Math 145	Tufts University Dept of Mathematics	Oct 28, 2013
	Midterm Exam	McNinch/Duchin

Throughout this exam, the symbol $\mathbf{Z}$ represents the set of all *integers*. If $n \in \mathbf{Z}_{>0}$, then $\mathbf{Z}_n$ denotes the set of congruence classes of integers mod n, and $\mathbf{Z}_n^\times$ denotes the set of *units* of $\mathbf{Z}_n$.

The point values add up to 80. This is an 80-minute exam, so the point values double as time recommendations (e.g, 8 minutes on an 8-point problem).

You must justify/show work for all answers.

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1. (4 points) Give careful statements of the following:

~~(a) the *fundamental theorem of arithmetic*.~~

~~(b) the definition of an *associative binary operation* on a set S.~~

2. (12 points) Decide whether each of the following assertions is *True or False*. Briefly justify your conclusion.

~~(a) Let $n \in \mathbf{Z}$ with $n > 1$. If $a \in \mathbf{Z}$ and $3a \equiv 3 \pmod{3n}$, then $a \equiv 1 \pmod{n}$.~~

(b) The groups $\mathbf{Z}_8^\times$ and $\mathbf{Z}_4$ are isomorphic.

(c) The set X of *all* functions $f : \{1,2,3\} \to \{1,2,3\}$ is a group with operation given by composition of functions.

3. (8 points) If x is a real number, write $\lfloor x \rfloor$ for the *greatest integer* $n \in \mathbf{Z}$ with the property that $n \le x$. Consider the function $f : \mathbf{R}^2 \to \mathbf{Z}^2$ given by the rule that f maps $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} \lfloor x \rfloor \\ \lfloor y \rfloor \end{pmatrix}$. Then f determines an equivalence relation via $\begin{pmatrix} x \\ y \end{pmatrix} \sim \begin{pmatrix} u \\ v \end{pmatrix} \iff f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = f\left(\begin{pmatrix} u \\ v \end{pmatrix}\right)$.

(a) Give three distinct elements of $\mathbf{R}^2$ that are in the equivalence class $\left[\begin{pmatrix} 13/3 \\ \sqrt{10} \end{pmatrix}\right]$.

(b) Describe geometrically how $\sim$ partitions the domain $\mathbf{R}^2$ into equivalence classes.

4. (8 points) Let G be a finite group and suppose it has elements $a, b \in G$ with $o(a) = 10$ and $o(b) = 15$. Let $O(G) = \{m : o(g) = m \text{ for some } g \in G\}$ be the set of all numbers that are orders of elements of G.

(a) Find a subgroup of G having order 3, and show that $3 \in O(G)$.

(b) For each of the following numbers n, decide whether or not $O(G)$ **must** contain n. Be sure to indicate your reasoning.

$$n = 7, \quad n = 5, \quad \text{and} \quad n = 25.$$

5. (12 points)

(a) If p is a prime number bigger than n, show that S_n has no element of order p.

(b) Decide whether or not S_7 has an element of order 12. Either give an example, or give a proof that no such element exists.

(c) Give an example of an infinite set S and an element σ of the group $\mathrm{Sym}(S)$ such that $o(\sigma) = 3$.

EXAM CONTINUES ON REVERSE.

6. (18 points) Consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 6 & 5 & 4 \end{pmatrix} \in S_6$$

(a) Write σ as a product of disjoint cycles.

(b) Write σ as a product of transpositions.

(c) What is the parity of σ? (That is, even or odd?)

(d) What is the order $o(\sigma)$?

(e) Suppose that the vertices of a regular hexagon are labeled in counterclockwise order with the numbers 1 through 6. Then the dihedral group D_6 of symmetries of the hexagon can be identified with a subgroup of S_6. Is $\sigma \in D_6$?

7. (18 points) Let us abbreviate $0 = [0]$ and $1 = [1]$ in $\mathbf{Z}_5$. Let $G = \mathrm{GL}_2(\mathbf{Z}_5)$.

(a) Let $H = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x, y \in \mathbf{Z}_5^\times \right\}$. Show that H is a *subgroup* of G and find its order.

(b) Let $K = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$. Write out all the elements of K and show that $K \cong \mathbf{Z}_m$ for some m.

(c) If $A \in H$, carefully check that $A \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot A^{-1} \in K$.

END OF EXAM

EXTRA CREDIT:

(a) Write $d = u \cdot 27 + v \cdot 100$ for $u, v \in \mathbf{Z}$, where $d = \gcd(27, 100)$.

(b) Find $[27]^{-1}$ in the group $\mathbf{Z}_{100}^\times$.

(c) Find all incongruent solutions to the equation $100x \equiv 2 \pmod{27}$.