and Momentum Balances .................................................. 12
  4.1 Mass and Energy Balance ................................................................. 12
    4.1.1 Mass-Energy Balance ................................................................. 12
    4.1.2 Example .................................................................................... 12
  4.2 Bernoulli Equation ........................................................................... 13
  4.3 Momentum Balance ......................................................................... 13

5 Differential Equations of Fluid Mechanics ............................................... 14
  5.1 Vectors and Operators ..................................................................... 14
    5.1.1 Dot Product ............................................................................... 14
    5.1.2 Cross Product ........................................................................... 14
    5.1.3 Gradient ................................................................................... 14
    5.1.4 Divergence ............................................................................... 14
    5.1.5 Curl ........................................................................................ 14
    5.1.6 Laplacian ............................................................................... 14
  5.2 Solution of the Equations of Motion .................................................... 14
  5.3 Procedure for Using Navier-Stokes Equation ....................................... 15
  5.4 Flow Through a Circular Tube Using Navier-Stokes ........................... 15
  5.5 Flow Through a Heat-Exchanger ........................................................ 16
6 Velocity Distributions with More Than One Variable 17
6.1 Time-Dependent Flow of Newtonian Fluids ............................................. 17
6.1.1 Definitions ......................................................................................... 17
6.1.2 Flow near a Wall Suddenly Set in Motion ........................................... 17
6.2 The Potential Flow and Streamfunction .................................................. 19
6.3 Solving Flow Problems Using Streamfunctions ......................................... 19
6.3.1 Overview ........................................................................................... 19
6.3.2 Creeping Flow Around a Sphere ......................................................... 20
6.4 Solving Flow Problems with Potential Flow ............................................. 23
6.4.1 Overview ........................................................................................... 23
6.4.2 Steady Potential Flow Around a Stationary Sphere ............................... 23
6.5 Boundary Layer Theory ........................................................................... 25
7 Flow in Chemical Engineering Equipment ................................................. 28
8 Appendix .............................................................................................. 29
8.1 Newton’s Law of Viscosity ..................................................................... 29
8.1.1 Cartesian ............................................................................................. 29
8.1.2 Cylindrical .......................................................................................... 29
8.1.3 Spherical ............................................................................................. 29
8.2 Gradient .................................................................................................. 29
8.3 Divergence ............................................................................................... 29
8.4 Curl ......................................................................................................... 30
8.5 Laplacian ................................................................................................. 30
8.6 Continuity Equation ................................................................................ 30
8.7 Navier-Stokes Equation ......................................................................... 31
8.8 Stream Functions and Velocity Potentials .............................................. 31
8.8.1 Velocity Components ......................................................................... 31
8.8.2 Differential Equations ....................................................................... 32
1 Dimensional Analysis and Scale-Up

1.1 Procedure

1. To solve a problem using dimensional analysis, write down all relevant variables, their corresponding units, and the fundamental dimensions (e.g. length, time, mass, etc.)

   (a) Some important reminders: a newton (N) is equivalent to kg m s$^{-2}$, a joule (J) is equivalent to kg m$^2$ s$^{-2}$, and a pascal (Pa) is equivalent to kg m$^{-1}$ s$^{-2}$

   (b) $\mu = [ML^{-1}t^{-1}]$, $P = [ML^{-1}t^{-2}]$, $F = [MLt^{-1}]$, Power = $[ML^2t^{-3}]$

   (c) This is a helpful conversion: 32.2 \( \frac{lb \cdot ft/s^2}{lbf} \)

2. The number of dimensionless groups that will be obtained is equal to the number of variables minus the number of unique fundamental dimensions.

3. For each fundamental dimension, choose the simplest reference variable

   (a) No two reference variables can have the same fundamental dimensions.

4. Solve for each fundamental dimension using the assigned reference variables.

5. Solve for the remaining variables using the previously defined dimensions.

6. The dimensionless groups can then be computed by manipulating the algebraic equation created in Step 5.

7. If scaling is desired, one can manipulate the constant dimensionless equations.

1.2 Example

Consider a fan with a diameter $D$, rotational speed $\omega$, fluid density $\rho$, power $P$, and volumetric flow rate of $Q$. To solve for the dimensionless groups that can be used for multiplicative scaling, we implement the steps from Section 1.1:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Unit</th>
<th>Fundamental Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D$</td>
<td>m</td>
<td>$L$</td>
</tr>
<tr>
<td>$\omega$</td>
<td>rad/s</td>
<td>$t^{-1}$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>kg/m$^3$</td>
<td>$m \cdot L^{-3}$</td>
</tr>
<tr>
<td>$Q$</td>
<td>m$^3$/s</td>
<td>$L^3 \cdot t^{-1}$</td>
</tr>
<tr>
<td>$P$</td>
<td>W</td>
<td>$m \cdot L^2 \cdot t^{-3}$</td>
</tr>
</tbody>
</table>

1. Create a table of the variables, as shown above.

2. The number of dimensionless groups can be found by: 5 variables - 3 fundamental dimensions = 2 dimensionless groups.

3. The reference variables will be chosen as $D$, $\omega$, and $\rho$ for length, time, and mass, respectively.

4. Each fundamental dimension can be represented by $L = [D]$, $m = [\rho D^3]$, and $t = [\omega^{-1}]$.

5. The remaining variables can be solved by $[Q] = [D^3 \omega] = L^3 \cdot t^{-1}$ and $[P] = [\rho P^3 D^5 \omega^3] = m \cdot L^2 \cdot t^{-3}$.

6. The two dimensionless groups can therefore be found as $N_1 = QD^{-3} \omega^{-1} \equiv Q^{-1} D^3 \omega$ and $N_2 = P \rho^{-1} D^{-5} \omega^{-3} \equiv P^{-1} \rho D^3 \omega^3$.

---

1 An important note about variables: I shall define anything with a carat (e.g. $\hat{m}$) as a quantity per unit mass and anything with an overdot (e.g. $\dot{m}$) as a quantity per time. The only exception is that $Q$ shall be used in place of $\dot{V}$ for volumetric flow rate. I will define vectors with overarrows (e.g. $\vec{v}$) and tensors with boldfaces (e.g. $\varphi$).
2 Introduction to Fluid Mechanics

2.1 Definitions and Fundamental Equations

- Newtonian fluids exhibit constant viscosity but virtually no elasticity whereas a non-Newtonian fluid does not have a constant viscosity and/or has significant elasticity
- Pressure is also equal to a force per unit area but is involved with the compression of a fluid, as is typically seen in hydrostatic situations
  - Pressure is independent of the orientation of the area associated with it
  - Additionally, \( F = P \, dA \)
- For a velocity, \( v \), the volumetric flow, \( Q \), through a plane must be,
  \[
  Q = \int v \, dA \rightarrow \dot{v} \, A
  \]
- A mass flow rate can be defined as
  \[
  \dot{m} = \int \rho \, v \, dA \rightarrow \rho \, v \, A = \rho \, Q
  \]
- Similarly, a mass of a vertical column of liquid with height \( z \) can be found as
  \[
  m = \rho \, V = \rho \, A \, z
  \]
- Specific gravity with a water reference is defined as
  \[
  s = \frac{\rho_i}{\rho_{H2O}}
  \]

2.2 Hydrostatics

2.2.1 Pressure Changes with Elevation

- For a hydrostatic situation, it is important to note that \( \sum F = ma = 0 \)
- Since pressure changes with elevation,
  \[
  \frac{dP}{dz} = -\rho g
  \]
  - At constant \( \rho \) and \( g \), the above equation becomes the following when integrated from 0 to \( z \)
    \[
    P = \rho g \, z + P_{\text{surface}}
    \]
- The potential, \( \Phi \), of a fluid is defined as
  \[
  \Phi = P + \rho g \, z = \text{constant}
  \]
  - This is only true for static liquids with free motion (no barriers) and where \( z \) is in the opposite direction of \( g \)
- For a multiple fluid U-tube system, the force on the magnitude of the force on the left side must equal the magnitude of the force on the right side
  - Therefore, for U-tube with the same area on both sides, the pressure on the left column must equal the pressure on the right column\(^2\)
  - Another way to solve this problem is to realize that \( \Phi \) at the top of a liquid is the same as \( \Phi \) at the bottom of the same liquid
  - The density of air is very small, so \( \rho_{\text{air}} \) can be assumed to be approximately zero if needed

\(^2\)Don’t forget to include surface pressures (typically atmospheric) if necessary
2.2.2 U-Tube Example

Find \( \rho_{\text{oil}} \) in the diagram below if \( z_{1\rightarrow2} = 2.5 \text{ ft}, \ z_{1\rightarrow3} = 3 \text{ ft}, \ z_{1\rightarrow\text{bottom}} = 4 \text{ ft}, \) and \( z_{4\rightarrow\text{bottom}} = 3 \text{ ft} \):

1. Since \( \sum F = 0 \) for this system and the area is the same on both sides of the U-tube, \( P_1 = P_4 \)
2. Equating the two sides yields
   \[ P_1 + \rho_{\text{oil}} g (2.5 \text{ ft}) + \rho_{\text{air}} g (3 \text{ ft} - 2.5 \text{ ft}) + \rho_{\text{water}} g (4 \text{ ft} - 3 \text{ ft}) = \rho_{\text{water}} g (3 \text{ ft}) + P_4 \]
   (a) Note that the depths here are not depths from the surface but, rather, the vertical height of each individual fluid
3. Since \( P_1 = P_4 \) and \( s = \frac{\rho_{\text{species}}}{\rho_{\text{water}}} \),
   \[ s_{\text{oil}} g (2.5 \text{ ft}) + s_{\text{air}} g (0.5 \text{ ft}) + s_{\text{water}} g (1 \text{ ft}) = s_{\text{water}} g (3 \text{ ft}) \]
4. Canceling the \( g \) terms, substituting \( s_{\text{water}} \equiv 1 \), and assuming \( s_{\text{air}} \approx 0 \),
   \[ s_{\text{oil}} = \frac{3 \text{ ft} - 1 \text{ ft}}{2.5 \text{ ft}} = 0.8 \]

2.2.3 Force on a Dam

- The force on a dam can be given by the following equation where the \( c \) subscript indicates the centroid
  \[ F = \rho g h_c A = P_c A \]
- For a rectangle of depth \( D \),
  \[ h_c = \frac{1}{2} D \]
- For a vertical circle of diameter \( D \) or radius \( r \),
  \[ h_c = \frac{1}{2} D = r \]
- For a vertical triangle with one edge coincident with the surface of the liquid and a depth \( D \),
  \[ h_c = \frac{1}{3} D \]
- The centroid height can be used on any shaped surface. What matters is the shape of the projection of the surface. For instance, a curved dam will have a rectangular projection, and \( h_c \) for a rectangle can be used.
2.2.4 Archimedes’ Law

- An object submerged in water will experience an upward buoyant force (has the same magnitude of the weight of the displaced fluid), and at equilibrium it will be equal to the weight of the object downward.
- For a submerged object of volume $V_o$ in a fluid of density $\rho_f$, 
  \[ F_{\text{buoyant}} = \rho_f g V_o = \rho g A \]

2.2.5 Buoyancy Example

Consider the system below, which is a system of two immiscible fluids, one of which is water (w) and the other is unknown (u). At the interface is a cylinder, which is one-third submerged in the water layer. If the specific gravity of the cylinder is 0.9, find the specific gravity of the unknown:

1. The buoyant force is due to both fluids, so 
  \[ F_b = \rho_w g A_o \left( \frac{1}{3} \right) + \rho_u g A_o \left( \frac{2}{3} \right) \]

2. The weight of the object is 
  \[ F_o = m_o g = \rho_o A_o \left( \frac{1}{3} + \frac{2}{3} \right) \]

3. Equating Step 1 and Step 2 yields, dividing by $\rho_w$ to create specific gravity yields, and canceling out $A_o$ and $g$ yields 
  \[ \left( \frac{1}{3} \right) s_w + \left( \frac{2}{3} \right) s_u = s_o \]

4. The problem statement said that $s_o = 0.9$ and $s_w \equiv 1$, so solving for $s_u$ yields $s_u = 0.85$

2.2.6 Pressure Caused by Rotation

- The height of a spinning fluid with angular velocity $\omega$ and radius $r$ can be given by 
  \[ z = \frac{\omega^2 r^2}{2g} \]

3 Shear Stress and the Shell Momentum Balance

3.1 Types of Stress

- Newton’s Law of Viscosity states that the viscous stress is given by 
  \[ \tau_{ij} = -\mu \left( \frac{\partial v_j}{\partial x} + \frac{\partial v_i}{\partial j} \right) \]

- A tensor of $\tau_{ij}$ simply indicates the stress on the positive $i$ face acting in the negative $j$ direction

- To visualize this easier, in Cartesian coordinates:
  \[ \tau_{xy} = \tau_{yx} = -\mu \left[ \frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right] \]
  \[ \tau_{yz} = \tau_{zy} = -\mu \left[ \frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right] \]
  \[ \tau_{zx} = \tau_{xz} = -\mu \left[ \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial x} \right] \]

3.2 Shell Momentum Balance

3.2.1 Procedure

This method produces the same results as the BSL Method albeit less rigorous:

1. Choose a coordinate system

2. Find the direction of fluid flow. This will be the direction that the momentum balance will be performed on as well as $j$ in $\tau_{ij}$. The velocity in the $j$ direction will be a function of $i$. This $i$ direction will be the dimension of the shell that will approach zero

3. Find what the pressure in the $j$ direction is a function of (the direction in which $\rho gh$ plays a role)

4. A momentum balance can be written as

$$\sum_{in} (\dot{m}v) - \sum_{out} (\dot{m}v) + \sum_{sys} F = 0$$

- Frequently, this is simplified to a force balance of

$$\sum_{sys} F = 0$$

- The relevant forces are usually $F_{\text{gravity}} = \rho g V$, $F_{\text{stress}} = A \tau_{ij}$, and $F_{\text{pressure}} = PA$
  - $F_{\text{gravity}}$ is positive if in the same direction of the fluid flow and negative otherwise

5. Divide out constants and let the thickness of the fluid shell approach zero

6. Use the definition of the derivative

$$f'(x_0) \equiv \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

7. Integrate the equation to get an expression for $\tau_{ij}$

   (a) Find the constant of integration by using the boundary condition

8. Insert Newton’s law of viscosity and obtain a differential equation for the velocity

9. Integrate this equation to get the velocity distribution and find the constant of integration by using the boundary condition

   (a) To find the average value of the velocity,

$$\langle v_i \rangle = \frac{\iint_D v_idA}{\iint_D dA}$$

3.2.2 Boundary Conditions

The following boundary conditions are used when finding the constant(s) of integration. Note that at this point, we are no longer dealing with an infinitesimal shell but, rather, the system as a whole

- At solid-liquid interfaces, the fluid velocity equals the velocity with which the solid surface is moving (which it’s frequently not)

- At a liquid-liquid interfacial plane of constant $i$ (assuming a coordinate system of $i$, $j$, and $k$), $v_j$ and $v_k$ are constant across the $i$ direction as well as any stress along this plane

- At a liquid-gas interfacial plane of constant $i$ (assuming a coordinate system of $i$, $j$, and $k$), $\tau_{ij}$ and $\tau_{ik}$ (and subsequently $\tau_{ji}$ and $\tau_{ki}$) are zero if the gas-side velocity gradient is not large
• If there is creeping flow around an object, analyze the conditions infinitely far out (e.g. \( r \to \infty \) for creeping flow around a sphere)

• It is also important to check for any unphysical terms. For instance, if it is possible for \( x \) to equal zero, and the equation has a \( C \ln(x) \) term in it, then \( C = 0 \) since \( \ln(0) \) is not possible and thus the term should not even exist

### 3.3 Flow of a Falling Film

Consider the following system where the \( z \) axis will be aligned with the downward sloping plane and its corresponding “shell” shown on the right:

1. Cartesian coordinates will be chosen. The direction of the flow is in the \( z \) direction, so this will be the direction of the momentum balance. \( v_z(x) \), so the relevant stress tensor is \( \tau_{xz} \), and \( \Delta x \) will be the differential element

2. \( P(x) \), so it is not included in the \( z \) momentum balance

3. Set up the shell balance as

\[
LW \left( \tau_{xz}\big|_x - \tau_{xz}\big|_{x+\Delta x} \right) + \rho \Delta x LW g \cos \beta = 0
\]

4. Dividing by \( LW \Delta x \) and then letting the limit of \( \Delta x \) approach zero yields

\[
\lim_{\Delta x \to 0} \left( \frac{\tau_{xz}\big|_{x+\Delta x} - \tau_{xz}\big|_x}{\Delta x} \right) = \rho g \cos \beta
\]

5. Note that the first term is the definition of the derivative of \( \tau_{xz} \), so

\[
\frac{d\tau_{xz}}{dx} = \rho g \cos \beta
\]

6. Integrating this equation yields \( \tau_{xz} = \rho g x \cos \beta + C_1 \), and \( C_1 = 0 \) since \( \tau_{xz} = 0 \) at \( x = 0 \) (liquid-gas interface)

\[
\tau_{xz} = \rho g x \cos \beta
\]

7. Inserting Newton’s law of viscosity of \( \tau_{xz} = -\mu \frac{dv_z}{dx} \) yields \( \frac{dv_z}{dx} = -\left( \frac{\rho g \cos \beta}{2\mu} \right) x \) for the velocity distribution

8. Integrating the velocity profile yields \( v_z = -\left( \frac{\rho g \cos \beta}{2\mu} \right) x^2 + C_2 \), and \( C_2 = \left( \frac{\rho g \cos \beta}{2\mu} \right) \delta^2 \) since \( v_z = 0 \) at \( x = \delta \) if \( \delta \) is the depth in the \( x \) direction of the fluid film

\[
v_z = -\left( \frac{\rho g \delta^2 \cos \beta}{2\mu} \right) \left[ 1 - \left( \frac{x}{\delta} \right)^2 \right]
\]

\(^3\)Note the rearrangement of terms required so that the definition of the derivative has \( \tau_{xz}\big|_{x+\Delta x} - \tau_{xz}\big|_x \) and not \( \tau_{xz}\big|_x - \tau_{xz}\big|_{x+\Delta x} \)
3.4 Flow Through a Circular Tube

Consider the following system

1. This problem is best done using cylindrical coordinates. The fluid is moving in the \( z \) direction, \( v_z = v_z(r) \), and thus the \( j \) tensor subscript is equal to \( z \) and \( i \) is \( r \). Therefore, the only relevant stress is \( \tau_{rz} \). Also, \( P(z) \). A \( z \) momentum balance will be performed with a differential in the \( r \) direction. The force of gravity is in the same direction as fluid flow, so it will be positive.

2. Setting up the shell balance yields

\[
2\pi r L \left( \tau_{rz}\bigg|_r - \tau_{rz}\bigg|_{r+\Delta r} \right) + 2\pi r \Delta r (P_0 - P_L) + 2\pi r \Delta r L \rho g = 0
\]

3. Dividing by \( 2\pi L \Delta r \), letting the thickness of the shell approach zero, and utilizing the definition of the derivative yields

\[
\frac{d (r \tau_{rz})}{dr} = \left( \frac{P_0 - P_L}{L} + \rho g \right) r
\]

4. While it’s algebraically equal to the above equation, a substitution of modified pressure\(^4\) can generalize the equation as

\[
\frac{d (r \tau_{rz})}{dr} = \left( \frac{\mathcal{P}_0 - \mathcal{P}_L}{L} \right) r
\]

5. Integrating the above equation yields \( \tau_{rz} = \frac{r (\mathcal{P}_0 - \mathcal{P}_L)}{2L} + \frac{C_1}{r} \), and \( C_1 = 0 \) since \( \tau_{rz} = \infty \) at \( r = 0 \), which is impossible, so

\[
\tau_{rz} = \frac{\mathcal{P}_0 - \mathcal{P}_L}{2L} r
\]

6. Using Newton’s law of viscosity of \( \tau_{rz} = -\mu \left( \frac{dv_z}{dr} \right) \) yields

\[
\frac{dv_z}{dr} = - \left( \frac{\mathcal{P}_0 - \mathcal{P}_L}{2\mu L} \right) r
\]

\(^4\)The modified pressure, \( \mathcal{P} \), is defined as \( \mathcal{P} = P + \rho gh \) where \( h \) is the distance in the direction opposite of gravity. In this problem, \( \mathcal{P} = P - \rho gz \) since height \( z \) is in the same direction as gravity.
7. Integrating the above equations yields $v_z = -\left(\frac{P_0 - P_L}{4\mu L}\right) r^2 + C_2$, and $C_2 = \frac{(P_0 - P_L) R^2}{4\mu L}$ since the solid-liquid boundary condition states that $v_z = 0$ at $r = R$, so

$$v_z = \frac{(P_0 - P_L) R^2}{4\mu L} \left[ 1 - \left(\frac{r}{R}\right)^2 \right]$$

3.5 Flow Through an Annulus

Consider the following system where the fluid is moving upward in an annulus of height $L$

1. This problem is best done using cylindrical coordinates. The fluid is moving in the $z$ direction, so the momentum balance is in the $z$ direction with a differential in the $r$ direction. $v_z(r)$, so the only relevant stress is $\tau_{rz}$. Also, $P(z)$, and the force of gravity is negative since it’s in the opposite direction of fluid flow.

2. The shell balance can be written as

$$2\pi r \Delta r P_0 - 2\pi r \Delta r P_L + 2\pi r \tau_{rz}|_r - 2\pi r L \tau_{rz}|_{r+\Delta r} - \rho 2\pi r L \Delta r g = 0$$

3. Dividing by $2\pi L \Delta r$ yields

$$-\frac{r (\tau_{rz}|_{r+\Delta r} - \tau_{rz}|_r)}{\Delta r} + \frac{r (P_0 - P_L)}{L} - \rho g r = 0$$

4. Letting the limit of $\Delta r$ approach zero, and applying the definition of the derivative yields

$$\frac{d (r \tau_{rz})}{dr} = \frac{r (P_0 - P_L - \rho g)}{L}$$

5. Substituting the modified pressure yields

$$\frac{d (r \tau_{rz})}{dr} = \left(\frac{P_0 - P_L}{L}\right) r$$

6. Integrating the above equation yields

$$\tau_{rz} = \left(\frac{P_0 - P_L}{2L}\right) r + C_1$$

$\text{Note here that } P = \mathcal{P} + \rho g z \text{ since the } z \text{ height is in the opposite direction of gravity}$
7. The constant of integration cannot be determined yet since we don’t know the boundary momentum flux conditions. We know that the velocity only changes in the \( r \) direction, so there must be a maximum velocity at some arbitrary width \( r = \lambda R \), and at this point there will be no stress. This is because \( \tau_{ij} \) is a function of the rate of change of velocity, and since the derivative is zero at a maximum, the stress term will go to zero. Therefore,

\[
0 = \left( \frac{p_0 - p_L}{2L} \right) \lambda R + \frac{C_1}{\lambda R}
\]

8. Solving the above equation for \( C_1 \) and substituting it in yields

\[
\tau_{rz} = \frac{(p_0 - p_L)R}{2L} \left[ \left( \frac{r}{R} \right) - \lambda^2 \left( \frac{R}{r} \right) \right]
\]

9. Using \( \tau_{rz} = -\mu \frac{dv_z}{dr} \) and integrating \( \frac{dv_z}{dr} \) yields

\[
v_z = -\frac{(p_0 - p_L)R^2}{4\mu L} \left[ \left( \frac{r}{R} \right)^2 - 2\lambda^2 \ln \left( \frac{r}{R} \right) + C_2 \right]
\]

10. The boundary conditions state that \( v_z = 0 \) at \( r = \kappa R \) and \( v_z = 0 \) at \( r = R \) (solid-liquid interfaces), which yields a system of equations that can be solved to yield \( C_2 = -1 \) and \( \lambda^2 = \frac{1 - \kappa^2}{2 \ln \left( \frac{1}{\kappa} \right)} \)

11. Substituting the results from above yields the general equations of

\[
\tau_{rz} = \frac{(p_0 - p_L)R}{2L} \left[ \left( \frac{r}{R} \right) - \frac{1 - \kappa^2}{2 \ln \left( \frac{1}{\kappa} \right)} \left( \frac{R}{r} \right) \right]
\]

\[
v_z = -\frac{(p_0 - p_L)R^2}{4\mu L} \left[ 1 - \left( \frac{r}{R} \right)^2 - \frac{1 - \kappa^2}{\ln \left( \frac{1}{\kappa} \right)} \ln \left( \frac{r}{R} \right) \right]
\]
4 Mass, Energy, and Momentum Balances

4.1 Mass and Energy Balance

4.1.1 Mass-Energy Balance

- For mass balances,
  \[ \sum m_{in} - \sum m_{out} = \frac{d}{dt} (m)_{\text{system}} = \frac{d}{dt} (\rho V)_{\text{system}} \]

- For an incompressible fluid, the continuity equation states that for a simple input-output system at steady state,
  \[ A_1 v_1 = A_2 v_2 = \frac{\dot{m}}{\rho} = Q \]

- For energy balances with heat \( q \) and work\(^6 \) \( w \),
  \[ q - w + \sum E_{\text{in}} - \sum E_{\text{out}} = \frac{d}{dt} (E_{\text{sys}}) \]

- Therefore, a general equation can be written as the following where \( \hat{e} \) is the internal energy per unit mass,
  \[ \sum m_{in} \left( \frac{\hat{e} + P + gz + \frac{1}{2} v^2}{\rho} \right)_{in} - \sum m_{out} \left( \frac{\hat{e} + P + gz + \frac{1}{2} v^2}{\rho} \right)_{out} + q - w = \frac{d}{dt} \left[ m_{\text{sys}} \left( \hat{e} + gz + \frac{1}{2} v^2 \right) \right]_{\text{sys}} \]

- Power can be written as
  \[ \text{Power} = \dot{m} \dot{w} = F v = Q \Delta P \]

4.1.2 Example

Consider a 1 m\(^3\) tank at 1 bar of pressure and isothermal conditions that is giving off ideal gas at a rate of 0.001 m\(^3\)/s. How long will it take for the pressure to fall to 0.0001 bar?

1. Set up a mass balance equation:
  \[ -Q \rho = \frac{d}{dt} (\rho V) = V \frac{d\rho}{dt} + \rho \frac{dV}{dt} \]

2. Realize that the tank volume is constant, so \( \frac{dV}{dt} = 0 \)

3. For an ideal gas, \( \rho = \frac{MP}{RT} \), so after a change of variables in the derivative term yields
  \[ -Q \frac{MP}{RT} = V \frac{dP}{dt} \]

4. The above equation can be simplified to
  \[ \frac{dP}{dt} = -\frac{Q}{V} P \rightarrow \ln \left( \frac{P}{P_0} \right) = -\frac{Qt}{V} \]

5. Solving for \( t \) yields
  \[ t = -\frac{V}{Q} \ln \left( \frac{P}{P_0} \right) = 9210 \text{ s} \]

\(^6\)A positive work value indicates work done by the system while a negative work indicates work done on the system.
4.2 Bernoulli Equation

- At steady state conditions, the general equation for the mass-energy balance of an inlet-outlet system can be written as the following:

\[
\dot{e}_1 + \frac{v_1^2}{2} + gz_1 + \frac{P_1}{\rho_1} + q = \dot{e}_2 + \frac{v_2^2}{2} + gz_2 + \frac{P_2}{\rho_2} + \dot{w}
\]

- The energy balance simplifies to the mechanical energy balance with constant \( g \) and at steady-state with \( \dot{F} \) representing frictional losses:

\[
\Delta \left( \frac{v^2}{2} \right) + g \Delta z + \int_1^2 \frac{dP}{\rho} + \dot{w} + \dot{F} = 0
\]

- \( \dot{F} \) is always positive, \( \dot{w} \) is positive if the fluid performs work on the environment, and \( \dot{w} \) is negative if the system has work done on it.

- Note that \( \dot{F} \) is a frictional energy per unit mass. Recall that energy is \( \text{kg} \cdot \text{m}^2 / \text{s}^2 \). Also, \( \dot{w} \) is energy.

  * Therefore, if one solves for \( \dot{F} \), this value can be divided by \( g \) to get the “friction head,” which has SI units of m.

- If the fluid is incompressible, \( \rho \) is constant, so it becomes

\[
\Delta \left( \frac{v^2}{2} \right) + g \Delta z + \frac{\Delta P}{\rho} + \dot{w} + \dot{F} = 0
\]

- For steady-state, no work, no frictional losses, constant \( g \), and constant density (incompressible), the Bernoulli Equation is obtained:

\[
\Delta \left( \frac{v^2}{2} \right) + g \Delta z + \frac{\Delta P}{\rho} = 0
\]

- For a problem that involves draining, if the cross sectional area of the tank is significantly larger than the siphon or draining hole, \( v_1 \) can be approximated as zero.

4.3 Momentum Balance

Of course, the momentum balance was already seen in the shell momentum balance section, but for clarity it is included here as well (without assuming steady state conditions):

- Recall that momentum is:

\[
\mathcal{M} \equiv mv
\]

- Therefore,

\[
\sum_{in} (\dot{mv}) - \sum_{out} (\dot{mv}) + \sum_{sys} F = \frac{d}{dt} (mv)_{sys} = \frac{d\mathcal{M}_{sys}}{dt}
\]

- While it may seem impossible to add a force and momentum together, realize that is not what is being done. Instead, it’s a mass flow rate times velocity, which happens to have the same units of force, so addition can be performed.

- Some forces that are relevant to fluid dynamics are

\[
\sum_{sys} F = \sum_{\text{gravity}} F + \sum_{\text{pressure}} F + \sum_{\text{visc.}} F + \sum_{\text{other}} F
\]
5 Differential Equations of Fluid Mechanics

5.1 Vectors and Operators

5.1.1 Dot Product
- If $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$, then the dot product is $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$

5.1.2 Cross Product
- The cross product is the following where $\theta$ is between 0 and $\pi$ radians
  $\vec{u} \times \vec{v} = (u_2 v_3 - u_3 v_2) \hat{i} - (u_1 v_3 - u_3 v_1) \hat{j} + (u_1 v_2 - u_2 v_1) \hat{k}$

5.1.3 Gradient
- The $\nabla$ operator is defined as
  $\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$
- The gradient of an arbitrary scalar function $f(x, y, z)$ is
  $\text{grad}(f) = \nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}$

5.1.4 Divergence
- For an arbitrary 3-D vector $\vec{v}$, the divergence is
  $\text{div}(\vec{v}) = \nabla \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

5.1.5 Curl
- The curl of an arbitrary 3-D vector $\vec{v}$ is
  $\text{curl}(\vec{v}) = \nabla \times \vec{v} = \left( \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{i} + \left( \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{j} + \left( \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{k}$

5.1.6 Laplacian
- The Laplacian operator, $\nabla^2$, acting on an arbitrary scalar function $f(x, y, z)$ is
  $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

5.2 Solution of the Equations of Motion
- A differential mass balance known as the continuity equation can be set up as
  $\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} = 0$
- If $\rho$ is constant, then $\frac{\partial \rho}{\partial t} = 0$, and the $\rho$ values can be factored out of the above equation to make
  $\nabla \cdot \vec{v} = 0$
- For constant $\rho$ and $\mu$, the Navier-Stokes Equation states that
  $\rho \left( \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} \right) = -\nabla P + \mu \nabla^2 \vec{v} + \rho \vec{g}$

\[7^\text{See Section 8 for equations written out for the various coordinate systems}\]
5.3 Procedure for Using Navier-Stokes Equation

1. Choose a coordinate system and find the direction of the flow

2. Use the continuity equation (making simplifications if $\rho$ is constant) to find out more information

3. Use the Navier-Stokes equation in the direction of the fluid flow and eliminate terms that are zero based on the direction of the flow and the results of the continuity equation
   (a) Be careful to make sure that $\vec{g}$ is in the correct direction

4. Integrate the resulting equation and solve for the boundary conditions

5.4 Flow Through a Circular Tube Using Navier-Stokes

The problem in Section 3.4 can be revisited, as the Navier-Stokes equation can be used on problems where the shell momentum balance method was originally used. For this problem, refer to the earlier diagram, and assume the pressure drops linearly with length. Also assume that $\rho$ and $\mu$ are constant:

1. The best coordinate system to use is cylindrical, and the continuity equation can be written as

   \[ \nabla \cdot \vec{v} = \frac{1}{r} \frac{\partial}{\partial r} (r v_r) + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \]

2. The continuity equation turns into the following since velocity is only in the $z$ direction:

   \[ \frac{\partial v_z}{\partial z} = 0 \]

3. Now the Navier-Stokes equation can be written in the $z$ direction as

   \[ \rho \left( \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + v_\theta \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \mu \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z \]

4. The Navier-Stokes equation simplifies to the following since $v_z(r), v_r = v_\theta = 0$, and the linear change in pressure means $\frac{\partial P}{\partial z} = \frac{P_L - P_0}{L}$:

   \[ 0 = -\frac{P_L - P_0}{L} + \frac{\mu}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) + \rho g_z \]

5. Rearranging the above equation yields

   \[ \frac{1}{\mu} \left[ \frac{P_L - P_0}{L} - \rho g_z \right] = \frac{1}{r} \frac{d}{dr} \left[ r \frac{\partial v_z}{\partial r} \right] \]

6. For simplicity’s sake, let the left-hand side equal some arbitrary constant $\alpha$ to make integration easier

   \[ \alpha = \frac{1}{r} \frac{d}{dr} \left[ r \frac{\partial v_z}{\partial r} \right] \]

7. Integrating the above equation yields

   \[ \frac{\alpha r^2}{2} = r \frac{\partial v_z}{\partial r} + C_1 \]

8. Integrating the above equation again yields

   \[ \frac{\alpha r^2}{4} - C_1 \ln r + C_2 = v_z \]
9. At $r = 0$, the log term becomes unphysical, so $C_1 = 0$. The equation is now

$$\frac{\alpha r^2}{4} + C_2 = v_z$$

10. There is a solid-liquid boundary at $r = R$, and $v_z = 0$ here, so $C_2 = -\frac{\alpha R^2}{4}$. Therefore,

$$v_z = \frac{\alpha r^2}{4} - \frac{\alpha R^2}{4} = \alpha (r^2 - R^2) = \frac{1}{4\mu} \left[ \frac{P_L - P_0}{L} - \rho g z \right] (r^2 - R^2)$$

11. Substituting $\mathcal{P} = P - \rho gz$ into this problem yields the same result as in Section 3.4

### 5.5 Flow Through a Heat-Exchanger

Consider a heat-exchanger as shown below with an inner radius of $R_1$ and outer radius of $R_2$ with $z$ pointing to the right. Fluid is moving to the right ($+z$) in the outer tube, and the fluid is moving to the left in the inner tube ($-z$). Assume that the pressure gradient is linear and that $R_1 < r < R_2$:

1. Cylindrical coordinates are best to use here, and velocity is only in the $z$ direction. The continuity equation is equivalent to the following, assuming constant $\rho$ and $\mu$

$$\nabla \cdot \vec{v} = 0 \rightarrow \frac{\partial v_z}{\partial z} = 0$$

2. Setting up the Navier-Stokes equation in the $z$ direction simplifies to the following (recall that $\rho g$ isn’t necessary since gravity isn’t playing a role)

$$\frac{\partial P}{\partial z} = \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) \right]$$

3. Since the pressure gradient is linear,

$$\frac{P_L - P_0}{\mu L} r = \frac{d}{dr} \left( r \frac{dv_z}{dr} \right)$$

4. Integrating the above equation yields

$$\frac{P_L - P_0}{2\mu L} r^2 = r \frac{dv_z}{dr} + C_1$$

5. Integrating the above equation yields

$$v_z = \frac{P_L - P_0}{4\mu L} r^2 - C_1 \ln r + C_2$$

6. Letting the constant term in front of the $r^2$ become $\alpha$ for simplicity yields

$$v_z = \alpha r^2 - C_1 \ln r + C_2$$
7. At $r = R_1$, $v_z = 0$ since it’s a solid-liquid boundary, so 
\[ C_2 = -\alpha R_1^2 + C_1 \ln R_1 \]

8. At $r = R_2$, $v_z = 0$ since it’s a solid-liquid boundary, so 
\[ C_1 = \frac{-\alpha (R_2^2 - R_1^2)}{\ln \left(\frac{R_1}{R_2}\right)} \]

9. Substituting back into the equation for $v_z$ yields 
\[
v_z = \alpha \left[ r^2 + \frac{(R_2^2 - R_1^2)}{\ln \left(\frac{R_1}{R_2}\right)} \ln r - R_1^2 - \frac{\ln R_1 (R_2^2 - R_1^2)}{\ln \left(\frac{R_1}{R_2}\right)} \right]
\]

10. Substituting back in for $\alpha$ and simplifying yields 
\[
v_z = \frac{P_L - P_0}{4\mu L} \left[ \frac{\ln \left(\frac{r}{R_2}\right)}{\ln \left(\frac{R_2}{R_1}\right)} (R_2^2 - R_1^2) + (R_2^2 - r^2) \right]
\]

6. Velocity Distributions with More Than One Variable

6.1 Time-Dependent Flow of Newtonian Fluids

6.1.1 Definitions

- To solve these problems, a few mathematical substitutions will be made and should be employed when relevant to make the final solution simpler

- The kinematic viscosity is 
\[ \nu = \frac{\mu}{\rho} \]

- For a wall-bounded flow, the dimensionless velocity is defined as the following where $v_i$ is the local velocity and $v_0$ is the friction velocity at the wall
\[ \phi = \frac{v_i}{v_0} \]

6.1.2 Flow near a Wall Suddenly Set in Motion

A semi-infinite body of liquid with constant density and viscosity is bounded below by a horizontal surface (the $xz$-plane). Initially, the fluid and the solid are at rest. Then at $t = 0$, the solid surface is set in motion in the $+x$ direction with velocity $v_0$. Find $v_x$, assuming there is no pressure gradient in the $x$ direction and that the flow is laminar.

1. Cartesian coordinates should be used. Also, $v_y = v_z = 0$, and $v_x = v_x(y, t)$ since it will change with height and over time

2. Using the continuity equation yields $\frac{\partial v_x}{\partial x} = 0$ at constant $\rho$, which we already know because $v_x$ is not a function of $x$

---

8Note that $C_1 \neq 0$ because $r$ is never actually zero. The outer tube cannot be smaller than the inner tube, so zero volume in the outer tube is actually $r = R_1$.
3. Using the Navier-Stokes equation in the \( x \) direction yields \( \rho \left( \frac{\partial v_x}{\partial t} \right) = \mu \left( \frac{\partial^2 v_x}{\partial y^2} \right) \). This can be simplified to \( \frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2} \).

4. Boundary and initial conditions must be set up:
   (a) There is a solid-liquid boundary in the \( x \) direction, and the wall is specified in the problem stating as moving at \( v_0 \). Therefore, at \( y = 0 \) and \( t > 0 \), \( v_x = v_0 \)
   (b) At infinitely high up, the velocity should equal zero, so at \( y = \infty \), \( v_x = 0 \) for all \( t > 0 \)
   (c) Finally, time is starting at \( t = 0 \), so \( v_x = 0 \) at \( t \leq 0 \) for all \( y \)

5. It is helpful to have the initial conditions cause solutions to be values of 1 or 0, so the dimensionless velocity of \( \phi(y,t) = \frac{v_x}{v_0} \) will be introduced such that \( \frac{\partial \phi}{\partial t} = \nu \frac{\partial^2 \phi}{\partial y^2} \)
   (a) It is now possible to say \( \phi(y,0) = 0 \), \( \phi(0,t) = 1 \), and \( \phi(\infty,t) = 0 \)

6. Since \( \phi \) is dimensionless, it must be related to \( \frac{y}{\sqrt{\nu t}} \) since this (or multiplicative scale factors of it) is the only possible dimensionless group from the given variables. Therefore, \( \phi = \phi(\eta) \) where \( \eta = \frac{y}{\sqrt{4 \nu t}} \). The \( \sqrt{4} \) term is included in the denominator for mathematical simplicity later on but is not necessary

7. With this new dimensionless quantity, the equation in Step 5 can be broken down from a PDE to an ODE
   (a) First, \( \frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial \eta} \frac{\partial \eta}{\partial t} \). The value for \( \frac{\partial \eta}{\partial t} \) can be found from taking the derivative with respect to \( t \) of \( \eta \) defined in Step 6. This yields \( \frac{\partial \phi}{\partial t} = -\frac{d \phi}{d \eta} \frac{1}{2} \frac{\eta}{t} \)
   (b) Next, \( \frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial \eta} \frac{\partial \eta}{\partial y} \). The value for \( \frac{\partial \eta}{\partial y} \) can be found from taking the derivative with respect to \( y \) of \( \eta \) defined in Step 6. This yields \( \frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial \eta} \frac{1}{\sqrt{4 \nu t}} \)
   (c) Therefore, \( \frac{d^2 \phi}{d \eta^2} + 2 \eta \frac{d \phi}{d \eta} = 0 \)

8. New sets of boundary conditions are needed for \( \eta \)
   (a) At \( \eta = 0 \), \( \phi = 1 \) since this is when \( y = 0 \), and it was stated earlier that \( \phi(0,t) = 1 \)
   (b) At \( \eta = \infty \), \( \phi = 0 \) since this is when \( y = \infty \), and it was stated earlier that \( \phi(\infty,t) = 0 \)

9. To solve this differential equation, introduce \( \psi = \frac{d \phi}{d \eta} \) to make the equation \( \frac{d^2 \psi}{d \eta^2} + 2 \eta \psi = 0 \), which will yield \( \psi = c_1 \exp(-\eta^2) \)

10. Integrating \( \psi \) yields \( \phi = c_1 \int_0^\eta \exp(-\eta^2) \, d\eta + c_2 \)
    (a) \( \eta \) is used here since it is a dummy variable of integration and should not be confused with the upper bound of \( \eta \)

11. The boundary conditions of \( \phi = 0 \) and \( \phi = 1 \) can be used here to find \( c_1 \) and \( c_2 \), which produces the equation \( \phi(\eta) = 1 - \text{erf}(\eta) \)
    (a) The error function is defined for an arbitrary \( z \) and \( \xi \) as \( \text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\xi^2} \, d\xi \)
6.2 The Potential Flow and Streamfunction

- The vorticity of a fluid is defined as
  \[ \vec{\omega} = \nabla \times \vec{v} \]
  - If the vorticity of a fluid is zero, then it is said to be irrotational

- The velocity potential, \( \phi \), can be defined as the following, which satisfies irrotationality\(^9\)
  \[ \vec{v} = -\nabla \phi \]
  - Note that Wilkes defines velocity potential as \( \vec{v} = \nabla \phi \). The first definition will be used here though.

- The stream function, \( \psi \), can be defined as the following, which satisfies continuity\(^10\)
  \[ \vec{v} = \hat{z} \times \nabla \psi \]
  - Note that Wilkes define the stream function as \( \vec{v} = \nabla \times \psi \). The first definition will be used here though.

- The Laplace equation is
  \[ \nabla^2 \phi = 0 \]
  - To obtain the Laplace equation from \( \phi \), substitute it into the continuity equation
  - To obtain the Laplace equation from \( \psi \), substitute it into the irrotationality condition

- A stagnation point is defined as the point where the velocity in all dimensions is zero

- The equipotential lines (\( \phi \)) will be perpendicular to the streamlines (\( \psi \))
  - To plot something such as a streamline, find the equation for \( \psi \), set it equal to an arbitrary constant, and solve the equation such that it can be plotted (e.g. \( y \) as a function of \( x \)). Repeat this process for multiple values of \( \psi \)
    * It is good to test the behavior of \( \psi \) and/or \( \phi \) at the axes

- The continuity equation for axisymmetric coordinates is typically rewritten as W
  \[ \sin \theta \frac{\partial (r^2 v_r)}{\partial r} + r \frac{\partial (v_\theta)}{\partial \theta} = 0 \]

6.3 Solving Flow Problems Using Streamfunctions

6.3.1 Overview

- To solve two-dimensional flow problems, the streamfunction can be used, and the equations listed in the Appendix for the differential equations of \( \psi \) that are equivalent to the Naiver-Stokes equation should be used

- For steady, creeping flow, the only term not equal to zero is the term associated with the kinematic viscosity, \( \nu \)
  - This is also true at \( \text{Re} \ll 1 \)

- When dealing with spherical coordinates, \( r \) and \( \theta \) will still be used, but \( z \) will be used in place of \( \phi \) for the third dimension

\(^9\) A table of velocity potentials can be found in the Appendix.
\(^10\) A table of stream functions can be found in the Appendix.
6.3.2 Creeping Flow Around a Sphere\textsuperscript{11}

**Problem Statement:** Obtain the velocity distributions when the fluid approaches a sphere in the positive $z$ direction (if $z$ is to the right). Assume that the sphere has radius $R$ and $Re \ll 1$.

**Solution:**

First, realize that this is two-dimensional flow, so a stream function should be used in place of the Navier-Stokes equation for (relative) mathematical simplicity. The $\psi$ equivalent for the Navier-Stokes equation in spherical coordinates can be found in the Appendix. Since $Re \ll 1$, the stream function differential equation becomes $0 = \nu E^4 \psi$, which is simplified to $0 = E^4 \psi$. Substituting in $(E^2)^2$ into this equation yields

$$
\left[ \frac{\partial^2}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right) \right]^2 \psi = 0
$$

(1)

The next step is to find the boundary conditions. Note that it is best to find all boundary conditions regardless of how many variables are actually needed. The first two boundary conditions are for the no-slip solid-liquid boundary condition where $r = R$. Here,

$$v_r = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \theta} = 0 \quad (r = R)
$$

(2)

and

$$v_\theta = \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r} = 0 \quad (r = R)
$$

(3)

which are the definitions of the stream function in spherical coordinates. The velocity components at $r = R$ equal zero since the sphere itself is not moving. For the last boundary condition, analyze the system at a point far away from the sphere: $r \to \infty$. To do this, first note that $v_\theta$ is tangential to the streamlines around the sphere. Due to mathematical convention (quadrants are numbered counterclockwise), this is graphically shown as

![Diagram showing velocity around a sphere](image)

The diagram shown above indicates the following. The projection of the sphere (a circle) is drawn such that it is similar to a unit circle with the angles ($\theta$) drawn in at the appropriate quadrants. The cyan lines represent the streamlines. $v_\infty$ represents some velocity at a point infinitely far out to the right. The yellow vectors represent the direction of $v_\theta$, which is always tangential to the sphere and goes counterclockwise. Note how the yellow $v_\theta$ vector goes in the same direction of $v_\infty$ at $\theta = \frac{3\pi}{2}$ and against the direction of $v_\infty$ at $\theta = \frac{\pi}{2}$. Also notice that $v_\theta$ is completely perpendicular to $v_\infty$ at $\theta = 0, \pi$.

With this diagram available, one should find $v_\theta$ at each quadrant when $r \to \infty$. At $\theta = 0$, $v_\theta = 0$ since the flow of the fluid is perpendicular to $v_\theta$ and thus none of the fluid is actually flowing in the $v_\theta$ direction.

\textsuperscript{11}This problem has not been explained in full detail with regards to both concepts and the math in all the textbooks and websites I could find, so effort has gone into a thorough explanation here.
At $\theta = \frac{\pi}{2}$, $v_{\theta} = -v_{\infty}$ since the flow is parallel (but in the opposite direction) to $v_{\theta}$, assuming that $v_{\infty}$ is the velocity of the fluid infinitely far to the right of the sphere. At $\theta = \pi$, $v_{\theta} = 0$ since the flow is perpendicular to $v_{\theta}$. At $\theta = \frac{3\pi}{2}$, $v_{\theta} = v_{\infty}$ since the flow is parallel (and in the same direction) to $v_{\theta}$.

The next step is to come up with a function for $v_{\theta}$ given the values we previously defined. The most reasonable function that has the values of $v_{\theta}$ shown earlier at each axis is

$$v_{\theta} = -v_{\infty} \sin \theta$$

This can be seen by analyzing the sine function. Sine goes from 0 to 1 to 0 to $-1$ in intervals of $\frac{\pi}{2}$ around the unit circle. Multiplying sine by a factor of $-v_{\infty}$ simply changes the magnitude of the function and ensures the correct value at each point.

As a quick aside, one might care to know what $v_{r}$ is equal to at $r \to \infty$ since we have just found $v_{\theta}$ at $r \to \infty$ (Eq. 4). To do this, analyze the vector triangle shown below. Note that $v_{\infty}$ is horizontal since it is defined as the velocity infinitely far out to the right of the sphere, and $v_{r}$ is at some arbitrary angle $\theta$ since it represents a radial value, which can be oriented at infinitely many angles. The triangle is therefore

![Vector Triangle Diagram]

Using trigonometry, one can state the following (note the location of the right angle and thus the hypotenuse):

$$v_{r} = v_{\infty} \cos \theta$$

With (Eq. 4), we can equate $v_{\theta}$ to the corresponding definition of the stream function. Recall that $v_{\theta} = \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r}$ in spherical coordinates. This can be rearranged to $d\psi = -v_{\infty} r \sin^{2} \theta dr$ by cross-multiplication and substitution of (Eq. 4) for $v_{\theta}$. This can then be integrated to yield the third and final boundary condition of

$$\psi = -\frac{1}{2} v_{\infty} r \sin^{2} \theta \quad (r \to \infty)$$

A solution must now be postulated for $\psi$. To do this, look at (Eq. 5). It is clear that $\psi$ has an angular component that is solely a function of $\sin^{2} \theta$. There is also a radial component, which can be assigned some arbitrary $f(r)$. Therefore, a postulated solution is of the form

$$\psi(r, \theta) = f(r) \sin^{2} \theta$$

Now, substitute (Eq. 6) into (Eq. 1) and solve. This is a mathematically tedious process, but the bulk of the work has been shown below. Recognize that it is probably easiest to find $E^{2} \psi$ and then do $E^{2} (E^{2} \psi)$ instead of trying to do $E^{4} \psi$ all in one shot. Therefore,

$$E^{2} \psi = \left[ \frac{\partial^{2}}{\partial r^{2}} + \frac{\sin \theta}{r^{2}} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right) \right] f(r) \sin^{2} \theta = 0$$

Multiply $f(r) \sin^{2} \theta$ through and be careful of terms that are and are not influenced by the partial derivatives. Also recall that differential operators are performed from right to left. This yields

$$E^{2} \psi = \sin^{2} \theta \frac{d^{2} f(r)}{dr^{2}} + \frac{f(r) \sin \theta}{r^{2}} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \cdot 2 \cos \theta \sin \theta \right) = 0$$

After applying the right-most derivative,

$$E^{2} \psi = \sin^{2} \theta \frac{d^{2} f(r)}{dr^{2}} - \frac{2 f(r) \sin^{2} \theta}{r^{2}} = 0$$
Factoring $\sin^2 \theta$ out of the equation yields

$$E^2 \psi = \frac{d^2 f(r)}{dr^2} - \frac{2f(r)}{r^2} = 0 \tag{7}$$

This is close to the final solution of $E^4 \psi = 0$, but recall that this was just $E^2 \psi$ that was evaluated. We must now evaluate $E^2 (E^2 \psi)$, or, equivalently, $E^2$ of (Eq. 7). This is equivalent to saying

$$E^4 \psi = \left[ \frac{\partial^2}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right) \right] \left[ \frac{d^2 f(r)}{dr^2} - \frac{2f(r)}{r^2} \right] = 0$$

This simplifies to

$$E^4 \psi = \left( \frac{d^2}{dr^2} - \frac{2}{r^2} \right) \left( \frac{d^2}{dr^2} - \frac{2}{r^2} \right) f = 0 \tag{8}$$

The above differential equation is referred to as an equidimensional equation, and equidimensional equations should be tested with trial solutions of the form $Cr^n$. An equidimensional equation has the same units throughout. For instance, $\frac{d^2}{dr^2}$ has units of $m^{-2}$ and so does $-\frac{2}{r^2}$. Therefore, since all terms have the same units, this is an equidimensional equation. By substituting $Cr^n$ into (Eq. 8), $n$ may have values of $-1, 1, 2, \text{ and } 4$. This can be seen by the following:

$$\left( \frac{d^2}{dr^2} - \frac{2}{r^2} \right) \left( \frac{d^2}{dr^2} - \frac{2}{r^2} \right) Cr^n = 0$$

Applying the right-hand parenthesis yields

$$\left( \frac{d^2}{dr^2} - \frac{2}{r^2} \right) n (n-1) Cr^{n-2} - 2Cr^{n-2} = 0$$

This simplifies to

$$\left( \frac{d^2}{dr^2} - \frac{2}{r^2} \right) Cr^{n-2} [n (n-1) - 2] = 0$$

Applying the left-hand parentheses yields

$$(n-2) (n-3) Cr^{n-4} [n (n-1) - 2] - 2Cr^{n-4} [n (n-1) - 2]$$

Simplifying this yields

$$[(n-2) (n-3) - 2] [n (n-1) - 2] Cr^{n-4} = 0$$

Finally, one more algebraic simplification yields

$$(n-4) (n-2) (n-1) (n+1) Cr^{n-4} = 0$$

The solutions to the above equation are $n = -1, 1, 2, 4$. This makes $f(r)$ the following general equation with four terms:

$$f(r) = C_1 r^{-1} + C_2 r + C_3 r^2 + C_4 r^4 \tag{9}$$

The constant $C_4$ must equal zero. This is because $C_4$ is multiplied by $r^4$, and $\psi$ (Eq. 5/6) does not have an $r^4$ term in it. $C_3$ must equal $-\frac{1}{2} v_\infty$ because $C_3$ is multiplied by an $r^2$ term, and the equation for $\psi$ (Eq. 5/6) has a $-\frac{1}{2} v_\infty r^2$ in it. The reason that $C_4 = 0$ and $C_1$ and $C_2$ do not is that we know that the highest term in the equation for $\psi$ is an $r^2$ term. Any term that converges faster than the maximum order ($r^2$) must

---

I was having an issue simplifying this on my own.
be zero. Therefore, the $C_1$ term and $C_2$ term do not have to go to zero, but the $C_4$ term does. This makes the equation for $\psi$ the following by plugging (Eq. 9) into (Eq. 6):

$$\psi(r, \theta) = \left( C_1 r^{-1} + C_2 r - \frac{1}{2} v_\infty r^2 \right) \sin^2 \theta \quad (10)$$

Using the definition of $v_r$ in relation to the stream function, one can rewrite this equation as the following by plugging the equation for $\psi$ (Eq. 10) into $\frac{\partial \psi}{\partial \theta}$ in the equation for $v_r$:

$$v_r = -\frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} = \left( v_\infty - 2 \frac{C_2}{r} - 2 \frac{C_1}{r^3} \right) \cos \theta = -\frac{2}{r^2} \left[ -\frac{1}{2} v_\infty r^2 + \frac{C_1}{r} + C_2 r \right] \quad (11)$$

Similarly for $v_\theta$,

$$v_\theta = \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r} = \left( -v_\infty + \frac{C_2}{r} - \frac{C_1}{r^3} \right) \sin \theta \quad (12)$$

By setting $v_r = 0$ and $v_\theta = 0$ at the $r = R$ no-slip boundary conditions and solving for $C_1$ and $C_2$, we get a system of 2 linear equations with $C_1 = -\frac{1}{4} v_\infty R^3$ and $C_2 = \frac{3}{4} v_\infty R$. One somewhat easy way to do this is to add (Eq. 11) and (Eq. 12), which cancels a lot of terms in the system of equations. However, any method to solve the system is sufficient. With these constants, one can rewrite the equation for $\psi$ (Eq. 10) as

$$\psi(r, \theta) = \left( -\frac{1}{4} v_\infty \frac{R^3}{r} + \frac{3}{2} v_\infty R - \frac{3}{2} v_\infty R^2 \right) \sin^2 \theta \quad (13)$$

Therefore, rewriting (Eq. 11) and (Eq. 12) with the values for the constants substituted in yields the following

$$v_r = v_\infty \left( 1 - \frac{3}{2} \left( \frac{R}{r} \right) + \frac{1}{2} \left( \frac{R}{r} \right)^3 \right) \cos \theta$$

$$v_\theta = -v_\infty \left( 1 - \frac{3}{4} \left( \frac{R}{r} \right) - \frac{1}{4} \left( \frac{R}{r} \right)^3 \right) \sin \theta$$

6.4 Solving Flow Problems with Potential Flow
6.4.1 Overview

- To solve two-dimensional problems with fluids that have very low viscosities, are irrotational, are incompressible, and are at steady-state, the potential flow method can be used

6.4.2 Steady Potential Flow Around a Stationary Sphere

Problem Statement: Consider the flow of an incompressible, inviscid fluid in irrotational flow around a sphere. Solve for the velocity components.

Solution:

This problem is best done using spherical coordinates. The boundary conditions should then be found. There is a no-slip boundary condition at $r = R$. Here,

$$v_r = -\frac{\partial \phi}{\partial r} = 0 \quad (r = R) \quad (14)$$

and

$$v_\theta = -\frac{1}{r} \frac{\partial \phi}{\partial \theta} = 0 \quad (r = R) \quad (15)$$

Now, infinitely far away should be analyzed. At this point,

$$v_r = v_\infty \cos \theta \quad (r \to \infty) \quad (16)$$
and
\[ v_\theta = -v_\infty \sin \theta \quad (17) \]

For explanation of how to obtain (Eq. 16/17), see the previous example problem with the stream function.

Now, analyze \( \phi \) in order to come up with a trial expression. This is done by substituting (Eq. 16) or (Eq. 17) into (Eq. 14) or (Eq. 15) and solving for \( \phi \) via integration. Whether you choose to substitute \( v_r \) or \( v_\theta \), the reasonable trial solution is of the form
\[ \phi(r, \theta) = f(r) \cos \theta \quad (18) \]

Next, substitute (Eq. 18) into the Laplace equation such that
\[ \nabla^2 f(r) \cos \theta = 0 \]

This expression becomes the following in spherical coordinates
\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \cos \theta \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) = 0 \]

Doing some of the derivatives yields
\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{df}{dr} \cos \theta \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( -f(r) \sin^2 \theta \right) = 0 \]

Performing the right-most derivative and factoring out the cosine yields the following. Note that the left-most derivative is retained since doing a product rule is just extra work.
\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{df(r)}{dr} \right) - \frac{2f(r)}{r^2} = 0 \quad (19) \]

Note that (Eq. 19) is an equidimensional equation (see previous example problem with the stream function for more detail). Therefore, a test solution of \( Cr^n \) should be inputted for \( f(r) \) in (Eq. 19). Doing this yields
\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d(Cr^n)}{dr} \right) - \frac{2(Cr^n)}{r^2} = 0 \]

This should be some fairly simple calculus, which will result in the algebraic expression
\[ (n + 2) (n - 1) = 0 \]

This has roots of \( n = -2 \) and \( n = 1 \), so the trial expression \( f(r) \) can be written as
\[ f(r) = C_1 r + C_2 r^{-2} \]

Substituting \( f(r) \) into (Eq. 18) yields
\[ \phi(r, \theta) = (C_1 r + C_2 r^{-2}) \cos \theta \quad (20) \]

Apply the boundary conditions to solve for the constants. For instance, at \( r = R \), we know that \( v_r = -\frac{\partial \phi}{\partial r} = 0 \), so substituting (Eq. 20) into this expression yields
\[ C_1 - \frac{2C_2}{r^3} = 0 \]

Evaluating this at \( r = R \) and rearranging yields
\[ C_2 = \frac{1}{2} C_1 R^3 \quad (21) \]
At $r \to \infty$, we know that $v_r = v_\infty \cos \theta = -\frac{\partial \phi}{\partial r}$, so substituting (Eq. 20) into this expression yields

$$-(C_1 - 2C_2r^{-3}) \cos \theta = v_\infty \cos \theta$$

Evaluating this at $r \to \infty$ and rearranging yields

$$C_1 = -v_\infty$$

Plugging the above expression into (Eq. 21) will yield

$$C_2 = -\frac{1}{2}v_\infty R^2$$

Plugging the newly found expressions for $C_1$ and $C_2$ into $f(r)$ yields

$$f(r) = -v_\infty \left(1 + \frac{R^2}{2r^2}\right)$$

and substituting this $f(r)$ into (Eq. 20) yields

$$\phi(r, \theta) = -v_\infty \left(1 + \frac{R^2}{2r^2}\right) \cos \theta$$

(22)

Now that an expression for $\phi$ is found, we can find the velocity components using $\mathbf{v} = -\nabla \phi$ (or the equivalent definitions in the Appendix). For instance, we know that

$$v_r = -\frac{\partial \phi}{\partial r}, \quad v_\theta = -\frac{1}{r} \frac{\partial \phi}{\partial \theta}$$

Therefore, plugging (Eq. 22) into the above expressions will yield $v_r$ and $v_\theta$ after simplification. They will come to

$$v_r = v_\infty \left(1 - \left(\frac{R}{r}\right)^3\right) \cos \theta$$

and

$$v_\theta = -v_\infty \left(1 + \frac{1}{2} \left(\frac{R}{r}\right)^3\right) \sin \theta$$

6.5 Boundary Layer Theory

- Boundary layer theory\textsuperscript{13} involves the analysis of a very thin region (the boundary layer), which, due to its thinness, can be modeled in Cartesian coordinates despite any apparent curvature. For consistency, $x$ will indicate downstream and $y$ will indicate a direction perpendicular to a solid surface

- To be clear, a boundary layer is the following. Consider a solid object in a fluid. The area we are investigating is near this solid - not on the edge of the solid yet not very far away from the solid. Therefore, we will consider a hypothetical thin boundary layer that surrounds the solid

- Let $v_\infty$ be the approach velocity on the surface (arbitrary dimension), $I_0$ be the length of the object, and $\delta_0$ be the thickness of the thin boundary layer

  – Since we define this as a very thin boundary layer, we know that $\delta_0 \ll I_0$

- The scenario described above is depicted below for reference. The dashed line is the boundary layer:

\textsuperscript{13}All of this is derivation. If you want the meat and potatoes of boundary layer theory, see the Karman momentum balance equation at the end of the section.
• The continuity equation for this system is $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$, and the N-S equation in the $x$ and $y$ directions can be written as

$$(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y}) = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \left( \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right)$$

and

$$(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_y}{\partial y}) = -\frac{1}{\rho} \frac{\partial P}{\partial y} + \nu \left( \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} \right)$$

• To solve a boundary layer problem, approximations need to be made that will simplify the N-S equations

• We know that $v_x = 0$ at the solid-liquid boundary (no-slip condition). Therefore, $v_x$ varies from 0 to $v_\infty$ from the solid’s edge to the edge of the hypothetical boundary layer. Also, $\delta_0$ is the thickness, which is in the $y$ direction. As such\textsuperscript{14},

$${\frac{\partial v_x}{\partial y} = O \left( \frac{v_\infty}{\delta_0} \right)}$$

• In the length direction, $I_0$, the fluid can only slow down once it hits the solid. Therefore, $v_x$ has a maximum of $v_\infty$ such that

$${\frac{\partial v_x}{\partial x} = O \left( \frac{v_\infty}{I_0} \right)}$$

– Integrating (Eq. 26) yields

$$v_x = O \left( \frac{v_\infty}{I_0} \int_0^{I_0} dx \right) = O \left( v_\infty \right)$$

• The continuity equation can be performed to find out that

$${\frac{\partial v_y}{\partial y} = O \left( \frac{v_\infty}{I_0} \right)}$$

– Integrating (Eq. 27) yields

$$v_y = O \left( \frac{v_\infty}{I_0} \int_0^{\delta_0} dy \right) = O \left( \frac{v_\infty \delta_0}{I_0} \right)$$

* This means that $v_y \ll v_x$ since we stated earlier that $\delta_0$ is a very small quantity

• Looking at the N-S equations listed earlier, the terms can be replaced with their order of magnitude equivalents. First, the $x$ direction N-S equation will be analyzed (Eq. 23):

\textsuperscript{14}The $O$ operator indicates “order of magnitude of.”
– First, the following relationship holds due to (Eq. 26) and the fact that \( v_x = O(v_\infty) \)

\[
v_x \frac{\partial v_x}{\partial x} = O\left(\frac{v_\infty^2}{I_0}\right)
\]

(28)

– Second, the following relationship holds due to (Eq. 25) and the fact that \( v_y = O\left(\frac{v_\infty \delta_0}{I_0}\right) \) found earlier

\[
v_y \frac{\partial v_x}{\partial y} = O\left(\frac{v_\infty^2}{I_0}\right)
\]

(29)

– Next, the following relationship holds due to (Eq. 26):

\[
\frac{\partial^2 v_x}{\partial x^2} = O\left(\frac{v_\infty}{I_0}\right)
\]

(30)

* Note that the it is not \( v_x^2 \) in the numerator because a second-derivative indicates two instances of \( dx \) and not two instances of \( v_x \)

– This also means that the following relationship holds due to (Eq. 25):

\[
\frac{\partial^2 v_x}{\partial y^2} = O\left(\frac{v_\infty}{\delta_0}\right)
\]

(31)

– Additionally,

\[
\frac{\partial^2 v_x}{\partial x^2} \ll \frac{\partial^2 v_x}{\partial y^2}
\]

(32)

* This is because we already stated \( I_0 \gg \delta_0 \), and \( \delta_0 \) is in the denominator of (Eq. 31) while \( I_0 \) is in the denominator of (Eq. 30)

• Now all portions of the \( x \) direction N-S equation have been replaced piece-by-piece

• According to boundary layer theory, the left velocity components of N-S equations should have the same order of magnitude as the velocity components on the right side of the N-S equations at the boundary layer

– Therefore, rewriting (Eq. 23) with the previously defined order of magnitude analogues and the fact that \( v_x \gg v_y \) as well as the relationship in (Eq. 32) yields

\[
\frac{v_\infty^2}{I_0} = O\left(\nu \frac{v_\infty}{\delta_0^2}\right)
\]

– Rearranging the above equation and substituting in the Reynolds number yields the more frequently used relationship of

\[
\frac{\delta_0}{I_0} = O\left(\frac{1}{\sqrt{Re}}\right)
\]

• All information from the \( x \) direction N-S equation has now been extracted. To be completely mathematically rigorous, the \( y \) direction N-S equation should be analyzed, but this will simply yield that the N-S equation in the \( y \) direction is much less significant than the \( x \) direction N-S equation (see Eq. 4.4-9 in Bird for a mathematical justification). This does not really come as a surprise since we stated \( v_y \ll v_x \) earlier. This also means that \( \frac{\partial \mathcal{P}}{\partial y} \ll \frac{\partial \mathcal{P}}{\partial x} \) such that the modified pressure is only a function of \( x \).
Collectively, this information leads us to the Prandtl boundary layer equations, which are

\[ \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \]

and

\[ v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho} \frac{d\rho}{dx} + \nu \frac{\partial^2 v_x}{\partial y^2} \]

- The first Prandtl boundary layer equation is simply the continuity equation stated at the beginning of this section, and the second equation is the simplified N-S equation in the \( x \) direction taking into account (Eq. 32)

With these assumptions in place, the Prandtl boundary layer equations can be solved for and will yield the Karman momentum balance of

\[ \mu \frac{\partial v_x}{\partial y} \bigg|_{y=0} = \frac{d}{dx} \int_0^\infty \rho v_x (v_e - v_x) dy + \frac{dv_e}{dx} \int_0^\infty \rho (v_e - v_x) dy \]

- Here, \( v_e \) indicates the external velocity at the outer edge of the boundary layer such that \( v_x (x, y) \rightarrow v_e (x) \) as \( y \rightarrow \infty \)

7 Flow in Chemical Engineering Equipment
8 Appendix

8.1 Newton’s Law of Viscosity

8.1.1 Cartesian

\[ \tau_{xy} = \tau_{yx} = -\mu \left[ \frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right] \]

\[ \tau_{yz} = \tau_{zy} = -\mu \left[ \frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right] \]

\[ \tau_{zx} = \tau_{xz} = -\mu \left[ \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right] \]

8.1.2 Cylindrical

\[ \tau_{r\theta} = \tau_{\theta r} = -\mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right] \]

\[ \tau_{\theta z} = \tau_{z\theta} = -\mu \left[ \frac{1}{r} \frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \right] \]

\[ \tau_{zr} = \tau_{r z} = -\mu \left[ \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right] \]

8.1.3 Spherical

\[ \tau_{r \theta} = \tau_{\theta r} = -\mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right] \]

\[ \tau_{\theta \phi} = \tau_{\phi \theta} = -\mu \left[ \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} \frac{\sin \theta}{\sin \theta} + \frac{1}{r} \frac{\partial v_\phi}{\partial \theta} \right] \]

\[ \tau_{\phi r} = \tau_{r \phi} = -\mu \left[ \frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} + \frac{1}{r} \frac{\partial v_\phi}{\partial r} \right] \]

8.2 Gradient

\[ \nabla f = \frac{\partial f}{\partial x} \hat{x} + \frac{\partial f}{\partial y} \hat{y} + \frac{\partial f}{\partial z} \hat{z} \] (Cartesian)

\[ \nabla f = \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\theta} + \frac{\partial f}{\partial z} \hat{z} \] (Cylindrical)

\[ \nabla f = \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\phi} \] (Spherical)

8.3 Divergence

\[ \nabla \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \] (Cartesian)

\[ \nabla \cdot \vec{v} = \frac{1}{r} \frac{\partial}{\partial r} \left( rv_r \right) + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_\phi}{\partial z} \] (Cylindrical)

\[ \nabla \cdot \vec{v} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 v_r \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( v_\theta \sin \theta \right) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} \] (Spherical)
8.4 Curl
\[ \nabla \times \vec{v} = \left( \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{x} + \left( \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{y} + \left( \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{z} \text{ (Cartesian)} \]
\[ \nabla \times \vec{v} = \left( \frac{1}{r} \frac{\partial v_z}{\partial \theta} - \frac{\partial v_y}{\partial z} \right) \hat{r} + \left( \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial r} \right) \hat{\theta} + \frac{1}{r} \left( \frac{\partial (rv_y)}{\partial r} - \frac{\partial v_r}{\partial \theta} \right) \hat{\phi} \text{ (Cylindrical)} \]
\[ \nabla \times \vec{v} = \frac{1}{r \sin \theta} \left( \frac{\partial (v_\phi \sin \theta)}{\partial \theta} - \frac{\partial v_\theta}{\partial \phi} \right) \hat{r} + \left( \frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{1}{r} \frac{\partial (rv_\theta)}{\partial r} \right) \hat{\theta} + \frac{1}{r} \left( \frac{\partial (rv_\phi)}{\partial r} - \frac{\partial v_r}{\partial \theta} \right) \hat{\phi} \text{ (Spherical)} \]

8.5 Laplacian
\[ \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \text{ (Cartesian)} \]
\[ \nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \text{ (Cylindrical)} \]
\[ \nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \phi^2} \text{ (Spherical)} \]

8.6 Continuity Equation

\[ [\frac{\partial \rho}{\partial t} + (\nabla \cdot \rho \vec{v}) = 0] \]

<table>
<thead>
<tr>
<th>Cartesian coordinates (x, y, z):</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho v_x) + \frac{\partial}{\partial y} (\rho v_y) + \frac{\partial}{\partial z} (\rho v_z) = 0 ]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Cylindrical coordinates (r, θ, z):</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (\rho v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\rho v_\theta) + \frac{\partial}{\partial z} (\rho v_z) = 0 ]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Spherical coordinates (r, θ, φ):</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (\rho r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\rho v_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (\rho v_\phi) = 0 ]</td>
</tr>
</tbody>
</table>

Note that at constant ρ, the above equations simply become \( \nabla \cdot \vec{v} = 0 \)
8.7 Navier-Stokes Equation

Cartesian coordinates (x, y, z):

\[
\rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left[ \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right] + \rho g_x.
\]

\[
\rho \left( \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left[ \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right] + \rho g_y.
\]

\[
\rho \left( \frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left[ \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right] + \rho g_z.
\]

\[
v_r = \frac{1}{r} \frac{\partial \psi}{\partial r}, \quad v_\theta = \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \theta} \quad \text{radians,}
\]

\[
v_r = \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial r}, \quad v_\theta = \frac{1}{r \sin \theta \cos \phi} \frac{\partial \phi}{\partial \theta} \quad \text{radians,}
\]

\[
v_r = \frac{1}{r^2 \sin \theta} \frac{\partial \phi}{\partial r}, \quad v_\theta = \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \theta} \quad \text{radians}.
\]

8.8 Stream Functions and Velocity Potentials

8.8.1 Velocity Components

**Note:** The Bird definitions have been used here. The Wilkes definition has all velocity potentials as positive and all stream functions with swapped signs (except for the spherical definition).

\[
v_x = -\frac{\partial \psi}{\partial y} = -\frac{\partial \phi}{\partial x}, \quad v_y = \frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y} \quad \text{(Cartesian)}
\]

\[
v_r = \frac{1}{r} \frac{\partial \psi}{\partial r}, \quad v_\theta = \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \theta} \quad \text{(Cylindrical with } v_z = 0) \]

\[
v_r = \frac{1}{r} \frac{\partial \phi}{\partial r}, \quad v_\theta = \frac{1}{r \cos \phi} \frac{\partial \phi}{\partial \theta} \quad \text{(Cylindrical with } v_\theta = 0) \]

\[
v_r = \frac{1}{r^2 \sin \theta} \frac{\partial \phi}{\partial r}, \quad v_\theta = \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \theta} \quad \text{(Spherical)}
\]
8.8.2 Differential Equations

1. Planar Flow

(a) For Cartesian with $v_z = 0$ and no $z$-dependence:

\[
\frac{\partial}{\partial t} (\nabla^2 \psi) + \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = \nu \nabla^4 \psi
\]

\[
\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}
\]

\[
\nabla^4 \psi \equiv \left( \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \right) \psi
\]

(b) For cylindrical coordinate with $v_z = 0$ and no $z$-dependence:

\[
\frac{\partial}{\partial t} (\nabla^2 \psi) + \frac{1}{r} \left[ \frac{\partial \psi}{\partial r} \frac{\partial^2 \psi}{\partial \theta \partial r} - \frac{\partial \psi}{\partial \theta} \frac{\partial^2 \psi}{\partial r \partial \theta} \right] = \nu \nabla^4 \psi
\]

\[
\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}
\]

2. Axisymmetrical

(a) For cylindrical with $v_z = 0$ and no $z$-dependence:

\[
\frac{\partial}{\partial t} (E^2 \psi) - \frac{1}{r} \left[ \frac{\partial \psi}{\partial r} \frac{\partial (E^2 \psi)}{\partial z} - \frac{\partial \psi}{\partial z} \frac{\partial (E^2 \psi)}{\partial r} \right] - \frac{2}{r^2} \frac{\partial \psi}{\partial z} E^2 \psi = \nu E^2 (E^2 \psi)
\]

\[
E^2 \equiv \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}
\]

(b) For spherical with $v_\phi = 0$ and no $\phi$-dependence:

\[
\frac{\partial}{\partial t} (E^2 \psi) + \frac{1}{r^2 \sin \theta} \left[ \frac{\partial \psi}{\partial r} \frac{\partial (E^2 \psi)}{\partial \theta} - \frac{\partial \psi}{\partial \theta} \frac{\partial (E^2 \psi)}{\partial r} \right] - \frac{2E^2 \psi}{r^2 \sin^2 \theta} \left( \frac{\partial \psi}{\partial r} \cos \theta - \frac{1}{r} \frac{\partial \psi}{\partial \theta} \sin \theta \right) = \nu E^2 (E^2 \psi)
\]

\[
E^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right)
\]