## MATH 145-02: WORKSHEET

This worksheet, which is your last problem set of the semester, covers the two post-midterm 2 topics: (A-B) all groups are quotients of free groups, and (C-D) free groups are "paradoxical."

Problem A Let $G$ be the quotient $F_2 \,/\, \langle b \rangle$. (a) What is a simplified form of $ab^8a^5b^{10}$? (b) What is a normal form for the elements of $G$? (In other words, what is a full set of representatives for the cosets?) (c) What familiar group is $G$ isomorphic to?

Problem B Let $G$ be the quotient $F_2 \,/\, \langle a^4, b^4, aba^{-1}b^{-1} \rangle$. In other words, this quotient is formed by the equivalence relations $a^4 \equiv e$, $b^4 \equiv e$, $ab \equiv ba$. Now answer (a),(b),(c) just like in the last question!

Problem C Recall that the word length in $G = \langle S|R\rangle$ is defined by letting $|g|_S$ equal the shortest "spelling" of $g$ in letters from the alphabet $S$. The corresponding distance function is $d(g,h) = \left|g^{-1}h\right|_S$.

(a) Consider the group $G = \mathbb{Z}^2$ generated by $S = \left\{\left[\begin{smallmatrix}\pm 2\\ \pm 1\end{smallmatrix}\right], \left[\begin{smallmatrix}\pm 1\\ \pm 2\end{smallmatrix}\right]\right\}$. In other words, the generators are the "moves" that can be made by a chess knight. For this generating set, what is $\left|\left[\begin{smallmatrix}1\\ 0\end{smallmatrix}\right]\right|_S$? (Prove it.)

(b) Consider the group $G = \mathbb{Z}^2$ generated by $S = \{\left[\begin{smallmatrix}1\\ 0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\ 1\end{smallmatrix}\right]\}$. How many group elements $g$ have $|g|_S \leq n$? (Work your way up to this with $n = 2, 3, 4$.) Sketch a picture that shows the words of length $\leq n$ in the Cayley graph.

Problem D Show that left-multiplication by any group element is a rigid motion by showing that $d(g,h) = d(jg, jh)$ for any $j \in G$. (This is needed to show that the group $F_2$ rigidly covers itself two times over!)