QUIZ 4

Math 145, Abstract Algebra, Duchin

(1) Using the definition of $F[x]$ as all polynomials in x with coefficients in F, explain why $\mathbb{Q}[\sqrt{3}] = \{a + b\sqrt{3} : a, b \in \mathbb{Q}\}$.

> Any monomial of the form $a_k(\sqrt{3})^k$ where $a \in \mathbb{Q}$ can be rewritten: if $k = 2m$ is even, then $(\sqrt{3})^k = 3^m$ is rational, and if $k = 2m+1$ is odd, then $(\sqrt{3})^k = 3^m\sqrt{3}$. So either $a_k(\sqrt{3})^k$ is rational, or it's rational times $\sqrt{3}$. That means that every polynomial in $\sqrt{3}$ can be reduced to $a + b\sqrt{3}$ by combining terms.

(2) Using the definition of $F(x)$ as the smallest field containing F and x, explain why $\mathbb{Q}(\sqrt{3}) = \mathbb{Q}[\sqrt{3}]$.

> I just need to show that $\mathbb{Q}[\sqrt{3}]$ is already a field, so it does not have to be expanded any more. To do that, I need to show that every element is invertible. We have
> $$\frac{1}{a + b\sqrt{3}} = \frac{a - b\sqrt{3}}{a^2 - 3b^2} = \frac{a}{a^2 - 3b^2} + \frac{-b}{a^2 - 3b^2}\sqrt{3},$$
> which means that this ring is already a field. (Note that the denominator of this expression has no solutions in rationals, so the expression is well-defined.)

(3) How many polynomials of degree three are there in $\mathbb{Z}_2[x]$? How many of them are monic?

> The coefficients are all either 0 or 1, so the polynomials are of the form $x^3 + a_2x^2 + a_1x + a_0$. So there are three coefficients that can be 0 or 1, which means $2 \cdot 2 \cdot 2 = 8$ choices.
> All of these polynomials are monic.