Final Exam Practice Problems Math 145, Abstract Algebra, Duchin

(1) Consider a regular octagon with vertices labeled cyclically. Consider $C_8 \hookrightarrow D_8 \hookrightarrow S_8$, where the copy of C_8 is generated by a counterclockwise rotation r, the copy of D_8 is all the (rigid) symmetries of the octagon, and the symmetric group is given by all the permutations of the labels, whether or not they can be rigidly achieved.

What is $|D_8/C_8|$? What does an arbitrary element of D_8/C_8 look like? Name two distinct elements. Is D_8/C_8 a group?

What is $|S_8/D_8|$? What does an arbitrary element of S_8/D_8 look like? Name two distinct elements. Is S_8/D_8 a group?

(2) Show that every finite integral domain is a field by considering all the powers of an arbitrary element $a \in D$ and using that to find an inverse of a. Give an example of a finite ring that is not a field and name some of its zero divisors.

(3) Recall that $Q(D)$ is the "fraction field" associated to an integral domain D. Formally, its elements are denoted as fractions $\frac{a}{b}$ where $a, b \in D$ and $b \neq 0$. Explain how addition is defined and why the definition requires that D be an integral domain.

(4) How many distinct elements in the free group $F_3 = \langle a, b, c\rangle$ have word length ≤ 2? How about the Heisenberg group $H(\mathbb{Z}) = \langle a, b, c\rangle$ (where a, b, c represent the elementary matrices, as usual)?