# A “CRASH” COURSE ON BLOW-UPS

CHRISTOPHER GUEVARA

Abstract. Blow-ups are useful tools in algebraic geometry. We will look at one specific yet common example, namely the blow-up of the affine plane at the origin. The result is a space which is isomorphic to the affine plane, except at the origin; the origin is replaced with a projective line. We will see that blow-ups are primarily useful in resolving (i.e. smoothing) singularities, in this case those on curves in the plane. We will assume no prerequisite knowledge in algebraic geometry.

## 1. Motivation

For this talk, we will work over the real numbers. Let the *affine plane* be

$$\mathbb{A}^2 := \{(a, b) : a, b \in \mathbb{R}\}.$$

This is essentially $\mathbb{R}^2$, we just use different notation so as not to confuse with the two-dimensional real vector space (i.e. we forget the vector space structure). Let $f = y^2 - x^3 - x^2$, and consider the curve $C \subset \mathbb{A}^2$ given by the points satisfying the equation $f = 0$. By looking at its graph, we would agree that the curve is not smooth at the origin $P = (0, 0)$, because two different branches intersect there and form a *node*. One way to verify this is via the Jacobian (which you may be familiar with from differential geometry). Consider the matrix

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} -3x^2 - 2x & 2y \end{bmatrix}.$$

A point $(a, b) \in C$ will be smooth if this matrix, when evaluated at the point $(a, b)$, has rank at least $2-1 = 1$, where the 2 is the dimension of the ambient plane we are in, and 1 is the codimension of the point in the curve $C$. If $(a, b) \neq (0, 0)$, then one of the coordinates is nonzero, so one of the entries in the matrix is nonzero, thereby giving it rank 1. So $C$ is smooth at all of these points. However, at the origin, this matrix has no rank, so it is not smooth here. We call this point *singular*, or a *singularity*.

Ideally, we want to smooth this curve, or *resolve* its singularity. To be more precise, what we want is a map $\pi : \tilde{C} \to C$, such that

(1) $\tilde{C}$ is a smooth curve (i.e. smooth at all points),
(2) $\pi$ is surjective,
(3) at every point $Q \neq P \in C$, $\pi^{-1}(Q)$ consists of exactly one point (together with condition (2), this says that $\pi^{-1}(C \setminus P) \to C \setminus P$ is an bijection).

You can envision adding a third dimension, i.e. a “$z$-axis”, and then stretching the curve out along that new dimension. The resulting curve is something like a helix shape, and should now be smooth everywhere since we have separated those two branches at the node. We will describe this method in algebraic terms, and call it a *blow-up*.

## 2. Brief Introduction to Varieties and the Projective Line

Let $\mathbb{R}[x, y]$ denote the ring of polynomials in the indeterminates $x, y$ with coefficients in $\mathbb{R}$. Given a polynomial $f \in \mathbb{R}[x, y]$, we can ask what its *vanishing locus* is, i.e. $\{(a, b) \in \mathbb{A}^2 : f(a, b) = 0\}$. This will define a subset of $\mathbb{A}^2$, which for our purposes will be called a variety. More formally, given any finite collection of polynomials $f_1, \dots, f_r \in \mathbb{R}[x, y]$, an *affine variety in* $\mathbb{A}^2$ is

$$V(f_1, \dots, f_r) := \{(a, b) \in \mathbb{A}^2 : f_1(a, b) = \cdots = f_r(a, b) = 0\}.$$

The variety $V(x,y)$ is simply the origin $(0,0)$ in $\mathbb{A}^2$. The variety $V(x^2-y)$ is the set of points $\{(a,a^2) : a \in \mathbb{R}\}$, which corresponds to the standard parabola. The affine plane $\mathbb{A}^2$ is itself a variety, since it can be viewed as $V(0)$, where 0 is the zero polynomial in $\mathbb{R}[x,y]$. If $X, Y$ are varieties and $X \subset Y$, we say $X$ is a *subvariety* of $Y$. Hence every variety in $\mathbb{A}^2$ is a subvariety of $\mathbb{A}^2$. If $f, g$ are polynomials, then $V(f,g)$ is a subvariety of $V(f)$.

For our purposes, a variety of the form $V(f)$ will determine a *curve*. Hence when $f = y^2 - x^3 - x^2$ in our motivating example, we in fact saw that $C = V(f)$ was a curve. If $f, g$ are "different enough" polynomials, meaning one is not a multiple of the other, then $V(f,g)$ should consist of finitely many points, namely the points of intersection of $V(f)$ and $V(g)$. For example, $V(x^2-y, y-4)$ consists of the points $(2,4)$ and $(-2,4)$, which is where the curves $x^2-y=0$ and $y-4=0$ intersect. In general, if $f, g$ are "different enough" polynomials, then $\mathbb{A}^2$ is two-dimensional, the curve $V(f)$ is a one-dimensional subvariety of $\mathbb{A}^2$, and the finitely-many points in $V(f,g)$ is a zero-dimensional subvariety of $\mathbb{A}^2$. Every time we add a "different enough" polynomial, the dimension goes down by one.

We will also need to know about the projective line. We will define this to be

$$\mathbb{P}^1 := (\mathbb{A}^2 \setminus \{(0,0)\})/\sim$$

where $(a,b) \sim (c,d)$ if and only if there is a nonzero constant $\lambda \in \mathbb{C}$ such that $(c,d) = (\lambda a, \lambda b)$. So for example, $(2, \frac{3}{2}) \sim (4,3)$, hence they define the same point in $\mathbb{P}^1$. To notationally distinguish points in $\mathbb{P}^1$ and points in $\mathbb{A}^2$, we will write $(a : b) \in \mathbb{P}^1$ to denote the equivalence class of $(a,b) \in \mathbb{A}^2$. Note also that there is no such thing as the point $(0:0)$ in the projective line; there is always at least one nonzero coordinate in every projective point.

Suppose $(a:b) \in \mathbb{P}^1$. One of $a, b$ must be nonzero. If $a \neq 0$, then we may assume $a = 1$ (since we can multiply by the appropriate scaling factor to make $a = 1$) and then $b$ is free to be any value in $\mathbb{R}$. If $a = 0$, then it must be the case that $b \neq 0$, so by similar reasoning $b = 1$. Hence we obtain the following points:

$$\mathbb{P}^1 = \{(1:b) : b \in \mathbb{R}\} \cup \{(0:1)\}.$$

Note that $\mathbb{P}^1$ "looks like" the real line $\mathbb{R}$ with an added point, which we call the "point at infinity". Hence we can interpret $\mathbb{P}^1$ as a "compactification" of the real line obtained by adding this point at infinity.

We can also think of $\mathbb{P}^1$ as being the space that parametrizes all of the lines in $\mathbb{A}^2$ through the origin. Two points $(a,b), (c,d) \in \mathbb{A}^2$ are equivalent in $\mathbb{P}^1$ if they differ by a scaling factor; equivalently $\frac{b}{a} = \frac{d}{c}$. Hence points which are equivalent lie on the same line through the origin in $\mathbb{A}^2$, since the slope is constant. If they are represented by the point $(1:b) \in \mathbb{P}^1$, then this corresponds to all points on the line $y = bx$ in $\mathbb{A}^2$. In particular, $(1:0)$ corresponds to the $x$-axis. If they are represented by the point $(0:1) \in \mathbb{P}^1$, then this corresponds to the $y$-axis, as this has "no slope".

## 3. Blow-Up of $\mathbb{A}^2$ at the Origin

The most straightforward example of a blow-up is the blow-up of $\mathbb{A}^2$ at the origin $(0,0)$. As we noted earlier, this is the subvariety $V(x,y)$. Let's now consider $\mathbb{A}^2 \times \mathbb{P}^1$. Think of this as just $\mathbb{P}^1$-many copies of $\mathbb{A}^2$ stacked on top of each other. Points in $\mathbb{A}^2 \times \mathbb{P}^1$ are of the form $((a,b),(c:d))$, where $(a,b) \in \mathbb{A}^2$ and $(c:d) \in \mathbb{P}^1$. We will let $x, y$ denote the indeterminates for points in $\mathbb{A}^2$ (i.e. if $(a,b) \in \mathbb{A}^2$, then $a$ is the $x$-coordinate and $b$ is the $y$-coordinate), and let $z, w$ denote the indeterminates for points in $\mathbb{P}^1$ (i.e. if $(c:d) \in \mathbb{P}^1$, then $c$ is the $z$-coordinate and $d$ is the $w$-coordinate).

Consider the variety $X = V(xw - yz) \subset \mathbb{A}^2 \times \mathbb{P}^1$. What points are contained in this variety? They firstly should be of the form $((a,b),(c:d))$, but moreover these coordinates must satisfy the polynomial relation

$$ad = bc.$$

We consider several cases depending on the values of $a, b$.

*Case 1:* Suppose $a, b \neq 0$. Then $d = \frac{bc}{a}$. Note that if $c = 0$, then this implies $d = 0$, contradicting $(c:d)$ being a point in $\mathbb{P}^1$. So $c \neq 0$, and since this is a projective coordinate, we may assume $c = 1$.

Thus $d = \dfrac{b}{a}$. So for each $a, b \neq 0$, we obtain a point in $X$ of the form $\left((a, b), \left(1 : \dfrac{b}{a}\right)\right)$. Note that $\dfrac{b}{a}$ is the slope of the line through the origin and $(a, b)$.

*Case 2:* Suppose $a \neq 0$ but $b = 0$. Then $ad = 0$; since $a \neq 0$, this means $d = 0$, but this then forces $c \neq 0$, hence $c = 1$. So for each $a \neq 0$, we obtain the point $((a, 0), (1 : 0)) \in X$.

*Case 3:* Suppose $a = 0$ but $b \neq 0$. Then $bc = 0$; since $b \neq 0$, this means $c = 0$, but this then forces $d \neq 0$, hence $d = 1$. So for each $b \neq 0$, we obtain the point $((0, b), (0 : 1)) \in X$.

*Case 4:* Suppose $a = b = 0$. Then the polynomial equation is trivially reduced to $0 = 0$, and there are no restrictions on $(c : d)$. So we obtain $\{((0, 0), (1 : d)) : d \in \mathbb{R}\} \cup \{((0, 0), (0 : 1))\} \subset X$.

Now consider the projection map $\pi$ from $X$ to $\mathbb{A}^2$, given by

$$\begin{aligned} \pi : X \subset \mathbb{A}^2 \times \mathbb{P}^1 &\to \mathbb{A}^2 \\ ((a, b), (c : d)) &\mapsto (a, b). \end{aligned}$$

Given a point $(a, b) \in \mathbb{A}^2$, what is its preimage $\pi^{-1}((a, b))$? Well, based on our computation above, over each point $(a, b) \neq (0, 0)$, there was a unique point in $X$, so $\pi$ is a bijection outside of the origin. However, over the origin, we obtained an entire $\mathbb{P}^1$-worth of points.

The picture to have in mind looks like this. Envision the $\mathbb{P}^1$ in the vertical direction. At $(1 : 0) \in \mathbb{P}^1$, we have all the points in Case 2; the points in the plane are of the form $(a, 0)$, i.e. the $x$-axis. At $(0 : 1) \in \mathbb{P}^1$, i.e. the point at infinity, we have the points in Case 3; the points in the plane are of the form $(0, b)$, i.e the $y$-axis. At $(1 : d) \in \mathbb{P}^1$ with $d \neq 0$, we have all the points in Case 1; the points in the plane are of the form $(a, b)$ where $\dfrac{b}{a} = d$; these are all points on the same line through the origin, as they all have that same slope $d$. So in effect $\mathbb{P}^1$ is (once again) parametrizing the different slopes through the origin, but this time carrying with it that entire line in $\mathbb{A}^2$, creating a spiral of lines rotating once fully around a “backbone” of a $\mathbb{P}^1$. This $\mathbb{P}^1$ “backbone” lies entirely over the origin. Another cool thing to notice: this resulting variety is the Möbius strip!

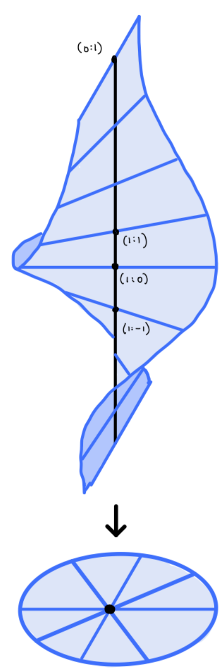


FIGURE 1. The blow-up of $\mathbb{A}^2$ at the origin.

We call $X$ the *blow-up of* $\mathbb{A}^2$ *at the origin*, often denoted $\mathrm{Bl}_{(0,0)}\mathbb{A}^2$. We call $(0,0)$ the *center* of the blow-up. The blow-up comes equipped with the surjective projection map $\pi : \mathrm{Bl}_{(0,0)}\mathbb{A}^2 \to \mathbb{A}^2$. Note that $\pi^{-1}(\mathbb{A}^2 \setminus \{(0,0)\}) \to \mathbb{A}^2 \setminus \{(0,0)\}$ is a bijection. Note also that the blow-up is virtually identical to $\mathbb{A}^2$, except that the codimension-2 subvariety which was the origin was now replaced with a codimension-1 subvariety in bijection with $\mathbb{P}^1$. We call $\pi^{-1}((0,0)) \cong \mathbb{P}^1$ the *exceptional divisor* of the blow-up, which we will denote by $E$.

## 4. Using Blow-Ups to Resolve Singularities

We return to our motivating example, where our singular curve $C$ was given by $V(y^2 - x^3 - x^2)$. Recall that $\mathbb{P}^1$ parametrizes slopes. If we think of drawing tangent lines as we move around the curve, we get different slopes. In fact, at the origin, there are two tangent lines, namely $y = x$ and $y = -x$. Every other (smooth) point on $C$ has one tangent line.

We know how to blow-up $\mathbb{A}^2$ at the origin, and coincidentally the origin is where $C$ has a singularity. If we consider the preimage of $C \subset \mathbb{A}^2$ under the map $\pi : \mathrm{Bl}_{(0,0)}\mathbb{A}^2 \to \mathbb{A}^2$, then we should hope that this preimage contains a smooth version of $C$.

So, which points $((a,b),(c:d)) \in \mathrm{Bl}_{(0,0)}\mathbb{A}^2$ are in $\pi^{-1}(C)$? Being in the blow-up, these points must satisfy the relation $ad - bc = 0$. Moreover, they must also project onto the curve $C$, so they must also satisfy $b^2 - a^3 - a^2 = 0$. In other words, we need to solve the system of equations

$$\begin{aligned} xw - yz &= 0 \\ y^2 - x^3 - x^2 &= 0. \end{aligned}$$

We will restrict our attention to the points whose $\mathbb{P}^1$-coordinates are of the form $(1 : d)$. Then $z \neq 0$ (i.e. $z = 1$), so our first equation is reduced to $y = xw$. Think of this as a change of variables, so that in this portion of the blow-up, the only two variables governing the two-dimensional space are $x$ and $w$. We can substitute this into the second equation to get

$$\begin{aligned} (xw)^2 - x^3 - x^2 &= 0 \\ x^2w^2 - x^3 - x^2 &= 0 \\ x^2(w^2 - x - 1) &= 0. \end{aligned}$$

Keep in mind that the coordinate $w$ corresponds to the value $d$ in the projective point $(1 : d)$, so $w$ tells us where along the vertical $\mathbb{P}^1$ direction our point is. So when is this equation satisfied? When either $x = 0$, or when $w^2 - x - 1 = 0$. If $((a,b),(c:d))$ satisfies $x = 0$, then $a = 0$, but then also $b = 0$ since $y = xw$. Also $c = 1$ since $z \neq 0$, so we get points of the form $((0,0),(1:d))$ for each $d \in \mathbb{R}$. This is precisely the $\mathbb{P}^1$ "backbone" of the blow-up, i.e. the exceptional divisor $E$. The second equation gives us the parabola $x = w^2 - 1$. Note that this intersects the exceptional divisor $x = 0$ when $w = 1$ and $w = -1$.

Hence $\pi^{-1}(C)$ consists of two components: the exceptional divisor $E$, and what we will call the *strict transform* $\tilde{C}$, which is given by $V(\tilde{f})$, where $\tilde{f} = w^2 - x - 1$. Note that $\tilde{C}$ is smooth: we can verify that the Jacobian matrix

$$\begin{bmatrix} \dfrac{\partial \tilde{f}}{\partial x} & \dfrac{\partial \tilde{f}}{\partial w} \end{bmatrix} = \begin{bmatrix} -1 & 2w \end{bmatrix}$$

will always have rank 1, so every point is smooth. If we restrict $\pi$ to be the map $\tilde{C} \to C$, we see that $\pi^{-1}(C \setminus P) \to C \setminus P$ is still a bijection; the preimage of $(0,0)$ consists of two points, so it is not a bijection here. These preimage points are $((0,0),(1:1))$ and $((0,0),(1:-1))$, and the projective coordinates correspond to the lines $y = x$ and $y = -x$, which were exactly the tangent lines we expected! So therefore the strict transform $\tilde{C}$, together with the projection map from the blow-up, precisely gives us a resolution of the singularity of $C$.

Let us try another example. Let $D = V(g) \subset \mathbb{A}^2$ where $g = y^2 - x^3$. Checking the Jacobian

$$\begin{bmatrix} \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} -3x^2 & 2y \end{bmatrix}$$

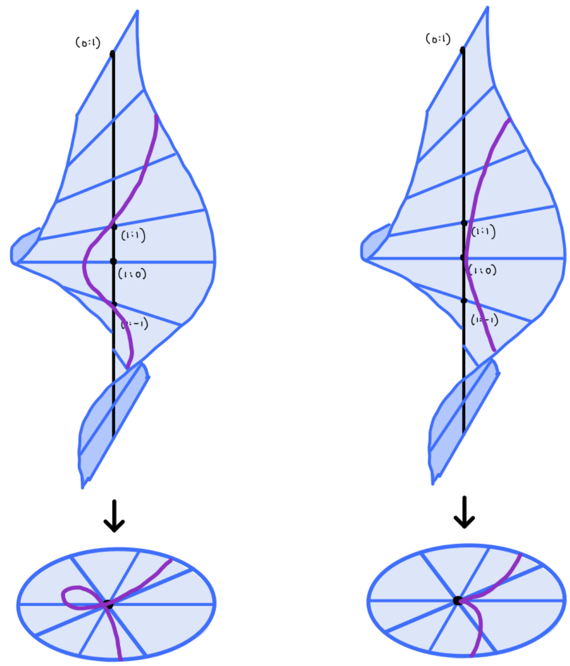


FIGURE 2. The strict transforms when blowing up $y^2 - x^3 - x^2$ and $y^2 - x^3$.

we again see that this is smooth at all points except for the origin. We hope to resolve the singularity by looking at the preimage of $D$ under the blow-up. Before we do that, let us use some geometric intuition. If we look at tangent lines, the curve $D$ will only have one tangent line at the origin, and this is precisely the $x$-axis. So we should expect $\pi^{-1}(D) = E \cup \tilde{D}$, where these two components intersect at the point $((0,0),(1:0))$, as this projective point corresponds to the $x$-axis.

In fact, $\pi^{-1}(D)$ consists of all the points $((a,b),(c:d))$ satisfying the equations

$$xw - yz = 0$$
$$y^2 - x^3 = 0.$$

We can again only look at the projective points of the form $(1:d)$, so that $z = 1$. Again $y = xw$ in this portion of the blow-up, so our second equation becomes

$$(xw)^2 - x^3 = 0$$
$$x^2w^2 - x^3 = 0$$
$$x^2(w^2 - x) = 0.$$

The points satisfying this equation must satisfy either $x = 0$ (which again gives us the exceptional divisor) or the points satisfying $w^2 - x = 0$, which is the standard parabola. Note that this intersects $x = 0$ precisely when $w = 0$, i.e. at the point $((0,0),(1:0))$, confirming our intuition earlier.

Thus $\pi^{-1}(D) = E \cup \tilde{D}$, where $\tilde{D}$ is the strict transform given by $V(\tilde{g})$, where $\tilde{g} = w^2 - x$. We can easily check that this is a smooth curve, so once again we have resolved the singularity of $D$.

For one last example, consider the curve $V(y^2 - x^5)$, which will once again be singular at the origin. If we do the same type of computation, we see that in the portion of the blow-up where $y = xw$, we obtain the following curve:

$$(xw)^2 - x^5 = 0$$
$$x^2w^2 - x^5 = 0$$
$$x^2(w^2 - x^3) = 0.$$

Again $x = 0$ will give us the exceptional divisor $E$, but now the strict transform is given by the equation $w^2 - x^3$, which we just saw was singular at the origin. The blow-up didn't resolve the singularity! But, we just saw that a blow-up can resolve the singularity on this curve, so we didn't entirely fail; it simply takes two total blow-ups to resolve the singularity at $y^2 - x^5$. This leads to the more general, and incredibly valuable theorem: any curve singularity can be resolved in finitely many successive blow-ups.