ES-2 Lecture 14: Linear Algebraic Eqns and Matrices

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Spring 2015
Today’s Outline

- In-class RAT
- HW8 preview
- Writing systems of equations as matrices
- Engineering examples
- A look ahead: how to solve these
- Last 10 min: Quiz 1 overview
Reminder: Matrix Multiplication

- The elements in the matrix $[C]$ that results from multiplying matrices $[A]$ and $[B]$ are:

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$$
Matrix Multiplication in Pictures

\[
\begin{align*}
\mathbf{A} \, (5 \times n) & \quad \times \quad \mathbf{B} \, (n \times 3) \\
\begin{bmatrix}
\vdots \\
\vdots \\
\vdots \\
\end{bmatrix} & \quad \times \quad \begin{bmatrix}
\vdots \\
\vdots \\
\end{bmatrix} \\
\end{align*}
\]

\[
\mathbf{C} \, (5 \times 3) = 7
\]

We get element in row 3, col 2 of \( \mathbf{C} \) by multiplying row 3 of \( \mathbf{A} \) by col 2 of \( \mathbf{B} \)
Matlab multiplication using (math) variables

- We can rewrite this system of equations in a matrix, in form $Ax = b$

\[
\begin{align*}
y + 2z &= 4 \\
9y + 3z &= 6 \\
\end{align*}
\]

\[
\begin{bmatrix}
1 & 2 \\
9 & 3 \\
\end{bmatrix}
\begin{bmatrix}
y \\
z \\
\end{bmatrix}
=
\begin{bmatrix}
4 \\
6 \\
\end{bmatrix}
\]
How do matrix problems arise in engineering?

- Three main ways
  - conservation laws (force balance, fluid or current flow, etc.)
  - data fitting
  - mixing problems: colors or sound tones
- Today: setting up $Ax=b$ problems
- Thursday and next Tuesday: solving them (and understanding when we can’t)
Traffic flow (an example of conservation)

- Assume all streets are one-way
- For each intersection, cars in = cars out
- Assume we have traffic flow measurements at four locations; want to find the other four
Step 1) Define unknowns
Step 2) @intersections, cars in = cars out

\[ 85 = x_1 + x_2 \]
\[ x_1 + 45 = 120 \]
\[ x_2 + 70 = x_3 \]

\[ x_3 = 45 + x_4 \]
Step 3) Write coupled equations in a matrix

\[ x_1 + x_2 + 0x_3 + 0x_4 = 85 \]
\[ x_1 + 0x_2 + 0x_3 + 0x_4 = 120 \cdot 45 \]

e tc.

\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
=
\begin{bmatrix}
85 \\
75 \\
45
\end{bmatrix}
\]
Pair work: set up $Ax=b$ form for this network

$Z_0 = x_1 + x_2$

I’ll randomly call on pairs after you talk
Pair work answer

\[ \text{det} \left( \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right) \]

\[ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \\ 20 \\ 10 \\ 10 \end{bmatrix} \]
Examples of conservation laws

(a) Chemical engineering

(b) Civil engineering

(c) Electrical engineering

(d) Mechanical engineering
How do matrix problems arise in engineering?

• Three main ways
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### Fitting rocket velocity to a model

(idea from numericalmethodsguy)

<table>
<thead>
<tr>
<th>Time</th>
<th>Speed</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>106.8</td>
</tr>
<tr>
<td>8</td>
<td>167.2</td>
</tr>
<tr>
<td>12</td>
<td>280.4</td>
</tr>
</tbody>
</table>

**Goal:** fit data to:

\[ v(t) = a \ t^2 + b \ t + c \]

**Time 1**

\[ v(5) = a \times 25 + b \times 5 + c = 106.8 \]

\[ v(8) = a \times 64 + b \times 8 + c = 167 \]

\[ a \times 144 + b \times 12 + c = 280.4 \]
Fitting rocket velocity to a model

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<td>280.4</td>
</tr>
</tbody>
</table>

\[
\begin{bmatrix}
25 & 5 & 1 \\
64 & 8 & 1 \\
144 & 12 & 1 \\
\end{bmatrix}
\begin{bmatrix}
[a] \\
[b] \\
[c] \\
\end{bmatrix}
= \begin{bmatrix}
data_1 \\
data_2 \\
data_3 \\
\end{bmatrix}
\]
How do matrix problems arise in engineering?

- Three main ways
  - conservation laws (force balance, fluid or current flow, etc.)
  - data fitting
  - mixing problems: colors or sound tones

For color table lookup, see
http://www.rapidtables.com/web/color/RGB_Color.htm
Color mixing: Tissue staining for breast cancer biopsy

- In cancer biopsies, tissues are stained using several stains with known colors that bind to different cell types
- Color is encoded in red/green/blue on computers
- Goal: quantify the amount of each stain

Data courtesy of S. Bogen, Tufts Med
Matrix problem: color mixing (unmixing is possible final project)

\[
\begin{bmatrix}
0.18 & 0.01 & 0.10 \\
0.20 & 0.13 & 0.21 \\
0.08 & 0.01 & 0.29 \\
\end{bmatrix}
\begin{bmatrix}
C_H \\
C_E \\
C_D \\
\end{bmatrix}
= 
\begin{bmatrix}
R \\
G \\
B \\
\end{bmatrix}
\]

Color matrix: Column = stain 
Row = color 

Dye concentration 

Color channel (per pixel)
Tufts alumni office sets up the matrix problem below:

\[
\begin{bmatrix}
0.8 & 0.1 \\
0.2 & 0.9
\end{bmatrix}
\begin{bmatrix}
D_1 \\
ND_1
\end{bmatrix}
= 
\begin{bmatrix}
D_2 \\
ND_2
\end{bmatrix}
\]

where
\[D = \# \text{ donors},\]
\[ND = \# \text{ non-donors}\]

Subscript is year number (1 or 2)

Questions:
1) If you have 100 donors and 0 non-donors in year 1, what happens in year 2? How about 0 and 100?
2) What do the matrix elements represent?
3) Could you predict year 3, if matrix stays same?
Pair work answer
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Reminder: Special Matrices

Matrices where $m=n$ are called **square matrices**. Special square matrices include:

<table>
<thead>
<tr>
<th>Symmetric</th>
<th>Diagonal</th>
<th>Identity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[A]=\begin{bmatrix} 5 &amp; 1 &amp; 2 \ 1 &amp; 3 &amp; 7 \ 2 &amp; 7 &amp; 8 \end{bmatrix}$</td>
<td>$[A]=\begin{bmatrix} a_{11} \ a_{22} \ a_{33} \end{bmatrix}$</td>
<td>$[A]=\begin{bmatrix} 1 \ 1 \ 1 \end{bmatrix}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Upper Triangular</th>
<th>Lower Triangular</th>
<th>Banded</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[A]=\begin{bmatrix} a_{11} &amp; a_{12} &amp; a_{13} \ a_{22} &amp; a_{23} &amp; \cdot \cdot \cdot \ a_{33} &amp; &amp; \end{bmatrix}$</td>
<td>$[A]=\begin{bmatrix} a_{11} &amp; \cdot \cdot \cdot &amp; \cdot \cdot \cdot \ a_{22} &amp; a_{22} &amp; \cdot \cdot \cdot \ a_{33} &amp; a_{32} &amp; a_{33} \end{bmatrix}$</td>
<td>$[A]=\begin{bmatrix} a_{11} &amp; a_{12} &amp; \cdot \cdot \cdot &amp; \cdot \cdot \cdot \ a_{21} &amp; a_{22} &amp; a_{23} &amp; \cdot \cdot \cdot \ a_{31} &amp; a_{32} &amp; a_{33} &amp; a_{34} \ a_{41} &amp; a_{42} &amp; a_{43} &amp; a_{44} \end{bmatrix}$</td>
</tr>
</tbody>
</table>
Back to original problem for
\[ Ax = b \]

\[
\begin{align*}
y + 2z &= 4 \\
9y + 3z &= 6
\end{align*}
\]

Set up matrices

\[ A = \begin{bmatrix} 1 & 2 \\ 9 & 3 \end{bmatrix}; \quad \text{and} \quad b = \begin{bmatrix} 4 \\ 6 \end{bmatrix}; \]

- Option 1:
  \[ x = A\backslash b \]

- Option 2:
  \[ x = \text{inv}(A) \cdot b \]

This is basically it, for the Matlab syntax!

... but \`\backslash\` is very complicated
Option 1: \( x = A\backslash b \)

Think about solving the problem by hand:

1. \( y + 2z = 4 \)
2. \( 9y + 3z = 6 \)

\[ \Rightarrow \text{solve } \odot 2 \text{ for } y \text{ in terms of } z \]

\[ \Rightarrow \text{plug into } \odot 1 \text{ to find } z \]

\[ \Rightarrow \text{solve for } z \]

\[ \Rightarrow \text{plug back in} \]
Option 1: $x = A\backslash b$

Think about solving the problem by hand:

\[
y + 2z = 4 \\
9y + 3z = 6
\]
Naïve Gauss Elimination

• **Forward elimination**
  – Add or subtract multiples of row 1 to eliminate the first coefficient from rows 2, 3, ...
  – Continue for row 2, etc
  – Stop when an upper triangular matrix remains.

• **Back substitution**
  – Starting with the last row, solve for the unknown, then substitute back into the next highest row..
  – Keep going until reach top

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & | & b_1 \\
a_{21} & a_{22} & a_{23} & | & b_2 \\
a_{31} & a_{32} & a_{33} & | & b_3 \\
\end{bmatrix}
\]

(a) **Forward elimination**

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & | & b_1 \\
a'_{22} & a'_{23} & | & b'_2 \\
a''_{33} & | & b''_3 \\
\end{bmatrix}
\]

(b) **Back substitution**

\[
x_3 = \frac{b''_3}{a''_{33}}
\]

\[
x_2 = \frac{(b_2' - a'_{23}x_3)}{a'_{22}}
\]

\[
x_1 = \frac{(b_1 - a_{13}x_3 - a_{12}x_2)}{a_{11}}
\]
Pivoting

- Problems arise if a coefficient along the diagonal is 0 (problem: division by 0) or close to 0 (problem: round-off error)

- *Pivoting*: check for this problem and switch the order of rows (and if needed, columns) in the matrix to always divide by the largest element in the column you’re working on.
Example: Forward elimination

\[ \begin{align*}
4x_0 + 6x_1 + 2x_2 &= 8 \\
2x_0 + 10x_1 + 5x_2 &= 4 \\
-4x_0 - 3x_1 - 5x_2 &= 1
\end{align*} \]

Basic idea:
- leave top equation alone
- replace equation 2 by (equation 2 + a scaled version of equation 1). This "works" because we are adding (or subtracting) the same amount to both sides of equation 2, so equality holds
- replace equation 3 by (equation 3 + a scaled version of equation 1)
Example: Forward elimination

\[
\begin{align*}
4x_0 + 6x_1 + 2x_2 &= 8 \\
-3x_1 + 4x_2 &= 0 \\
3x_1 - 3x_2 &= 9
\end{align*}
\]

Same as before, but now:
- leave 2nd equation alone
- replace 3rd eqn with (3rd eqn + a scaled version of Eq 2)
Example: Done w/ Forward elimination

\[4x_0 + 6x_1 + 2x_2 = 8\]

\[-3x_1 + 4x_2 = 0\]

\[x_2 = 9\]
Example: Back substitution

\[ 4x_0 + 6x_1 + 2x_2 = 8 \]
\[ -3x_1 + 4 \times 9 = 0 \]
\[ x_2 = 9 \]
Example: Back substitution

\[4x_0 + 6x_1 + 2x_2 = 8\]

\[x_1 = 12\]

\[x_2 = 9\]
Example: Back substitution

\[ 4x_0 + 6 \times 12 + 2 \times 9 = 8 \]

- \[ x_1 = 12 \]
- \[ x_2 = 9 \]
Example: Back substitution

\[ x_0 = \frac{-82}{4} = -20.5 \]

\[ x_1 = 12 \]

\[ x_2 = 9 \]