Lemma 1. *If $\boldsymbol{A}$ has unit-norm columns and coherence $\mu = \mu(\boldsymbol{A})$, then $\boldsymbol{A}$ satisfies the RIP of order k with $\delta_k = (k-1)\mu$ for all $k < \frac{1}{\mu}$.*

Proof. If $\boldsymbol{A}$ has unit-norm columns and coherence $\mu = \mu(\boldsymbol{A})$, it needs to be shown that

$$(1-\delta_k)||\boldsymbol{x}||_2^2 \leq ||A\boldsymbol{x}||_2^2 \leq (1+\delta_k)||\boldsymbol{x}||_2^2 \qquad \forall \boldsymbol{x} \in \Sigma_k$$

with $\delta_k = (k-1)\mu$ and $k < \frac{1}{\mu}$. Let $S \in \{1, ..., n\}$ denote the support of $\boldsymbol{x}$ with $|S| = k$. Denote the restricted matrix by A_s and consider $||\boldsymbol{Ax}||_2^2$.

$$||\boldsymbol{Ax}||_2^2 = ||\boldsymbol{A}_s\boldsymbol{x}_s||_2^2|| = \langle \boldsymbol{A}_s\boldsymbol{x}_s \, , \boldsymbol{A}_s\boldsymbol{x}_s \rangle = \boldsymbol{x}_s^T \boldsymbol{A}_s^T \boldsymbol{A}_s \boldsymbol{x}_s \tag{1}$$

Before proceeding further, we state and prove the claim below which concerns the matrix $\boldsymbol{A}_s^T\boldsymbol{A}_s$.

Claim 1. *$\boldsymbol{A}_s^T\boldsymbol{A}_s$ is symmetric and positive semi-definite. The eigenvalues of $\boldsymbol{A}_s^T\boldsymbol{A}_s$ are real and positive. The eigenvectors of $\boldsymbol{A}_s^T\boldsymbol{A}_s$ are orthogonal.*

Proof. Symmetry is trivial

$$(\boldsymbol{A}_s^T\boldsymbol{A}_s)^T = \boldsymbol{A}_s^T\boldsymbol{A}_s$$

To show positive semi-definiteness, consider $\boldsymbol{y}^T(\boldsymbol{A}_s^T\boldsymbol{A}_s)\boldsymbol{y}$ for any $\boldsymbol{y} \in \mathcal{R}^n$.

$$\boldsymbol{y}^T(\boldsymbol{A}_s^T\boldsymbol{A}_s)\boldsymbol{y} = (\boldsymbol{y}^T\boldsymbol{A}_s^T)(\boldsymbol{A}_s\boldsymbol{y}) = (\boldsymbol{A}_s\boldsymbol{y})^T(\boldsymbol{A}_s\boldsymbol{y}) = ||\boldsymbol{A}_s\boldsymbol{y}||_2^2 \geq 0$$

Since $\boldsymbol{A}_s^T\boldsymbol{A}_s$ is symmetric, it follows that it has real eigenvalues and orthogonal eigenvectors. □

Let $\{\boldsymbol{\beta}_i\}_{i=1}^n$ denote the orthonormal eigenvectors of $\boldsymbol{A}_s^T\boldsymbol{A}_s$. The corresponding eigenvalues will be denoted by $\boldsymbol{\lambda} = \{\lambda_i\}_{i=1}^n$. Any vector $\boldsymbol{x} \in \mathcal{R}^n$ can be expanded in the basis $\{\boldsymbol{\beta}_i\}_{i=1}^n$. With this, (1) can be written as

$$\begin{aligned}
||\boldsymbol{Ax}||_2^2 = \boldsymbol{x}_s^T\boldsymbol{A}_s^T\boldsymbol{A}_s\boldsymbol{x}_s &= \left(\sum_i c_i\boldsymbol{\beta}_i^T\right)\boldsymbol{A}_s^T\boldsymbol{A}_s\left(\sum_j c_j\boldsymbol{\beta}_j\right) \\
&= \left(\sum_i c_i\boldsymbol{\beta}_i^T\right)\left(\sum_j c_j\boldsymbol{A}_s^T\boldsymbol{A}_s\boldsymbol{\beta}_j\right) = \left(\sum_i c_i\boldsymbol{\beta}_i^T\right)\left(\sum_j c_j\lambda_j\boldsymbol{\beta}_j\right) \\
&= \sum_i\sum_j c_ic_j\lambda_j\boldsymbol{\beta}_i^T\boldsymbol{\beta}_j = \sum_i\sum_j c_ic_j\lambda_j\delta_{i,j} \\
&= \sum_i c_i^2\lambda_i
\end{aligned}$$

To bound $||\boldsymbol{Ax}||_2^2$, we note that

$$\min(\boldsymbol{\lambda})\sum_i c_i^2 \leq \sum_i c_i^2\lambda_i \leq \max(\boldsymbol{\lambda})\sum_i c_i^2 \Longrightarrow \min(\boldsymbol{\lambda})||\boldsymbol{x}||^2 \leq ||\boldsymbol{Ax}||_2^2 \leq \max(\boldsymbol{\lambda})||\boldsymbol{x}||_2^2 \tag{2}$$

It remains to find the minimum and maximum eigenvalues of $\boldsymbol{A}_s^T\boldsymbol{A}_s$ via Grešgorin theorem. Let $\boldsymbol{A}_s^i$ denote the i column of $\boldsymbol{A}_s$. With the assumption that $\boldsymbol{A}$ has unit-norm columns in consideration, The (i,j) the entry of $\boldsymbol{A}_s^T\boldsymbol{A}_s$ is given by

$$(\boldsymbol{A}_s^T\boldsymbol{A}_s)_{(i,j)} = \begin{cases} 1 & i = j \\ \langle \boldsymbol{A}_s^i \, , \boldsymbol{A}_s^j \rangle & i \neq j \end{cases}$$

From Grešgorin theorem, the eigenvalues of $\boldsymbol{A}_s^T \boldsymbol{A}_s$ with entries $(\boldsymbol{A}_s^T \boldsymbol{A}_s)_{(i,j)}$, $1 \leq i, j \leq n$, lie in the union of disks $d_i = d_i(c_i, r_i)$, $1 \leq i \leq n$, centered at 1 with radius

$$r_i = \sum_{j \neq i} |\boldsymbol{A}_s^T \boldsymbol{A}_s|_{(i,j)} = \sum_{j \neq i} |\langle \boldsymbol{A}_s^i \, , \boldsymbol{A}_s^j \rangle| \leq (k-1)\mu$$

where the last bound follows from the coherence of $\boldsymbol{A}$ i.e $\mu = \max_{1 \leq i,j \leq n} |\langle \boldsymbol{A}_s^i \, , \boldsymbol{A}_s^j \rangle|$. With this, $\min(\boldsymbol{\lambda}) \geq 1 - (k-1)\mu$ and $\max(\boldsymbol{\lambda}) \leq 1 + (k-1)\mu$. Using these bounds in (2) results

$$(1 - (k-1)\mu) \, ||\boldsymbol{x}||^2 \leq ||\boldsymbol{A}\boldsymbol{x}||_2^2 \leq (1 + (k-1)\mu) \, ||\boldsymbol{x}||_2^2 \tag{3}$$

With this, A satisfies the RIP of order k with $\delta_k = (k-1)\mu$

$$(1 - \delta_k) \, ||\boldsymbol{x}||^2 \leq ||\boldsymbol{A}\boldsymbol{x}||_2^2 \leq (1 + \delta_k) \, ||\boldsymbol{x}||_2^2 \tag{4}$$

RIP requires $\delta_k \in (0, 1)$ which implies that $k > 1$ and $k < \frac{1}{\mu} + 1$. □