Moduli of Finite-Codimensional Subalgebras of $\bigoplus_{i=1}^{m} k[[t_i]]$

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Objective

Describe the moduli space of complete local k-subalgebras of the direct sum $\bigoplus_{i=1}^{m} k[[t_i]]$. These subalgebras arise as complete local rings at m-branch curve singularities. For $m = 1$, this moduli space has already been described [2].

The Moduli Space

Let A be a Noetherian k-algebra, δ a natural number. The **δ-territory** $\text{ter}^{\delta} A$ parametrizes δ-codimensional k-subalgebras of A. This is a projective closed subscheme of a Grassmannian. $\text{ter}^{\delta}(\bigoplus_{i=1}^{m} k[[t_i]])$ is connected if and only if $m = 1$.

Constructing Subalgebras

(Goursat's Lemma.) Every k-subalgebra S of the direct sum $k[[t_1]] \oplus k[[t_2]]$ is uniquely determined by the data

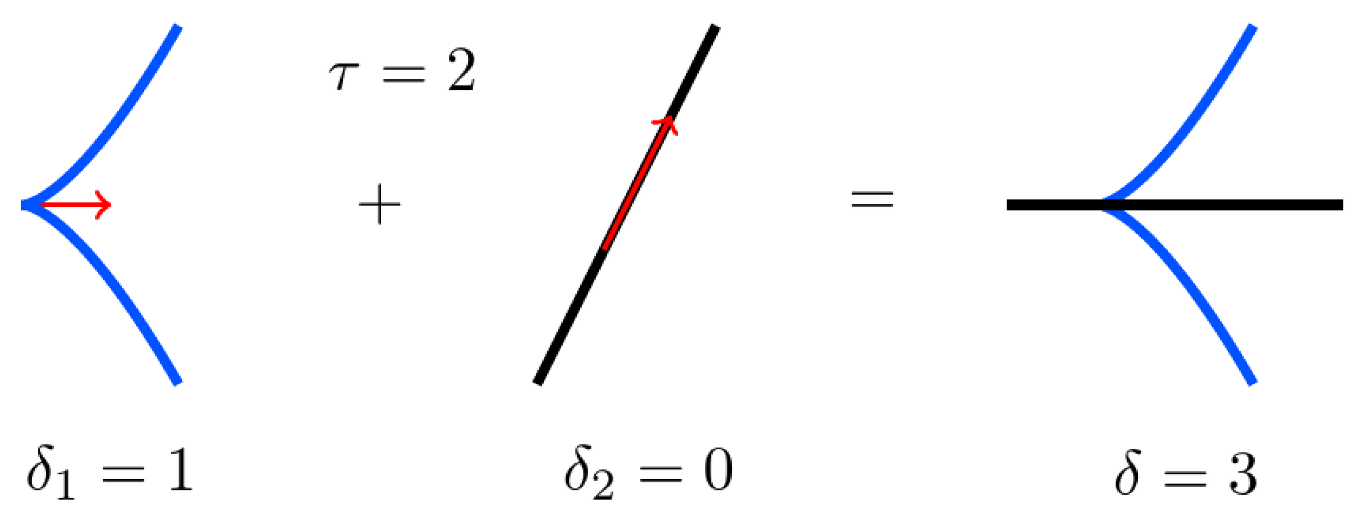
$$(S_1, I_1, S_2, I_2, \varphi)$$

where $S_1 \subset k[[t_1]]$, $S_2 \subset k[[t_2]]$ are k-subalgebras, $I_i \subset S_i$ are ideals, and $\varphi : S_1/I_1 \to S_2/I_2$ is a k-algebra isomorphism.

$$S = \{(s_1, s_2) \in S_1 \oplus S_2 : \varphi(s_1 + I_1) = s_2 + I_2\}.$$

Three quantities contribute to the codimension:

- δ_1 = codimension of S_1 in $k[[t_1]]$ $\Rightarrow$ delta-invariant of the singularity on branch 1
- δ_2 = codimension of S_2 in $k[[t_2]]$ $\Rightarrow$ delta-invariant of the singularity on branch 2
- τ = codimension of I_i in S_i $\Rightarrow$ intersection multiplicity of the branches



Invoke Goursat's Lemma recursively to construct k-subalgebras of a direct sum with m summands.

Glued Subalgebras

A k-subalgebra is **glued** if the corresponding curve singularity is connected. Not all k-subalgebras of $\bigoplus_{i=1}^{m} k[[t_i]]$ are glued when $m \geq 2$. $k[[(t_1, 0), (0, t_2)]]$ is glued (the ordinary node), but $k[[t_1]] \oplus k[[t_2^2, t_2^3]]$ is not.

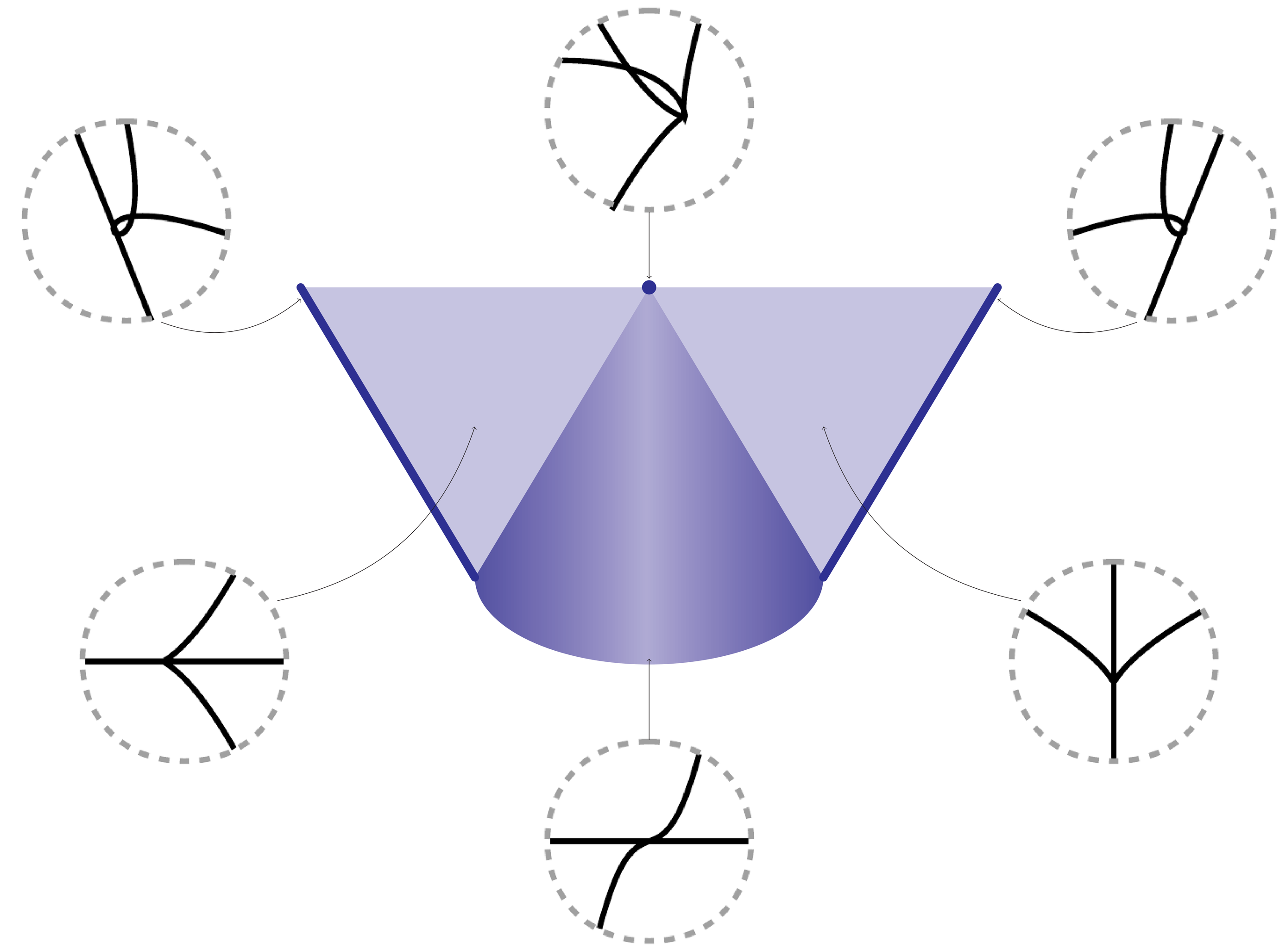


Figure: $\delta = 3, m = 2$

Main Results

Theorem. (G. [1]) The glued subalgebras in $\text{ter}^{\delta}(\bigoplus_{i=1}^{m} k[[t_i]])$ form a closed connected subscheme. It encodes the different ways an m-branch singularity of delta-invariant δ can be "glued" to a smooth curve.

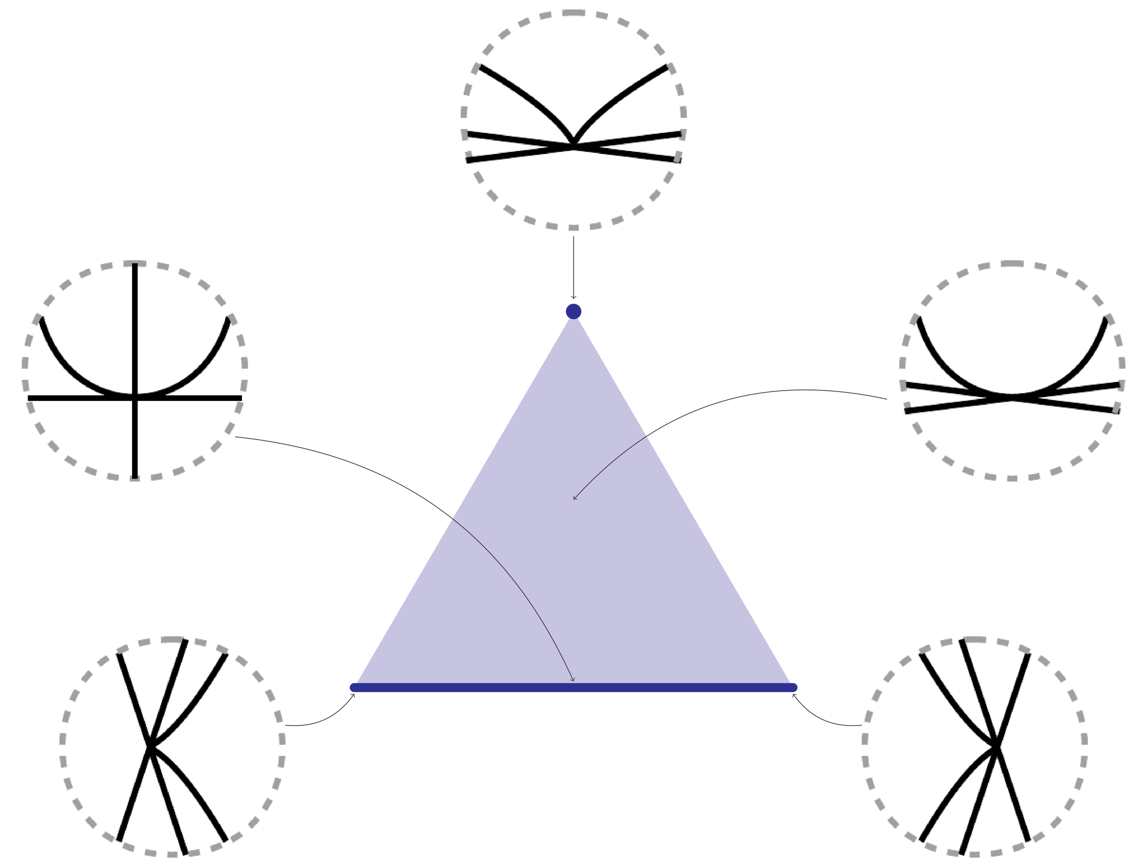


Figure: $\delta = 3, m = 3$

References

[1] C. Guevara, The Moduli Space of Finite-Codimensional k-Subalgebras of $\bigoplus_{i=1}^{m} k[[t_i]]$. (In preparation).

[2] S. Ishii, Moduli of Subrings of a Local Ring. *Journal of Algebra*, 67, 504-516, 1980.