THE MODULI SPACE OF GLUED SUBALGEBRAS

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Abstract. We address the problem of computing the moduli space which parametrizes subalgebras of a finite direct sum of formal power series rings in one variable over an algebraically closed field, which are of a fixed finite codimension as vector spaces over that field. This moduli space is a projective scheme and contains a connected closed subscheme called the glued territory which parametrizes the glued subalgebras, those which arise as complete local rings at curve singularities. As such, the glued territory equivalently parametrizes the different ways a singularity with fixed delta-invariant and branch number can be attached to a smooth curve. It also has tremendous utility in describing the moduli of all singular curves. Using Goursat's Lemma, we define a stratification of the glued territory, in which each component is a relative Isom scheme over a product of relative punctual Hilbert schemes. This stratification allows us to not only prove some results about the geometry of the glued territory, but also provides a recursive method for computing these moduli spaces explicitly.

Contents

1. Introduction

Let k be an algebraically closed field. Fix natural numbers δ, m such that $m \geq 1$. The goal of this paper is to construct the moduli space of k-subalgebras of the direct sum $\bigoplus_{i=1}^m k[[t_i]]$ which are δ-codimensional as k-vector spaces. This moduli space will be called the δ-*territory* of $\bigoplus_{i=1}^m k[[t_i]]$, and we will denote it as

$$\mathrm{ter}^\delta(m) = \left\{ k\text{-subalgebras } S \subset \bigoplus_{i=1}^m k[[t_i]] : \dim_k \left(\bigoplus_{i=1}^m k[[t_i]] \right) \Big/ S = \delta \right\}.$$

The motivation for examining this moduli space comes from curve singularities. Let C be a reduced curve over $\operatorname{Spec} k$, and let $\nu : \tilde{C} \to C$ be its normalization. The number of branches at P is $|\nu^{-1}(P)|$. If P has m branches, then the complete local ring $\widehat{\mathscr{O}}_{C,P}$ of C at P can be identified with a k-subalgebra

δ \ m	1	2	3	4	$\cdots$
1	$\mathrm{ter}^1(1)$ (Ex 1.9)	$\mathrm{ter}^1_{\mathcal{G}}(2)$ (Thm 2.13)			
2	$\mathrm{ter}^2(1)$ (Ex 1.10)	$\mathrm{ter}^2_{\mathcal{G}}(2)$ (Prop 4.4)	$\mathrm{ter}^2_{\mathcal{G}}(3)$ (Thm 2.13)		
3	$\mathrm{ter}^3(1)$ (Ex 1.11)	$\mathrm{ter}^3_{\mathcal{G}}(2)$ (Ex 5.3)	$\mathrm{ter}^3_{\mathcal{G}}(3)$ (Prop 4.4)	$\mathrm{ter}^3_{\mathcal{G}}(4)$ (Thm 2.13)	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 1.

of $\bigoplus_{i=1}^m k[[t_i]]$. The *delta-invariant* at P is the finite quantity

$$\dim_k(\nu_*\mathscr{O}_{\tilde{C}})_P/\mathscr{O}_{C,P} = \dim_k \left(\bigoplus_{i=1}^m k[[t_i]]\right) \Big/ \widehat{\mathscr{O}}_{C,P}.$$

Hence if P has delta-invariant δ, then $\widehat{\mathscr{O}}_{C,P}$ belongs to $\mathrm{ter}^\delta(m)$.

Little is known about the "multibranch" δ-territories (i.e. when $m \geq 2$), whereas the "unibranch" case ($m = 1$) has been more extensively studied, most notably by Ishii in [Ish80] and Hamilton in [Ham19]. In [Ish80], the δ-territory is viewed as the representing scheme of a functor, and $\mathrm{ter}^\delta(1)$ is given a stratification by affine varieties according to numerical semigroups. In [Ham19], an algorithm is provided to compute the polynomial equations defining these affine varieties.

Any k-subalgebra in $\mathrm{ter}^\delta(1)$ can be obtained as the complete local ring at a unibranch singularity. Unfortunately, the same is not true for $m \geq 2$; take, for instance, $k[[t_1]] \oplus k[[t_2^2, t_2^3]] \in \mathrm{ter}^1(2)$. Not only is this k-subalgebra not local, but its spectrum is disconnected, and would geometrically correspond to the disjoint union of a smooth branch and a cuspidal branch. We will restrict our attention to the k-subalgebras which do in fact arise as complete local rings at m-branch singularities, which will be referred to as *glued subalgebras*. We call the subset $\mathrm{ter}^\delta_{\mathcal{G}}(m)$ consisting of the glued subalgebras of $\mathrm{ter}^\delta(m)$ the *glued territory*.

It turns out that the glued territory encodes the different ways that an m-branch singularity with delta-invariant δ can be attached to a smooth point on a curve. In this way, the glued territory has tremendous utility in constructing global moduli of singular curves. Ishii uses her results about $\mathrm{ter}^\delta(1)$ in a subsequent paper [Ish82] to construct a global moduli of singular curves with unibranch singularities. Our goal in this paper and in future work is to expand upon these results to construct more explicitly the moduli of all curves.

One immediate application is in describing stable modular compactifications of $\mathcal{M}_{g,n}$. As shown in [Smy09], such compactifications are obtained by extremal assignments. For example, in $\overline{\mathcal{M}}_{g,1}$, with $g \geq 1$, there is the extremal assignment $\mathcal{Z}_{\mathrm{un}}$ which assigns to each stable curve C the subcurve $\mathcal{Z}_{\mathrm{un}}(C)$ consisting of all unmarked components. A $\mathcal{Z}_{\mathrm{un}}$-stable curve can be obtained from C by replacing each connected component C_i of $\mathcal{Z}_{\mathrm{un}}(C)$ with a singularity with m_i branches, where m_i is number of intersection points of C_i with the marked component, and with delta-invariant $p_a(C_i) + m_i - 1$. The compactification $\overline{\mathcal{M}}_{g,1}(\mathcal{Z}_{\mathrm{un}})$ obtained from this extremal assignment will therefore consist of various glued territory bundles over the boundary strata.

In fact, to determine $\overline{\mathcal{M}}_{g,1}(\mathcal{Z}_{\mathrm{un}})$, it is necessary to know all glued territories $\mathrm{ter}^\delta_{\mathcal{G}}(m)$, where $m-1 \leq \delta \leq g$ and $m \leq g+1$. Hence if we arrange all the glued territories by δ and m as in Table 1, then we would need to know all entries up to and including the gth row. This table provides the theorem, proposition, or example in which the corresponding glued territory is computed. Based solely on the computations in this paper, we could describe $\overline{\mathcal{M}}_{3,1}(\mathcal{Z}_{\mathrm{un}})$.

In the rest of this section we show that the δ-territory can be embedded into a Grassmannian variety as a closed subvariety, guaranteeing that this moduli space is in fact a projective k-scheme. We also review the relevant results from [Ish80] and [Ham19] for the unibranch case. In Section 2, we review Goursat's Lemma for k-algebras, which is essentially a recipe for building k-subalgebras of a direct sum, with the main ingredients being k-subalgebras of the summands, ideals of those k-subalgebras, and a k-algebra isomorphism between the resulting quotients. Since the territories of $k[[t]]$ are known, the k-subalgebras of $\bigoplus_{i=1}^m k[[t_i]]$ can therefore be recursively determined. We then formally define glued subalgebras and the glued territory, and show that the latter is a closed subscheme of $\mathrm{ter}^\delta(m)$.

In Section 3, we define a stratification of $\mathrm{ter}^\delta_{\mathcal{G}}(m)$ by *Isom-Hilb components*. By taking a functorial approach to Goursat's Lemma, the glued territory can be realized as a union of components, each of which is the image in the Grassmannian of a scheme obtained as a relative Isom scheme over a product of relative punctual Hilbert schemes over the glued territories of lower branch numbers. Using this, we are required to know about punctual Hilbert schemes (which have been independently studied for some curve singularities in [Fog68], [Lax00], [PS92], and [Ran02]), and automorphism groups of finite-dimensional k-algebras (which have been studied in [GS94]). This stratification helps us prove some statements about the geometry of the moduli space, including its connectedness, its irreducible components, and its dimension, which can be found in Section 4. We also discuss how the Isom-Hilb stratification comes with some obstructions. The Isom-Hilb stratification provides a procedure to compute the glued territories explicitly, and we provide some examples in Section 5.

1.1. **Embedding the Territory in a Grassmannian.** Here we show that $\mathrm{ter}^\delta(m)$ is a projective k-scheme by viewing it as a closed subscheme of a sufficient Grassmannian variety. To do this, we first show that there is a one-to-one correspondence between δ-codimensional k-subalgebras of $\bigoplus_{i=1}^m k[[t_i]]$ and δ-codimensional k-subalgebras of $\bigoplus_{i=1}^m k[[t_i]]/(t_i^{2\delta})$.

Proposition 1.1. *If $S \subset \bigoplus_{i=1}^m k[[t_i]]$ is a δ-codimensional k-subalgebra, then S contains the ideal $\mathfrak{a}^{2\delta}$, where $\mathfrak{a} = ((t_1, \dots, t_m))$.*

Proof. Let $A = \bigoplus_{i=1}^m k[[t_i]]$, and define the following gradation of k-modules: for $n \in \mathbb{N}$, let

$$(A/S)^n := \mathfrak{a}^n/((\mathfrak{a}^n \cap S) + \mathfrak{a}^{n+1})$$

where $\mathfrak{a}^0 = A$. It is clear that $\delta = \dim_k A/S = \sum_{n\in\mathbb{N}} \dim_k(A/S)^n$. The following fact will also become useful: for any $n, n' \in \mathbb{N}$, $(A/S)^n = (A/S)^{n'} = 0$ implies $(A/S)^{n+n'} = 0$. To see that this is true, take any $f \in \mathfrak{a}^{n+n'}$. Then $f = gh$ for some $g \in \mathfrak{a}^n$ and $h \in \mathfrak{a}^{n'}$. By assumption, $\mathfrak{a}^n = (\mathfrak{a}^n \cap S) + \mathfrak{a}^{n+1}$ and $\mathfrak{a}^{n'} = (\mathfrak{a}^{n'} \cap S) + \mathfrak{a}^{n'+1}$, so it follows that $f = gh \in (\mathfrak{a}^{n+n'} \cap S) + \mathfrak{a}^{n+n'+1}$. Hence $\mathfrak{a}^{n+n'} = (\mathfrak{a}^{n+n'} \cap S) + \mathfrak{a}^{n+n'+1}$, and thus $(A/S)^{n+n'} = 0$.

Assume $(A/S)^n \neq 0$ for some $n \geq 2\delta$. Then for any pair ℓ, ℓ' such that $\ell + \ell' = n$, we must have either $(A/S)^\ell \neq 0$ or $(A/S)^{\ell'} \neq 0$. We partition $\{1, \dots, n-1\}$ into subsets of the form $N_\ell = \{\ell, n-\ell\}$ for $\ell = 1, \dots, \lfloor \frac{n}{2} \rfloor$. Note that if n is even, then $N_{\lfloor \frac{n}{2} \rfloor} = \{\lfloor \frac{n}{2} \rfloor\}$. Hence there is some $n_\ell \in N_\ell$ such that $(A/S)^{n_\ell} \neq 0$, so $\dim_k(A/S)^{n_\ell} \geq 1$. But also $\dim_k(A/S)^n \geq 1$ by assumption, and since $n \geq 2\delta$, we obtain

$$\dim_k(A/S)^n + \sum_{\ell=1}^{\lfloor \frac{n}{2} \rfloor} \dim_k(A/S)^{n_\ell} \geq 1 + \left\lfloor \frac{n}{2} \right\rfloor \geq 1 + \delta.$$

However this contradicts $\dim_k A/S = \delta$. Therefore it must be the case that $(A/S)^n = 0$ for all $n \geq 2\delta$, but this means $\mathfrak{a}^{2\delta} = \sum_{n\geq 2\delta}(\mathfrak{a}^n \cap S)$, which implies $\mathfrak{a}^{2\delta} \subset S$. ■

Corollary 1.2. *There is a one-to-one correspondence between the δ-codimensional k-subalgebras of $\bigoplus_{i=1}^m k[[t_i]]$ and the δ-codimensional k-subalgebras of $\bigoplus_{i=1}^m k[[t_i]]/(t_i^{2\delta})$.*

We can now rephrase our moduli space in the following way:

$$\mathrm{ter}^\delta(m) = \left\{ k\text{-subalgebras } S \subset \bigoplus_{i=1}^m k[[t_i]]/(t_i^{2\delta}) : \dim_k \left(\bigoplus_{i=1}^m k[[t_i]]/(t_i^{2\delta}) \right) \Big/ S = \delta \right\}.$$

Observe that $\bigoplus_{i=1}^m k[[t_i]]/(t_i^{2\delta})$ is Artinian, and has dimension $2\delta m$ as a k-vector space. Hence we can consider these δ-codimensional k-subalgebras as δ-codimensional subspaces of $k^{2\delta m}$, with a multiplicative structure.

Definition 1.3. Let $\Lambda = \bigoplus_{i=1}^m k[[t_i]]/(t_i^{2\delta})$, let $\mathbf{1} = (1, \dots, 1) \in \Lambda$, and let $C = \mathrm{span}\{\mathbf{1}\}$. Let $\mu : \Lambda \times \Lambda \to \Lambda$ be the multiplication map. Then we can once again redefine the moduli space as

$$\mathrm{ter}^\delta(m) = \{S \in G(2\delta m - \delta, 2\delta m) : C \subset S \text{ and } \mu(S \times S) \subset S\}.$$

Note that the condition $C \subset S$ guarantees that the inclusion $k \hookrightarrow \Lambda$ factors through S, and the condition $\mu(S \times S) \subset S$ guarantees that S is closed under multiplication, so that S is in fact a k-subalgebra of Λ.

Proposition 1.4. *The moduli space* $\mathrm{ter}^\delta(m)$*, as defined in Definition 1.3, is a projective scheme. In particular, it is a closed subscheme of* $G(2\delta m - \delta, 2\delta m)$.

Proof. The condition $C \subset S$ defines a sub-Grassmannian of $G = G(2\delta m - \delta, 2\delta m)$, which can be identified as $G' = G(2\delta m - \delta - 1, 2\delta m - 1)$. For the second condition, we observe that μ is k-bilinear, hence induces a map $\tilde{\mu} : \Lambda \otimes \Lambda \to \Lambda$. Let $\tilde{\mu}_S : S \otimes S \to \Lambda$ be the restriction of $\tilde{\mu}$ to $S \otimes S$, and let $\nu_S : \Lambda \to \Lambda/S$ be the quotient map. Then the condition $\mu(S \times S) \subset S$ is equivalent to $\nu_S \circ \tilde{\mu}_S = 0$. If $\mathscr{U}$ is the universal subbundle over G, then the points of G for which this condition is satisfied will precisely form the vanishing locus of the morphism $\mathscr{U} \otimes \mathscr{U} \to (\Lambda \otimes_k \mathscr{O}_G)/\mathscr{U}$. This vanishing locus defines a closed subscheme Z of G. Thus $G' \cap Z = \mathrm{ter}^\delta(m)$. ∎

1.2. **Overview of the Unibranch Case.** Ishii showed that $\mathrm{ter}^\delta(1)$ can be stratified by affine varieties, each one consisting of k-subalgebras with the same numerical semigroup. Here we review semigroups, as well as this stratification.

Definition 1.5. A *numerical semigroup* Γ is a subset of $\mathbb{N}$ satisfying the following conditions:

(i) $0 \in \Gamma$,
(ii) Γ is closed under addition, and
(iii) $|\mathbb{N} \setminus \Gamma|$ is finite.

The *genus* of a numerical semigroup Γ is the quantity $g(\Gamma) = |\mathbb{N} \setminus \Gamma|$, and the elements of $\mathbb{N} \setminus \Gamma$ are called *gaps*. The *conductor* of Γ is the unique number c for which $c - 1 \notin \Gamma$ but $\{c, c+1, \dots\} \subset \Gamma$.

If Γ is a numerical semigroup, we say $n_1, \dots, n_s \in \Gamma$ is a *generating set* of Γ if every element can be expressed as a linear combination of the form $p_1 n_1 + \cdots + p_s n_s$ for some $p_i \in \mathbb{N}$. Every numerical semigroup has a unique minimal generating set [GR09, Theorem 2.7]. If $n_1, \dots, n_s$ is the minimal generating set of Γ, then we write $\Gamma = \langle n_1, \dots, n_s \rangle$.

Observe $k[[t]]$ is a discrete valuation ring, with valuation $v : k[[t]] \to \mathbb{N}$ given by orders, i.e. if $f = a_0 + a_1 t + a_2 t^2 + \cdots$, then $v(f)$ is the smallest i for which $a_i \neq 0$. If S is a finite-codimensional k-subalgebra of $k[[t]]$, then the set

$$\Gamma_S := \{n \in \mathbb{N} : v(f) = n \text{ for some } f \in S \setminus \{0\}\}$$

is a numerical semigroup. If $S \in \mathrm{ter}^\delta(1)$, then $g(\Gamma_S) = \delta$. For any numerical semigroup Γ with conductor c, we have $c \leq 2g(\Gamma)$ [GR09, Lemma 2.14]. Hence $g(\Gamma_S) = \delta$ implies the ideal $(t^{2\delta}) \subset k[[t]]$ is contained in S, confirming Proposition 1.1 when $m = 1$.

If a numerical semigroup Γ is fixed, then its *semigroup stratum* will be the set

$$T_\Gamma := \{k\text{-subalgebras } S \subset k[[t]] : \Gamma_S = \Gamma\}.$$

If $g(\Gamma) = \delta$, then $T_\Gamma \subset \mathrm{ter}^\delta(1)$, and in fact we have the following stratification:

$$\mathrm{ter}^\delta(1) = \bigcup_{g(\Gamma) = \delta} T_\Gamma.$$

Proposition 1.6. [Ish80, Proposition 6] *Let* $\Gamma = \langle n_1, \ldots, n_s \rangle \subset \mathbb{N}$ *be a numerical semigroup. Let*

$$N = \sum_{i=1}^{s} |\{g \in \mathbb{N} \setminus \Gamma : g > n_i\}|.$$

Then there is a closed immersion $T_\Gamma \hookrightarrow \mathbb{A}^N$.

Corollary 1.7. [Ish80, Corollary 4] *Let* c *be the conductor of* Γ. *If there are no relations among the generators smaller than the conductor, i.e. there are no nonzero vectors* $(p_1, \ldots, p_s) \neq (p'_1, \ldots, p'_s) \in \mathbb{N}^s$ *such that*

$$p_1 n_1 + \cdots + p_s n_s = p'_1 n_1 + \cdots + p'_s n_s < c,$$

then $T_\Gamma \cong \mathbb{A}^N$.

Definition 1.8. For any δ, the k-subalgebra $k[[t^{\delta+1}, \ldots, t^{2\delta+1}]]$ will be called the *maximum degree subalgebra.*

The numerical semigroup $\Gamma = \langle \delta + 1, \ldots, 2\delta + 1 \rangle$ has genus δ, since its gaps are $1, \ldots, \delta$. Hence the quantity N from above is zero, so $T_\Gamma = \{pt\}$, i.e. there is a unique k-subalgebra of codimension δ with numerical semigroup $\langle \delta + 1, \ldots, 2\delta + 1 \rangle$, and it is precisely the maximum degree subalgebra.

Example 1.9. $\mathrm{ter}^1(1)$. There is only one semigroup of genus 1, namely $\langle 2, 3 \rangle$, and by above $T_{\langle 2,3 \rangle} = \{k[[t^2, t^3]]\}$. Thus $\mathrm{ter}^1(1) = \{pt\}$.

Example 1.10. $\mathrm{ter}^2(1)$. There are two semigroups of genus 2, namely $\langle 2, 5 \rangle$ and $\langle 3, 4, 5 \rangle$, which have conductors 4 and 3, respectively. In each semigroup, there are no relations among the generators which are smaller than the respective conductor, so

$$T_{\langle 2,5 \rangle} = \{k[[t^2 + at^3, t^5]] : a \in k\} \cong \mathbb{A}^1$$
$$T_{\langle 3,4,5 \rangle} = \{k[[t^3, t^4, t^5]]\} \cong \{pt\}.$$

Since $\mathrm{ter}^2(1)$ is a subscheme of $G(2, 4)$, we compute the Plücker coordinates of these k-subalgebras. For instance, an arbitrary k-subalgebra $k[[t^2 + at^3, t^5]]$ in $T_{\langle 2,5 \rangle}$ can be viewed as the 2-dimensional k-vector subspace $k\{1, t^2 + at^3\}$ of $k\{1, t, t^2, t^3\}$. The Plücker coordinates of this subspace are the maximal minors of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & a \end{bmatrix}.$$

Similarly, $k[[t^3, t^4, t^5]]$ can be viewed as the k-subspace $k\{1, t^3\}$, and its Plücker coordinates are the maximal minors of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Altogether, we obtain a projective line inside this Grassmannian, so therefore $\mathrm{ter}^2(1) \cong \mathbb{P}^1$.

Example 1.11. $\mathrm{ter}^3(1)$. There are four semigroups of genus 3, namely $\langle 2, 7 \rangle, \langle 3, 4 \rangle, \langle 3, 5, 7 \rangle$, and $\langle 4, 5, 6, 7 \rangle$, which have conductors $6, 6, 5, 4$, respectively. Once again none of these semigroups have relations among the generators which are smaller than their conductors. Hence

$$T_{\langle 2,7 \rangle} = \{k[[t^2 + at^3 + bt^5, t^7]] : a, b \in k\} \cong \mathbb{A}^2$$
$$T_{\langle 3,4 \rangle} = \{k[[t^3 + ct^5, t^4 + dt^5]] : c, d \in k\} \cong \mathbb{A}^2$$
$$T_{\langle 3,5,7 \rangle} = \{k[[t^3 + et^4, t^5, t^7]] : e \in k\} \cong \mathbb{A}^1$$
$$T_{\langle 4,5,6,7 \rangle} = \{k[[t^4, t^5, t^6, t^7]]\} \cong \{pt\}.$$

Again, we consider $\mathrm{ter}^3(1)$ as a subscheme of $G(3, 6)$ and find the Plücker coordinates of these k-subalgebras. For instance, an arbitrary k-subalgebra $k[[t^2 + at^3 + bt^5, t^7]]$ in $T_{\langle 2,7 \rangle}$ can be viewed as the 3-dimensional k-vector subspace $k\{1, t^2 + at^3 + bt^5, t^4 + 2at^5\}$ of $k\{1, t, t^2, t^3, t^4, t^5\}$. The Plücker coordinates of this subspace are the maximal minors of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & a & 0 & b \\ 0 & 0 & 0 & 0 & 1 & 2a \end{bmatrix}$$

and computing them yields the affine patch $x_0 \neq 0$ on the projective quadric cone $V(x_0x_3 - x_1x_2, x_1 - 2x_2) \subset \mathbb{P}^4$. Continuing in this way, we conclude

$$\mathrm{ter}^3(1) \cong \mathbb{P}^2 \cup Q$$

where Q is the aforementioned quadric cone with vertex point V, and $\mathbb{P}^2 \cap Q = \mathbb{P}^1$. The k-subalgebras in $T_{\langle 3,4\rangle}$ form the affine plane $\mathbb{P}^2 \setminus (\mathbb{P}^2 \cap Q)$, the k-subalgebras in $T_{\langle 3,5,7\rangle}$ form the affine line $(\mathbb{P}^2 \cap Q) \setminus V$, and the unique k-subalgebra in $T_{\langle 4,5,6,7\rangle}$ forms the vertex V.

2. The Glued Territory

For $m \geq 2$, a k-subalgebra $S \in \mathrm{ter}^\delta(m)$ also has a corresponding semigroup of orders, contained in $\mathbb{N}^m$. However, we will choose not to stratify $\mathrm{ter}^\delta(m)$ by this invariant for many reasons. In the unibranch case, the genus of the numerical semigroup Γ was equal to $\dim_k(\bigoplus_{i=1}^m k[[t_i]])/S$. This is not true for $m \geq 2$, since the complements of these semigroups are rarely finite. Take, for instance, the k-subalgebra $k[[(t_1, 0), (0, t_2)]] \subset k[[t_1]] \oplus k[[t_2]]$. This is the complete local ring at an ordinary node, and it has codimension 1. However, its semigroup is

$$\Gamma_{k[[(t_1,0),(0,t_2)]]} = \{(0, 0)\} \cup \{(n_1, n_2) \in \mathbb{N}^2 : n_1, n_2 \geq 1\}$$

and $|\mathbb{N}^2 \setminus \Gamma_{k[[(t_1,0),(0,t_2)]]}| = \infty$. This example also shows that these multibranch semigroups are not necessarily finitely generated, as they were in the unibranch case.

2.1. Goursat's Lemma.

In certain categories with products, Goursat's Lemma describes subobjects of a product. It was initially stated in the category of groups to describe subgroups of a direct product. In [AC09], Goursat's Lemma was proven to hold for rings, and stated for modules. In [GM22], it was proven to hold for modules and, consequently, algebras over a ring. We state the theorem for k-algebras, where we assume the field k is algebraically closed.

Theorem 2.1. [GM22, Corollary 1] *Let A_1, A_2 be k-algebras.*

(i) Let $S \subset A_1 \times A_2$ be a k-subalgebra, and let $\pi_i : A_1 \times A_2 \to A_i$ be the canonical projection homomorphisms. Define

$$S_1 := \pi_1(S),$$
$$I_1 := \{s_1 \in S_1 : (s_1, 0) \in S\},$$
$$S_2 := \pi_2(S),$$
$$I_2 := \{s_2 \in S_2 : (0, s_2) \in S\}.$$

Then S_i is a k-subalgebra of A_i, and I_i is an ideal of S_i. Moreover, define $\varphi : S_1/I_1 \to S_2/I_2$ by $s_1 + I_1 \mapsto s_2 + I_2$ if $(s_1, s_2) \in S$. Then φ is a k-algebra isomorphism.

(ii) Conversely, suppose S_1 is a k-subalgebra of A_1, I_1 is an ideal of S_1, S_2 is a k-subalgebra of A_2, and I_2 is an ideal of S_2, such that there exists a k-algebra isomorphism $\varphi : S_1/I_1 \to S_2/I_2$. Then

$$S = \{(s_1, s_2) \in S_1 \times S_2 : \varphi(s_1 + I_1) = s_2 + I_2\}$$

is a k-subalgebra of $A_1 \times A_2$.

The constructions of (i) and (ii) are inverses. Equivalently, there is a one-to-one correspondence between k-subalgebras of $A_1 \times A_2$ and ordered quintuples of the form

$$(S_1, I_1, S_2, I_2, \varphi)$$

where each S_i is a k-subalgebra of A_i, I_i is an ideal of S_i, and $\varphi : S_1/I_1 \to S_2/I_2$ is a k-algebra isomorphism.

Definition 2.2. If $\varphi : A \to B$ is a k-algebra isomorphism of finite-dimensional k-algebras, then the *gluing dimension* of φ is the quantity $\dim_k \varphi := \dim_k A = \dim_k B$.

Proposition 2.3. *Let $S = (S_1, I_1, S_2, I_2, \varphi)$ be a k-subalgebra of $A_1 \oplus A_2$. If $\delta = \dim_k(A_1 \oplus A_2)/S$ and $\delta_i = \dim_k A_i/S_i$ then*

$$\delta = \delta_1 + \delta_2 + \dim_k \varphi.$$

Proof. By Goursat's Lemma, S is a k-submodule of $S_1 \oplus S_2$. First we show that $(S_1 \oplus S_2)/S \cong S_2/I_2$ as k-modules. Consider the k-linear map

$$\begin{aligned} \gamma : S_1 \oplus S_2 &\to S_2/I_2 \\ (s_1, s_2) &\mapsto (s_2 + I_2) - \varphi(s_1 + I_1). \end{aligned}$$

By the definition of S, it is clear that $(s_1, s_2) \in S$ if and only if $\varphi(s_1 + I_1) = s_2 + I_2$, if and only if $(s_1, s_2) \in \ker \gamma$. Moreover, γ is surjective, since for any coset $s_2 + I_2$, we have

$$\gamma((0, s_2)) = (s_2 + I_2) - \varphi(0 + I_1) = (s_2 + I_2) - (0 + I_2) = s_2 + I_2.$$

Thus $(S_1 \oplus S_2)/S \cong S_2/I_2$.

The inclusion of k-modules $S \subset S_1 \oplus S_2 \subset A_1 \oplus A_2$ allows us to conclude

$$\begin{aligned} \delta &= \dim_k(A_1 \oplus A_2)/S \\ &= \dim_k(A_1 \oplus A_2)/(S_1 \oplus S_2) + \dim_k(S_1 \oplus S_2)/S \\ &= \dim_k A_1/S_1 + \dim_k A_2/S_2 + \dim_k S_2/I_2 \\ &= \delta_1 + \delta_2 + \dim_k \varphi \end{aligned}$$

and thus the equality is proven. ■

Example 2.4. Consider the k-subalgebra $k[[(t_1, 0), (0, t_2)]] \subset k[[t_1]] \oplus k[[t_2]]$. This is identified with the data

$$(k[[t_1]], (t_1), k[[t_2]], (t_2), \mathrm{id}_k).$$

The quotients are $k[[t_i]]/(t_i) \cong k$, so the isomorphism id_k is completely determined by $1 \mapsto 1$. Hence the two-dimensional space of constants spanned by $\{(1, 0), (0, 1)\}$ in $k[[t_1]] \oplus k[[t_2]]$ is "glued" into the one-dimensional space spanned by $\{(1, \mathrm{id}_k(1))\} = \{(1, 1)\}$. By Proposition 2.3, the codimension of $k[[(t_1, 0), (0, t_2)]]$ in $k[[t_1]] \oplus k[[t_2]]$ is 1.

To construct k-subalgebras of a direct product of finitely many k-algebras $A_1 \times \cdots \times A_m$, we can simply invoke Goursat's Lemma $m-1$ times. A direct product of three k-algebras $A_1 \times A_2 \times A_3$ can be viewed as either $(A_1 \times A_2) \times A_3$ or as $A_1 \times (A_2 \times A_3)$. Since these are isomorphic as k-algebras, there is a natural bijection between k-subalgebras of $(A_1 \times A_2) \times A_3$ and k-subalgebras of $A_1 \times (A_2 \times A_3)$. This naturally extends to any finite number of k-algebras. Therefore without loss of generality, we will use the following convention: for $j \geq 2$,

$$A_1 \times \cdots \times A_j := (A_1 \times \cdots \times A_{j-1}) \times A_j.$$

The following result is clear under this recursive definition. It mimics the "asymmetric" version of Goursat's Lemma for groups [BSZ11].

Proposition 2.5. *Fix $m \geq 2$, and let $\theta_j = \{1, \ldots, j\}$. There is a one-to-one correspondence between k-subalgebras of $\prod_{i=1}^m A_i$ and ordered $(4m-3)$-tuples of the form*

$$(S_1, I_1, \ldots, S_m, I_m, \varphi_2, I_{\theta_2}, \varphi_3, I_{\theta_3}, \ldots, I_{\theta_{m-1}}, \varphi_m)$$

where each S_i is a k-subalgebra of A_i, I_i is an ideal of S_i, and for each $j \geq 2$, S_{θ_j} is the k-subalgebra of $\prod_{i=1}^j A_i$ identified with the data $(S_{\theta_{j-1}}, I_{\theta_{j-1}}, S_j, I_j, \varphi_j)$.

From this, we can generalize Proposition 2.3.

Corollary 2.6. *Let S be a k-subalgebra of $\bigoplus_{i=1}^m A_i$ identified with the data*

$$(S_1, I_1, \dots, S_m, I_m, \varphi_2, I_{\theta_2}, \varphi_3, I_{\theta_3}, \dots, I_{\theta_{m-1}}, \varphi_m).$$

If $\delta = \dim_k(\bigoplus_{i=1}^m A_i)/S$ and $\delta_i = \dim_k A_i/S_i$, then

$$\delta = \left(\sum_{i=1}^m \delta_i\right) + \left(\sum_{j=2}^m \dim_k \varphi_j\right).$$

While we still need some additional machinery to fully construct the territories of $\bigoplus_{i=1}^m k[[t_i]]$, we have sufficient information to compute $\mathrm{ter}^1(2)$.

Example 2.7. $\mathrm{ter}^1(2)$. If $S = (S_1, I_1, S_2, I_2, \varphi)$ is a k-subalgebra of $k[[t_1]] \oplus k[[t_2]]$ of codimension 1, then by Proposition 2.3,

$$\delta_1 + \delta_2 + \dim_k \varphi = 1.$$

Of course there are only three possible cases:

(i) $\delta_1 = 1, \delta_2 = 0, \dim_k \varphi = 0$. By Example 1.9, $S_1 = k[[t_1^2, t_1^3]]$, and $S_2 = k[[t_2]]$. Since φ has gluing dimension zero, this forces $I_i = S_i$, so that S is the direct product of S_1 and S_2, hence $S = k[[t_1^2, t_1^3]] \oplus k[[t_2]]$.

(ii) $\delta_1 = 0, \delta_2 = 1, \dim_k \varphi = 0$. Similar to the previous case, the only k-subalgebra obtained is $S = k[[t_1]] \oplus k[[t_2^2, t_2^3]]$.

(iii) $\delta_1 = 0, \delta_2 = 0, \dim_k \varphi = 1$. Then $S_i = k[[t_i]]$, and I_i must be an ideal of codimension 1. But $k[[t_i]]$ is local, so $I_i = (t_i)$. There is only one choice for the k-algebra isomorphism $\varphi : S_1/I_1 \to S_2/I_2$, namely id_k. Hence we uniquely obtain the k-subalgebra $k[[(t_1, 0), (0, t_2)]]$ from Example 2.4.

Since we have exhausted all cases, we conclude $\mathrm{ter}^1(2) \cong \{pt\} \cup \{pt\} \cup \{pt\}$. This example confirms that the multibranch territories are not necessarily connected.

2.2. **Glued Subalgebras and the Glued Territory.** As mentioned in the introduction, some (but not all) of the k-subalgebras of $\bigoplus_{i=1}^m k[[t_i]]$ are complete local rings at m-branch curve singularities. Those which are will be called *glued subalgebras*.

Definition 2.8. Let S be a k-subalgebra of $\bigoplus_{i=1}^m k[[t_i]]$, identified with the data

$$(S_1, I_1, \dots, S_m, I_m, \varphi_2, I_{\theta_2}, \varphi_3, I_{\theta_3}, \dots, I_{\theta_{m-1}}, \varphi_m).$$

The k-algebra isomorphisms $\varphi_2, \dots, \varphi_m$ are the *gluing isomorphisms* of S. Let $\Phi(S)$ denote the set of gluing isomorphisms of S. The *gluing dimension* of S is the quantity

$$\mathrm{gd}(S) := \sum_{j=2}^m \dim_k \varphi_j.$$

If $\dim_k \varphi \geq 1$ for all $\varphi \in \Phi(S)$, we will call S a *glued subalgebra*.

Definition 2.9. The *rational m-fold k-subalgebra* $*^m$ will refer to the local ring

$$k[[(t_1, 0, \dots, 0), \dots, (0, \dots, 0, t_m)]] \subset \bigoplus_{i=1}^m k[[t_i]]$$

or, depending on the context, it will also refer to the local ring

$$k[[(t_1, 0, \dots, 0), \dots, (0, \dots, 0, t_m)]]/((t_1, 0 \dots, 0)^{2\delta}, \dots, (0, \dots, 0, t_m)^{2\delta}) \subset \bigoplus_{i=1}^m k[[t_i]]/(t_i^{2\delta}).$$

Observe that $*^m \in \mathrm{ter}^{m-1}(m)$, and is the complete local ring at the singularity with m smooth branches intersecting transversely.

Proposition 2.10. *Suppose $S \in \mathrm{ter}^\delta(m)$. Let $D = \mathrm{span}\{(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)\}$ be the subspace of constants in $\bigoplus_{i=1}^m k[[t_i]]/(t_i^{2\delta})$, and let $C = \mathrm{span}\{\mathbf{1}\}$. The following are equivalent:*

(i) S *is glued;*
(ii) $S \cap D = C$*;*
(iii) $S \subset \ast^m$;
(iv) S *is local.*

Proof. (i) $\Leftrightarrow$ (ii): Clearly $C \subset S$ by virtue of S being a k-subalgebra. For any $j \geq 2$, we have the gluing isomorphism $\varphi_j : S_{\theta_{j-1}}/I_{\theta_{j-1}} \to S_j/I_j$. Since this a k-algebra isomorphism and $\dim_k \varphi_j \geq 1$ by assumption, $\varphi_j(\mathbf{1}) = 1$. Thus constants are always glued in a glued subalgebra, and therefore $S \cap D = C$. Conversely, if $\dim_k \varphi_j = 0$ for some j, then S contains constants of the form $(1, \dots, 1, 0, c_{j+1}, \dots, c_m)$ and $(0, \dots, 0, 1, c'_{j+1}, \dots, c'_m)$, so $S \cap D \not\subset C$.

(ii) $\Rightarrow$ (iii): The only elements in $\bigoplus_{i=1}^m k[[t_i]] \setminus \ast^m$ contain constant terms not in C. Hence $S \not\subset \ast^m$ implies $S \cap D \not\subset C$.

(iii) $\Rightarrow$ (iv): Let $\mathfrak{m} = ((t_1, 0, \dots, 0), \dots, (0, \dots, 0, t_m))$ be the maximal ideal of $\ast^m$, and let $\iota : S \to \ast^m$ be the inclusion. Observe $\ast^m$ is a finitely generated k-algebra, as it is isomorphic to $k[x_1, \dots, x_m]/(x_i^{2\delta}, x_i x_j : i \neq j \in \{1, \dots, m\})$. Hence $\iota^{-1}(\mathfrak{m}) = S \cap \mathfrak{m}$ is a maximal ideal of S. Suppose $\mathfrak{a}$ is any proper ideal in S. Let $\mathfrak{a}^e$ be the extension of $\mathfrak{a}$, i.e. the ideal generated by $\mathfrak{a}$ in $\ast^m$. Assume $\mathfrak{a}^e$ contains a unit, f. Then the constant term of f is nonzero, and this can only occur in $\mathfrak{a}^e$ if $\mathfrak{a}$ contained a unit, which would be a contradiction. So $\mathfrak{a}^e \subset \mathfrak{m}$, but then $\mathfrak{a} \subset \iota^{-1}(\mathfrak{a}^e) \subset \iota^{-1}(\mathfrak{m}) = S \cap \mathfrak{m}$. Thus S is local.

(iv) $\Rightarrow$ (ii): If $\mathfrak{m}$ is the unique maximal ideal of S, then $S \cong k \oplus \mathfrak{m}$ as a k-module. By virtue of S being a k-subalgebra of $\bigoplus_{i=1}^m k[[t_i]]/(t_i^{2\delta})$, we have the following commutative diagram

$$\begin{array}{ccc} k & \longrightarrow & k \oplus \mathfrak{m} \\ & \searrow & \big\downarrow \\ & & \displaystyle\bigoplus_{i=1}^m k[[t_i]]/(t_i^{2\delta}) \end{array}$$

from which it follows that $S \cap D = C$. ■

Let C_1, C_2 be two curves intersecting at a point P. Following the notation of [Hir57], let $\mathfrak{i}(P; C_1 \cdot C_2)$ denote the intersection multiplicity of C_1 and C_2 at P. More generally, let $\mathfrak{i}(P; \bigwedge_{i=1}^m C_i)$ denote the intersection multiplicity of m curves $C_1, \dots, C_m$ at P, which can be determined by

$$\mathfrak{i}\left(P; \bigwedge_{i=1}^m C_i\right) = \sum_{j=2}^m \mathfrak{i}\left(P; \left(\bigcup_{i=1}^{j-1} C_i\right) \cdot C_j\right).$$

Let $C = \bigcup_{i=1}^m C_i$. Then $\widehat{\mathscr{O}}_{C,P}$ is a k-subalgebra of $\bigoplus_{i=1}^m k[[t_i]]$. If P has delta-invariant δ when viewed as a point of C, and if P has delta-invariant δ_i when viewed as a point of C_i, then Hironaka showed [Hir57, Proposition 4] that

$$\delta = \left(\sum_{i=1}^m \delta_i\right) + \mathfrak{i}\left(P; \bigwedge_{i=1}^m C_i\right).$$

By our computation in Corollary 2.6, it follows that

$$\mathfrak{i}\left(P; \bigwedge_{i=1}^m C_i\right) = \mathrm{gd}(S).$$

Therefore the gluing dimension of $\widehat{\mathscr{O}}_{C,P}$ precisely measures the intersection multiplicity of $C_1, \dots, C_m$ at P. Moreover, it is clear that as we work recursively, for each $j \geq 2$,

$$\mathfrak{i}\left(P; \left(\bigcup_{i=1}^{j-1} C_i\right) \cdot C_j\right) = \dim_k \varphi_j$$

so that $\dim_k \varphi_j$ is precisely the intersection multiplicity of the jth branch with the first $j-1$ branches.

Corollary 2.11. *Let C be a curve with a connected m-branch singularity P. Then $\widehat{\mathscr{O}}_{C,P}$ is a glued subalgebra of $\bigoplus_{i=1}^m k[[t_i]]$.*

Definition 2.12. For $m \geq 1$, the *glued δ-territory* of $\bigoplus_{i=1}^m k[[t_i]]$ will be the subset $\operatorname{ter}^\delta_{\mathcal{G}}(m) \subset \operatorname{ter}^\delta(m)$ consisting of the glued subalgebras. Note that all finite-codimensional k-subalgebras of $k[[t]]$ are glued (by Proposition 2.10), so $\operatorname{ter}^\delta_{\mathcal{G}}(1) = \operatorname{ter}^\delta(1)$.

Theorem 2.13. *Take $m \geq 2$.*

(i) *If $\delta < m-1$, then $\operatorname{ter}^\delta_{\mathcal{G}}(m) = \varnothing$.*

(ii) *If $\delta \geq m-1$, then $\operatorname{ter}^\delta_{\mathcal{G}}(m)$ is a closed subscheme of $\operatorname{ter}^\delta(m)$. In particular, $\operatorname{ter}^{m-1}_{\mathcal{G}}(m) = \{pt\}$.*

Proof. (i) If $\delta < m-1$, then for any $S \in \operatorname{ter}^\delta(m)$, $\operatorname{gd}(S) < m-1$ by Corollary 2.6. But in order for S to be a glued subalgebra, $\operatorname{gd}(S) \geq m-1$.

(ii) We treat the case $\delta = m-1$ separately. Suppose

$$S = (S_1, I_1, \ldots, S_m, I_m, \varphi_2, I_{\theta_2}, \varphi_3, I_{\theta_3}, \ldots, I_{\theta_{m-1}}, \varphi_m)$$

is glued and of codimension $m-1$ in $\bigoplus_{i=1}^m k[[t_i]]$. Then by Corollary 2.6, $\delta_i = 0$ and $\dim_k \varphi_j = 1$ for all $j \geq 2$. This forces $S_i = k[[t_i]]$ and $I_i = (t_i)$, so that $S_i/I_i \cong k$, and $\varphi_j = \operatorname{id}_k$. It follows that S is uniquely the rational m-fold k-subalgebra $\ast^m$.

Now suppose $\delta \geq m$. As before, let $\Lambda = \bigoplus_{i=1}^m k[[t_i]]/(t_i^{2\delta})$, let D be the m-dimensional subspace of constants, and let $C = \operatorname{span}\{\mathbf{1}\}$. Let D^c denote the complement of D in Λ. We know $S \in \operatorname{ter}^\delta(m)$ is glued if and only if $S \cap D = C$. This is equivalent to saying S is a δ-codimensional subspace of the $(2\delta m - m + 1)$-dimensional subspace $C \oplus D^c$. Hence S/C is a $(\delta-1)$-codimensional subspace of the $(2\delta m - m)$-dimensional space $(C \oplus D^c)/C$. By the composition of the inclusion $C \hookrightarrow C \oplus D^c$ and the quotient $C \oplus D^c \to (C \oplus D^c)/C$, the sub-Grassmannian $G' = G(2\delta m - m - \delta + 1, 2\delta m - m)$ embeds as a closed subvariety of $G(2\delta m - \delta, 2\delta m)$. It naturally follows that $\operatorname{ter}^\delta_{\mathcal{G}}(m) = \operatorname{ter}^\delta(m) \cap G'$, so therefore the glued territory is a closed subscheme of $\operatorname{ter}^\delta(m)$. ■

3. Isom-Hilb Stratification

In what follows, a glued subalgebra S of $\bigoplus_{i=1}^m k[[t_i]]$ will be identified with the Goursat data $(S_1, I_1, S_2, I_2, \varphi)$, where S_1 is a glued subalgebra of $\bigoplus_{i=1}^{m-1} k[[t_i]]$, S_2 is a k-subalgebra of $k[[t_m]]$, each I_i is an ideal of S_i, and $\varphi : S_1/I_1 \to S_2/I_2$ is a k-algebra isomorphism. The benefit of using this quintuple of data instead of the $(4m-3)$-tuple from Proposition 2.5 is that it reveals the recursive procedure through which we can construct the glued territories.

Proposition 2.3 tells us that if $(S_1, I_1, S_2, I_2, \varphi)$ identifies a k-subalgebra of codimension δ, then $\dim_k S_1 + \dim_k S_2 + \dim_k \varphi = \delta$. Moreover, if this is a glued subalgebra, then there are additional restrictions, namely $\dim_k S_1 \geq m-2$ and $\tau \geq 1$.

Definition 3.1. Fix $m \geq 2$ and $\delta \geq m-1$. A *sufficient triple* for δ, m will refer to an ordered triple $(\delta_1, \delta_2, \tau) \in \mathbb{N}^3$ satisfying the following properties:

(i) $\delta_1 \geq m-1$,

(ii) $\tau \geq 1$, and

(iii) $\delta_1 + \delta_2 + \tau = \delta$.

Let $\operatorname{st}(\delta, m)$ denote the set of all sufficient triples for δ, m. Given a sufficient triple $(\delta_1, \delta_2, \tau) \in \operatorname{st}(\delta, m)$, we say a glued subalgebra $(S_1, I_2, S_2, I_2, \varphi)$ in $\operatorname{ter}^\delta_{\mathcal{G}}(m)$ is *of type* $(\delta_1, \delta_2, \tau)$ if $\dim_k S_i = \delta_i$ for each i, and $\dim_k \varphi = \tau$.

We will stratify the glued territory according to these sufficient triples.

Definition 3.2. Fix $m \geq 2$ and $\delta \geq m-1$. For each sufficient triple $(\delta_1, \delta_2, \tau) \in \operatorname{st}^\delta(m)$, let its corresponding *Isom-Hilb component* be the locally closed subscheme $\mathfrak{I}\mathfrak{H}_m(\delta_1, \delta_2, \tau)$ of $\operatorname{ter}^\delta_{\mathcal{G}}(m)$ whose closed points are the glued subalgebras of type $(\delta_1, \delta_2, \tau)$. By Goursat's Lemma, it is clear that

$$\operatorname{ter}^\delta_{\mathcal{G}}(m) = \bigsqcup_{(\delta_1,\delta_2,\tau)\in \operatorname{st}(\delta,m)} \mathfrak{I}\mathfrak{H}_m(\delta_1, \delta_2, \tau).$$

To show that each Isom-Hilb component is locally closed, we first need a result on the semicontinuity of the values δ_1, δ_2, τ.

Proposition 3.3. *Fix $m \geq 2$ and $\delta \geq m-1$. Identify each closed point $S \in \mathrm{ter}^{\delta}_{\mathcal{G}}(m)$ with the data $(S_1, I_1, S_2, I_2, \varphi)$ where $S_1 \in \mathrm{ter}^{\delta_1}_{\mathcal{G}}(m-1)$ and $S_2 \in \mathrm{ter}^{\delta_2}(1)$. Consider the following $\mathbb{N}$-valued functions on the set of closed points of $\mathrm{ter}^{\delta}_{\mathcal{G}}(m)$:*

$$f(S) = \dim_k \left(\bigoplus_{i=1}^{m-1} k[[t_i]] \right) \Big/ S_1;$$
$$g(S) = \dim_k k[[t_m]]/S_2;$$
$$h(S) = \dim_k \varphi.$$

Then f, g are upper semicontinuous, and h is lower semicontinuous.

Proof. Let $\Lambda_1 = \bigoplus_{i=1}^{m-1} k[[t_i]]/(t_i^{2\delta})$, let $\Lambda_2 = k[[t_m]]/(t_m^{2\delta})$, and identify each $S \in \mathrm{ter}^{\delta}_{\mathcal{G}}(m)$ with its corresponding k-vector subspace in $G(2\delta m - \delta, \Lambda_1 \oplus \Lambda_2)$. Then $f(S) = \dim_k \Lambda_1/\pi_1(S) = 2\delta m - \dim_k \pi_1(S)$, where $\pi_1(S)$ denotes the projection of S onto Λ_1. Decompose S into $(S \cap \Lambda_1) \oplus V$, so that $\Lambda_1 \cap V$ is trivial. Then $\pi_1(S) = \pi_1(V) \cong V$, hence

$$\dim_k \pi_1(S) = \min\{2\delta(m-1), \dim_k V\}.$$

From this, it is clear that for any $n \in \mathbb{Z}$, $\dim_k \pi_1(S) \leq n$ if and only if $\dim_k(S \cap \Lambda_1) \geq 2\delta m - \delta - n$. But

$$\{S \in G(2\delta m - \delta, \Lambda_1 \oplus \Lambda_2) : \dim_k \pi_1(S) \leq n\} = \{S \in G(2\delta m - \delta, \Lambda_1 \oplus \Lambda_2) : \dim_k(S \cap \Lambda_1) \geq 2\delta m - \delta - n\}$$

is a Schubert variety, hence is a closed subscheme of the Grassmannian. Moreover, its intersection with $\mathrm{ter}^{\delta}_{\mathcal{G}}(m)$ is a closed subscheme of $\mathrm{ter}^{\delta}_{\mathcal{G}}(m)$. This implies $\pi_1(S)$ is lower semicontinuous on the glued territory, and consequently $f(S)$ is upper semicontinuous.

A similar argument shows that $g(S) = \dim_k \Lambda_2/\pi_2(S)$ is upper semicontinuous. Hence so is the sum $f(S) + g(S)$. We know $h(S) = \dim_k \varphi = \delta - (f(S) + g(S))$ by Proposition 2.3. Since δ is clearly a constant value on $\mathrm{ter}^{\delta}_{\mathcal{G}}(m)$, this guarantees $h(S)$ is lower semicontinuous. ■

Corollary 3.4. *Every Isom-Hilb component in $\mathrm{ter}^{\delta}_{\mathcal{G}}(m)$ is locally closed.*

Proof. Let f, g, h be the functions from Proposition 3.3, and let $(\delta_1, \delta_2, \tau)$ be a sufficient triple. By semicontinuity, the subset U of $\mathrm{ter}^{\delta}_{\mathcal{G}}(m)$ for which $f(S) < \delta_1 + 1$ is open, the subset V for which $g(S) < \delta_2 + 1$ is open, and the subset Z for which $h(S) \leq \tau$ is closed. We clearly see that $\mathfrak{IH}_m(\delta_1, \delta_2, \tau) = U \cap V \cap Z$, therefore it is locally closed. ■

For a sufficient triple $(\delta_1, \delta_2, \tau) \in \mathrm{st}(\delta, m)$, $\mathfrak{IH}_m(\delta_1, \delta_2, \tau)$ should parametrize all of the glued subalgebras $(S_1, I_1, S_2, I_2, \varphi)$ of type $(\delta_1, \delta_2, \tau)$. Based on this quintuple of data, it becomes clear that three familiar moduli spaces are needed: the glued territories will parametrize the possible glued subalgebras S_1 and S_2, respectively; punctual Hilbert schemes will parametrize the possible ideals I_1 and I_2; and an Isom scheme will parametrize the possible isomorphisms φ. This is where the "Isom-Hilb" nomenclature comes from. More precisely, we will construct a scheme $\mathfrak{X}$ which is a relative Isom scheme over the product of relative punctual Hilbert schemes taken over glued territories. There is a morphism $f : \mathfrak{X} \to G(2\delta m - \delta, 2\delta m)$, and the image $f(\mathfrak{X})$ is our desired $\mathfrak{IH}_m(\delta_1, \delta_2, \tau)$.

We will now overview the construction of the scheme $\mathfrak{X}$ and the morphism f just described. Again fix the sufficient triple $(\delta_1, \delta_2, \tau) \in \mathrm{st}(\delta, m)$. Clearly the glued territories $\mathrm{ter}^{\delta_1}_{\mathcal{G}}(m-1)$ and $\mathrm{ter}^{\delta_2}(1)$ will parametrize all possible S_1 and S_2 needed in the Goursat data. To find the τ-codimensional ideals of each S_i, we make use of the punctual Hilbert scheme, in the sense of [Iar77], [Fog68], and [Ber08]. More precisely, given a finite-dimensional local k-algebra A and natural number τ, the punctual Hilbert scheme $\mathrm{Hilb}(A, \tau)$, which parametrizes the zero-dimensional subschemes of $\mathrm{Spec}\, A$ of degree τ, is a projective connected scheme [Fog68, Proposition 2.2]. If A is local but not necessarily finite-dimensional, then there is a one-to-one correspondence between τ-codimensional ideals of A and τ-codimensional ideals of $A/\mathfrak{m}^\tau$ [Ber08, Lemma 2.22]. Hence, by abuse of notation, $\mathrm{Hilb}(A,\tau)$ will denote $\mathrm{Hilb}(A/\mathfrak{m}^\tau,\tau)$. In our case, S_1 and S_2 are glued subalgebras, hence local. Thus there exist punctual Hilbert schemes $\mathrm{Hilb}(S_1,\tau)$ and $\mathrm{Hilb}(S_2,\tau)$.

Since $\mathrm{ter}^{\delta_1}_{\mathcal{G}}(m-1)$ and $\mathrm{ter}^{\delta_2}(1)$ are projective, we can relativize the punctual Hilbert scheme over them. Let $\mathfrak{Hilb}_{m-1}(\delta_1,\tau)$ and $\mathfrak{Hilb}_1(\delta_2,\tau)$, respectively, denote the resulting schemes. We then take the product of these two relative punctual Hilbert schemes. Each closed point of $\mathfrak{Hilb}_{m-1}(\delta_1,\tau)\times\mathfrak{Hilb}_1(\delta_2,\tau)$ is of the form $((S_1,I_1),(S_2,I_2))$, where $S_1\in\mathrm{ter}^{\delta_1}_{\mathcal{G}}(m-1)$, $S_2\in\mathrm{ter}^{\delta_2}(1)$, $I_1\in\mathrm{Hilb}(S_1,\tau)$, and $I_2\in\mathrm{Hilb}(S_2,\tau)$.

Finally, to find the isomorphisms between S_1/I_1 and S_2/I_2, we make use of an Isom scheme. Given arbitrary flat projective k-schemes X,Y, there exists an Isom scheme $\mathrm{Isom}_k(X,Y)$ parametrizing k-isomorphisms $X\to Y$. This is an open subscheme of $\mathrm{Mor}_k(X,Y)$, the scheme parametrizing k-morphisms $X\to Y$ [FGI05]. In particular, if $X=\operatorname{Spec}B$ and $Y=\operatorname{Spec}C$, where B,C are finite-dimensional k-algebras, then $\mathrm{Mor}_k(X,Y)$ is an affine k-scheme [Sta22, Tag 0BL0], which equivalently classifies the k-algebra homorphisms $C\to B$; consequently, $\mathrm{Isom}_k(X,Y)$ in this case is also affine and classifies the k-algebra isomorphisms $C\to B$. If B,C are isomorphic, then the closed points of the Isom scheme form the affine algebraic group $\mathrm{Aut}_k(B)=\mathrm{Aut}_k(C)$; otherwise, there are no closed points. In our case, S_1/I_1 and S_2/I_2 are τ-dimensional k-algebras, so $\mathrm{Isom}_k(\operatorname{Spec}S_1/I_1,\operatorname{Spec}S_2/I_2)$ is an affine scheme.

Like the punctual Hilbert schemes, we want to relativize the Isom scheme over $\mathfrak{Hilb}_{m-1}(\delta_1,\tau)\times\mathfrak{Hilb}_1(\delta_2,\tau)$ [ACG11]. More precisely, let $\mathfrak{U}$ be the universal family over $\mathfrak{Hilb}_{m-1}(\delta_1,\tau)$, and let $\mathfrak{V}$ be the universal family over $\mathfrak{Hilb}_1(\delta_2,\tau)$. These are both projective and flat over their respective bases, hence the relative Isom scheme we desire is

$$\mathrm{Isom}_{\mathfrak{Hilb}_{m-1}(\delta_1,\tau)\times\mathfrak{Hilb}_1(\delta_2,\tau)}(\mathfrak{U}\times\mathfrak{Hilb}_1(\delta_2,\tau),\mathfrak{Hilb}_{m-1}(\delta_1,\tau)\times\mathfrak{V}).$$

This is precisely our scheme $\mathfrak{X}$. A closed point in this scheme is of the form $((S_1,I_1),(S_2,I_2),\varphi)$, where $((S_1,I_1),(S_2,I_2))\in\mathfrak{Hilb}_{m-1}(\delta_1,\tau)\times\mathfrak{Hilb}_1(\delta_2,\tau)$ and, assuming $S_1/I_1\cong S_2/I_2$, $\varphi\in\mathrm{Aut}_k(S_i/I_i)$.

The scheme $\mathfrak{X}$ comes equipped with a universal family, which by construction defines a subbundle of $(\bigoplus_{i=1}^m k[[t_i]]/(t_i^{2\delta}))\otimes_k\mathscr{O}_{\mathfrak{X}}$ of rank $2\delta m-\delta$. Hence we obtain a morphism $f:\mathfrak{X}\to G(2\delta m-\delta,2\delta m)$. It is clear that $f(\mathfrak{X})$ lies in $\mathrm{ter}^{\delta}_{\mathcal{G}}(m)$, and that f, when restricted to its image, is a bijection on closed points. This image is $\mathfrak{IH}_m(\delta_1,\delta_2,\tau)$.

For our purposes, it will only be necessary to know the closed points of the Isom-Hilb component, not its overall scheme structure. To this end, we will introduce the following concept.

Definition 3.5. Let X be a scheme. We say that a finite collection of schemes $X_1,\dots,X_r$ is a *parametrization* of X if there exists a morphism $g:\bigsqcup_{j=1}^r X_j\to X$ such that g is a bijection on closed points.

Note then that if $X_1,\dots,X_r$ define a parametrization of X via a morphism $g:\bigsqcup_{j=1}^r X_j\to X$, and $h:X\to Y$ is a morphism which is also a bijection on closed points, then $X_1,\dots,X_r$ will also define a parametrization of Y via $h\circ g$. In particular, a parametrization of the scheme $\mathfrak{X}$ constructed above will parametrize $\mathfrak{IH}_m(\delta_1,\delta_2,\tau)$. It is also easy to see that if $X_1,\dots,X_r$ parametrize X and $Y_1,\dots,Y_s$ parametrize Y, then the collection $X_i\times Y_j$ will parametrize $X\times Y$. We compute some relevant examples below.

Example 3.6. For any τ, $k[[t]]$ has a unique ideal of codimension τ, namely (t^τ). Hence $\{pt\}$ is a parametrization of $\mathrm{Hilb}(k[[t]],\tau)$. Since $k[[t]]$ is the unique point of $\mathrm{ter}^0(1)$, it follows that $\{pt\}$ is a parametrization of $\mathfrak{Hilb}_1(0,\tau)$.

Example 3.7. Any local k-algebra S has a unique ideal of codimension 1, namely its maximal ideal. Hence $\{pt\}$ is a parametrization of $\mathrm{Hilb}(S,1)$. It follows that for any δ,m, $\mathrm{ter}^{\delta}_{\mathcal{G}}(m)$ is a parametrization of $\mathfrak{Hilb}_m(\delta,1)$.

Example 3.8. The ideals of codimension $\tau\geq 2$ of $k[[t^2,t^3]]$ are of the form $(t^\tau+at^{\tau+1})$ for all $a\in k$, and $(t^{\tau+1},t^{\tau+2})$ [Lax00, Proposition 1.1]. This yields a family of ideals parametrized by $\mathbb{A}^1$

and $\{pt\}$, so there exist morphisms from each of these families to $\mathrm{Hilb}(k[[t^2,t^3]],\tau)$. Thus $\mathbb{A}^1 \sqcup \{pt\}$ is a parametrization of $\mathrm{Hilb}(k[[t^2,t^3]],\tau)$. Since $k[[t^2,t^3]]$ is the unique point of $\mathrm{ter}^1(1)$, it follows that $\mathbb{A}^1 \sqcup \{pt\}$ is also a parametrization for $\mathfrak{Hilb}_1(1,\tau)$.

Example 3.9. Let S be a local k-algebra with maximal ideal $\mathfrak{m}$, and let $r = \dim_k \mathfrak{m}/\mathfrak{m}^2$. If I is an ideal of codimension 2, then $\mathfrak{m}^2 \subset I$. Moreover, $S/I \cong k[x]/(x^2)$, as this is the unique two-dimensional k-algebra up to isomorphism. Hence $I = I' + \mathfrak{m}^2$ for some $I' \subset \mathfrak{m}$. Let $\{f_1,\dots,f_r\}$ be a basis for $\mathfrak{m}/\mathfrak{m}^2$ as a k-vector space. Let $\pi : S \to S/I$ be the quotient map. We consider all possible cases. If $f_1 \notin I$, then $S/I \cong k[f_1]/(f_1^2)$. This suggests $\pi(f_i) \in \mathrm{span}\{f_1\}$ for all $i \geq 2$. Hence for all $c_2,\dots,c_r \in k$, we obtain the following collection of ideals:

$$I_{1,c_2,\dots,c_r} = (f_2 + c_2 f_1, f_3 + c_3 f_1, \dots, f_r + c_r f_1) + \mathfrak{m}^2.$$

Now suppose $f_1 \in I$, but $f_2 \notin I$. Then again for all $c_3,\dots,c_r \in k$ we obtain the following ideals:

$$I_{0,1,c_3,\dots,c_r} = (f_1, f_3 + c_3 f_2, f_4 + c_4 f_2, \dots, f_r + c_r f_2) + \mathfrak{m}^2.$$

This pattern repeats until $f_1,\dots,f_{r-1} \in I$. If $f_r \in I$, then $I = \mathfrak{m}$, contradicting its codimension. So thus $f_r \notin I$, and we obtain the unique ideal

$$I_{0,\dots,0,1} = (f_1,\dots,f_{r-1}) + \mathfrak{m}^2.$$

It is clear from the subscripts of the ideals listed above that we have a $\mathbb{P}^{r-1}$-family of ideals, thus $\mathbb{P}^{r-1}$ parametrizes $\mathrm{Hilb}(S,2)$.

We now discuss an obstruction to the Isom-Hilb stratification, namely that fibers of Isom-Hilb components (and sometimes even the entire component itself) can be empty.

Definition 3.10. Let $((S_1,I_1),(S_2,I_2))$ be a closed point of $\mathfrak{Hilb}_{m-1}(\delta_1,\tau) \times \mathfrak{Hilb}_1(\delta_2,\tau)$. The *fiber* in the Isom-Hilb component $\mathfrak{IH}_m(\delta_1,\delta_2,\tau)$ over this closed point will refer to the Isom scheme $\mathrm{Isom}_k(\mathrm{Spec}\, S_1/I_1, \mathrm{Spec}\, S_2/I_2)$. The closed point will be called a *gluable pair* if $S_1/I_1 \cong S_2/I_2$ as k-algebras; equivalently, its fiber in the Isom-Hilb component is nonempty.

Example 3.11. Every closed point $((S_1,I_1),(S_2,I_2)) \in \mathfrak{Hilb}_{m-1}(\delta_1,1) \times \mathfrak{Hilb}_1(\delta_2,1)$ is a gluable pair, due to the fact that there is a unique one-dimensional k-algebra, namely k. Each fiber of the Isom-Hilb component $\mathfrak{IH}_m(\delta_1,\delta_2,1)$ consists of a point, since k has only the trivial k-automorphism id_k.

Example 3.12. Every closed point $((S_1,I_1),(S_2,I_2)) \in \mathfrak{Hilb}_{m-1}(\delta_1,2) \times \mathfrak{Hilb}_1(\delta_2,2)$ is a gluable pair, due to the fact that there is a unique two-dimensional k-algebra, namely $k[x]/(x^2)$. Each fiber of the Isom-Hilb component $\mathfrak{IH}_m(\delta_1,\delta_2,2)$ is isomorphic to $\mathbb{A}^1 \setminus \{0\}$, since $\mathrm{Aut}_k(k[x]/(x^2)) \cong \mathrm{GL}(1,k)$.

The next two examples show that empty fibers do appear in some Isom-Hilb components. These empty fibers will ultimately serve as an obstruction to making general statements about the geometry of the glued territory in Section 4.

Example 3.13. Consider $\mathfrak{IH}_2(0,1,3) \subset \mathrm{ter}^4_{\mathcal{G}}(2)$. We know $\mathfrak{Hilb}_1(0,3)$ is parametrized by a point, and $\mathfrak{Hilb}_1(1,3)$ is parametrized by $\mathbb{A}^1 \sqcup \{pt\}$. Hence $\mathfrak{Hilb}_1(0,3) \times \mathfrak{Hilb}_1(1,3)$ can be parametrized by $\mathbb{A}^1 \sqcup \{pt\}$, with the $\mathbb{A}^1$-family given by closed points of the form $((k[[t]],(t^3)),(k[[t^2,t^3]],(t^3+at^4)))$ for some $a \in k$, and the point given by $((k[[t]],(t^3)),(k[[t^2,t^3]],(t^4,t^5)))$. However, the point is not a gluable pair, since $k[[t^2,t^3]]/(t^4,t^5) \cong k[x,y]/(x^2,xy,y^2)$ is not isomorphic to $k[[t]]/(t^3) \cong k[x]/(x^3)$. Hence its fiber in the Isom-Hilb component will be empty.

Example 3.14. Consider $\mathfrak{IH}_2(0,1,4) \subset \mathrm{ter}^5_{\mathcal{G}}(2)$. Again, we know $\mathfrak{Hilb}_1(0,4)$ is parametrized by a point, and $\mathfrak{Hilb}_1(1,4)$ is parametrized by $\mathbb{A}^1 \sqcup \{pt\}$. Hence $\mathfrak{Hilb}_1(0,4) \times \mathfrak{Hilb}_1(1,4)$ can be parametrized by $\mathbb{A}^1 \sqcup \{pt\}$, with the $\mathbb{A}^1$-family given by closed points of the form $((k[[t]],(t^4)),(k[[t^2,t^3]],(t^4+at^5)))$ for some $a \in k$, and the point given by $((k[[t]],(t^4)),(k[[t^2,t^3]],(t^5,t^6)))$. None of the points in the $\mathbb{A}^1$-family are gluable pairs, since $k[[t^2,t^3]]/(t^4+at^5) \cong k[x,y](x^2,y^2)$ is not isomorphic to $k[[t]]/(t^4) \cong k[x]/(x^4)$. Similarly, the point is also not a gluable pair, since $k[[t^2,t^3]]/(t^5,t^6) \cong k[x,y]/(x^3,xy,y^2)$ is not isomorphic to $k[x]/(x^4)$. Therefore this Isom-Hilb component is entirely empty. The corresponding geometry fact is that a curve with a cusp and a smooth curve cannot meet at the cusp with intersection multiplicity 4.

Proposition 3.15. *The nonempty fibers of a nonempty Isom-Hilb component* $\mathfrak{IH}_m(\delta_1, \delta_2, \tau)$ *are projective if and only if* $\tau = 1$.

Proof. By Example 3.11, when $\tau = 1$, each fiber consists of one closed point, hence is projective. Now suppose $\tau \geq 2$, and assume, for the sake of contradiction, that a nonempty fiber was projective. If the fiber is over the closed point $((S_1, I_1), (S_2, I_2))$, then $S_1/I_1 \cong S_2/I_2$, and the fiber is the affine algebraic group $\mathrm{Aut}_k(S_i/I_i)$. If we assume the fiber is also projective, then $\mathrm{Aut}_k(S_i/I_i)$ must be a finite group. We show that this leads to a contradiction.

We have S_i/I_i is a commutative τ-codimensional k-algebra, necessarily local with maximal ideal $\mathfrak{m}$. Hence it is isomorphic to some quotient $k[x_1, \dots, x_n]/J$, where $J \subset (x_1, \dots, x_n)^2$ and $1 \leq n \leq \tau - 1$ [GS94]. The Loewy length ℓ of S_i/I_i is the smallest natural number for which $(x_1, \dots, x_n)^\ell \subset J$ (equivalently, $\mathfrak{m}^\ell = 0$). Note $\ell \neq 1$. If $\ell = 2$, then $S_i/I_i \cong k[x_1, \dots, x_{\tau-1}]/(x_1, \dots, x_{\tau-1})^2$, which has automorphism group $\mathrm{GL}(\tau - 1, k)$. Since k is algebraically closed, it is not finite, so this is never a finite group. If $\ell > 2$, then we can find a monomial basis $\{v_1, \dots, v_p\}$ for $\mathfrak{m}^{\ell-1}$. Let H be the subgroup of automorphisms determined by $x_j \mapsto x_j + c_{j,1}v_1 + \cdots + c_{j,p}v_p$. There are no restrictions on these coefficients, so $H \cong \mathbb{A}^{np}$, again showing that $\mathrm{Aut}_k(S_i/I_i)$ is not finite. ■

Corollary 3.16. *Let* $X = \mathfrak{IH}_m(\delta_1, \delta_2, \tau)$ *be an Isom-Hilb component in* $\mathrm{ter}^\delta_{\mathcal{G}}(m)$ *with* $\tau \geq 2$. *If* $\overline{X}$ *denotes the projective closure of* X, *then any closed point* $S \in \overline{X} \setminus X$ *belongs to an Isom-Hilb component of the form* $\mathfrak{IH}_m(\varepsilon_1, \varepsilon_2, \sigma)$ *where* $\sigma < \tau$.

Proof. By Proposition 3.15, the fibers of $\mathfrak{IH}_m(\delta_1, \delta_2, \tau)$ are not projective since $\tau \neq 1$, so there are in fact closed points in $\overline{X} \setminus X$. Since $\mathrm{ter}^\delta_{\mathcal{G}}(m)$ is projective, it contains $\overline{X}$. Hence if S is a closed point in $\overline{X} \setminus X$, then S must belong to some Isom-Hilb component of the form $\mathfrak{IH}_m(\varepsilon_1, \varepsilon_2, \sigma)$. By Proposition 3.3, gluing dimension is lower semicontinuous, hence $\sigma \leq \tau$. If $\sigma = \tau$, then $\delta_1 + \delta_2 = \varepsilon_1 + \varepsilon_2$. It cannot be the case that $\varepsilon_1 = \delta_1$ and $\varepsilon_2 = \delta_2$, since then this Isom-Hilb component is precisely X. But then either $\varepsilon_1 < \delta_1$ or $\varepsilon_2 < \delta_2$, and this would contradict the upper semicontinuity of these quantities. Thus $\sigma < \tau$. ■

Overall, we obtain a recursive procedure for computing all of the glued territories using the Isom-Hilb stratification.

Step 0: Take any sufficient triple $(\delta_1, \delta_2, \tau) \in \mathrm{st}(\delta, m)$. Assume that the closed points of $\mathrm{ter}^{\delta_1}_{\mathcal{G}}(m-1)$ and $\mathrm{ter}^{\delta_2}(1)$ are known.

Step 1: Find a parametrization of $\mathfrak{Hilb}_{m-1}(\delta_1, \tau)$ by finding the closed points of $\mathrm{Hilb}(S_1, \tau)$ for each $S_1 \in \mathrm{ter}^{\delta_1}_{\mathcal{G}}(m-1)$. Similarly, find a parametrization of $\mathfrak{Hilb}_1(\delta_2, \tau)$ by finding the closed points of $\mathrm{Hilb}(S_2, \tau)$ for each $S_2 \in \mathrm{ter}^{\delta_2}(1)$.

Step 2: Restrict the parametrization of $\mathfrak{Hilb}_{m-1}(\delta_1, \tau) \times \mathfrak{Hilb}_1(\delta_2, \tau)$ to its gluable pairs.

Step 3: Find a parametrization for $\mathfrak{IH}_m(\delta_1, \delta_2, \tau)$ by finding $\mathrm{Aut}_k(S_i/I_i)$ over each gluable pair $((S_1, I_1), (S_2, I_2))$. This provides sufficient data to explicitly write out all glued subalgebras appearing in this Isom-Hilb component. Repeat Steps 0-3 for each sufficient triple.

Step 4: View the glued subalgebras in each Isom-Hilb component as points in the sub-Grassmannian $G(2\delta m - m - \delta + 1, 2\delta m - m)$. That is, compute Plücker coordinates and find polynomial equations defining the closure of each Isom-Hilb component.

4. Global Geometry of the Glued Territory

Here we examine some geometric properties of the glued territory, including its connectedness, its irreducible components, and its dimension bounds. Throughout this section, we will make use of the following notation. For any nonempty subset $\Psi \subset \{1, \dots, \delta - m + 2\}$, we define the following subsets of $\mathrm{ter}^\delta_{\mathcal{G}}(m)$:

$$\mathcal{G}_\Psi := \bigcup_{\substack{(\delta_1, \delta_2, \tau) \in \mathfrak{st}(\delta, m) \\ \tau \in \Psi}} \mathfrak{IH}_m(\delta_1, \delta_2, \tau).$$

For instance $\mathcal{G}_{\{1\}}$ consists of the Isom-Hilb components with $\tau = 1$, and surely $\mathcal{G}_{\{1,\ldots,\delta-m+2\}} = \operatorname{ter}^{\delta}_{\mathcal{G}}(m)$. This will simply provide us with a way to arrange the Isom-Hilb components by gluing dimension.

4.1. **Connectedness and Irreducible Components.** In this section we prove that the glued territories are connected. First we require a lemma.

Lemma 4.1. *Suppose $S \in \operatorname{ter}^{\delta}_{\mathcal{G}}(m)$, where $m \geq 1$, and suppose $I \subset S$ is an ideal such that $\dim_k S/I = \tau \geq 1$. Then $k \oplus I \in \operatorname{ter}^{\delta+\tau-1}_{\mathcal{G}}(m)$.*

Proof. By assumption, S is glued, so it is local. Denote its maximal ideal by $\mathfrak{m}$. The k-module inclusions $k \hookrightarrow \bigoplus_{i=1}^m k[[t_i]]$ and $I \hookrightarrow \bigoplus_{i=1}^m k[[t_i]]$ induce an injective k-linear map $\psi : k \oplus I \to \bigoplus_{i=1}^m k[[t_i]]$, given by $(c, f) \mapsto \mathbf{c} + f$, where $\mathbf{c}$ represents the m-tuple $(c, \ldots, c)$. We claim $\operatorname{im}\psi$ is a subring of $\bigoplus_{i=1}^m k[[t_i]]$, by virtue of the fact that I is an ideal:

$$\begin{aligned}
\psi((c_1, f_1)) + \psi((c_2, f_2)) &= (\mathbf{c_1} + f_1) + (\mathbf{c_2} + f_2) \\
&= \mathbf{c_1} + \mathbf{c_2} + (f_1 + f_2) \\
&= \psi((c_1 + c_2, f_1 + f_2)), \\
\psi((c_1, f_1))\psi((c_2, f_2)) &= (\mathbf{c_1} + f_1)(\mathbf{c_2} + f_2) \\
&= \mathbf{c_1}\mathbf{c_2} + \mathbf{c_1} f_2 + \mathbf{c_2} f_1 + f_1 f_2 \\
&= \psi((c_1 c_2, \mathbf{c_1} f_2 + \mathbf{c_2} f_1 + f_1 f_2)).
\end{aligned}$$

Since $k \oplus I \cong \operatorname{im}\psi$ as k-modules, we will, by abuse of notation, continue to refer to $k \oplus I$ as a ring. To show that $k \oplus I$ is moreover a k-subalgebra, there is a natural module map (compatible as a ring map) $\iota : k \to k \oplus I$ given by $c \mapsto \psi((c, 0)) = \mathbf{c}$, so therefore the following diagram commutes:

$$\begin{array}{ccc}
k & \xrightarrow{\ \iota\ } & k \oplus I \\
& \searrow & \downarrow \\
& & \displaystyle\bigoplus_{i=1}^{m} k[[t_i]].
\end{array}$$

Note further that $k \oplus I$ has a unique maximal ideal, namely I, so it must be a glued subalgebra.

To compute the codimension of $k \oplus I$, observe $k \oplus I \subset S$, so

$$\dim_k \left(\bigoplus_{i=1}^m k[[t_i]] \right) \Big/ (k \oplus I) = \dim_k \left(\bigoplus_{i=1}^m k[[t_i]] \right) \Big/ S + \dim_k S/(k \oplus I) = \delta + \dim_k S/(k \oplus I).$$

Since S is local, $S \cong k \oplus \mathfrak{m}$, and $\dim_k S/I = \tau \geq 1$ by assumption. This forces $I \subset \mathfrak{m}$, so

$$\dim_k S/(k \oplus I) = \dim_k (k \oplus \mathfrak{m})/(k \oplus I) = \dim_k \mathfrak{m}/I = \tau - 1.$$

Therefore $k \oplus I$ has codimension $\delta + \tau - 1$ in $\bigoplus_{i=1}^m k[[t_i]]$. ∎

Theorem 4.2. *For $m \geq 1$ and $\delta \geq m - 1$, $\operatorname{ter}^{\delta}_{\mathcal{G}}(m)$ is connected.*

Proof. We already know $\operatorname{ter}^{\delta}(1)$ is connected for all δ. So we will inductively assume $\operatorname{ter}^{\delta}_{\mathcal{G}}(m-1)$ is connected for all δ, and use this to show $\operatorname{ter}^{\delta}_{\mathcal{G}}(m)$ is connected for all δ. If we look at any nonempty Isom-Hilb component $\mathfrak{IH}_m(\delta_1, \delta_2, \tau) \subset \operatorname{ter}^{\delta}_{\mathcal{G}}(m)$ with $\tau \geq 2$, then Corollary 3.16 tells us that its closure lies in $\mathcal{G}_{\{1,\ldots,\tau-1\}}$. Hence we obtain connected "towers" in which Isom-Hilb components are stacked by gluing dimension. All that needs to be checked is that the "foundation" of these towers is connected. In general, $\mathcal{G}_{\{1\}}$ will be disconnected (see Examples 5.1, 5.2, 5.3). Instead we prove $\mathcal{G}_{\{1,2\}}$ forms the necessary connected foundation for the towers.

Take any Isom-Hilb component $\mathfrak{IH}_m(\delta_1, \delta_2, 2)$ in $\mathcal{G}_{\{2\}}$, so that $\delta_1 + \delta_2 + 2 = \delta$, and take an arbitrary glued subalgebra $S = (S_1, I_1, S_2, I_2, \varphi)$ in this Isom-Hilb component. As a k-vector space, S looks like

$$k\{(\mathbf{1}, 1), (v_1, cv_2), (I_1, 0), (\mathbf{0}, I_2)\}$$

where $c \in k^*$ comes from $\mathrm{Aut}_k(k[x]/(x^2)) \cong \mathbb{A}^1 \setminus \{0\}$, $v_i \in S_i \setminus I_i$, and $\mathbf{0}$ and $\mathbf{1}$ represent the $(m-1)$-tuples $(0, \ldots, 0)$ and $(1, \ldots, 1)$, respectively. If we vary c, we get the $(\mathbb{A}^1 \setminus \{0\})$-fiber of the Isom-Hilb component over the point $((S_1, I_1), (S_2, I_2)) \in \mathfrak{Hilb}_{m-1}(\delta_1, 2) \times \mathfrak{Hilb}_1(\delta_2, 2)$. Taking the closure of this fiber means that there are glued subalgebras obtained at $c = 0, \infty$. When $c = 0$, we get the k-vector space

$$k\{(\mathbf{1}, 1), (v_1, 0)(I_1, 0), (\mathbf{0}, I_2)\}.$$

Observe $\dim_k S_1/I_1 = 2$ and $v_1 \notin I_1$, so $\dim_k S_1/((v_1)+I_1) = 1$, which forces $(v_1)+I_1$ to be the unique maximal ideal $\mathfrak{m}_{S_1}$ of S_1. Furthermore, $k \oplus I_2 \in \mathrm{ter}_{\mathcal{G}}^{\delta_2+1}(1)$ by Lemma 4.1. Therefore, this yields a glued subalgebra with the data $(S_1, \mathfrak{m}_{S_1}, k \oplus I_2, I_2, \mathrm{id}_k)$, and this is a closed point in $\mathfrak{IH}_m(\delta_1, \delta_2+1, 1)$. On the other hand, when $c = \infty$, we get the k-vector space

$$k\{(\mathbf{1}, 1), (\mathbf{0}, v_2), (I_1, 0), (\mathbf{0}, I_2)\}.$$

As before, $(v_2)+I_2$ must be the maximal ideal $\mathfrak{m}_{S_2}$ of S_2, and $k \oplus I_1 \in \mathrm{ter}^{\delta_1+1}(m-1)$. Therefore, this yields a glued subalgebra with the data $(k \oplus I_1, I_1, S_2, \mathfrak{m}_{S_2}, \mathrm{id}_k)$, and this is a closed point in $\mathfrak{IH}_m(\delta_1 + 1, \delta_2, 1)$. Thus any two Isom-Hilb components of the form $\mathfrak{IH}_m(\delta_1, \delta_2+1, 1)$ and $\mathfrak{IH}_m(\delta_1+1, \delta_2, 1)$ are connected via $\mathfrak{IH}_m(\delta_1, \delta_2, 2)$. ■

Remark 4.3. Alternatively, one can verify connectedness using Ishii's results (as pointed out by Sebastian Bozlee). For any Artinian k-algebra A and $\delta \in \mathbb{N}$, the contravariant functor $\mathbf{F}_A^\delta$ from the category of Noetherian k-schemes to the category of sets given by

$$\mathbf{F}_A^\delta(X) = \left\{ \begin{array}{c} \mathscr{O}_X\text{-subalgebras } \mathscr{S} \subset A \otimes_k \mathscr{O}_X \text{ such that} \\ (A \otimes_k \mathscr{O}_X)/\mathscr{S} \text{ is a locally free } \mathscr{O}_X\text{-module of rank } \delta \end{array} \right\}$$

is representable by a projective k-scheme $\mathrm{Ter}^\delta(A)$ [Ish80, Theorem 1]. If $\dim_k A = r$, then $\mathbf{F}_A^\delta$ is a subfunctor of the Grassmannian functor, so that $\mathrm{Ter}^\delta(A)$ is a closed subscheme of $G(r-\delta, r)$. This scheme with its reduced structure is given the notation $\mathrm{ter}^\delta(A)$. If A is local, then $\mathrm{ter}^\delta(A)$ is connected [Ish80, Corollary 2]. Now by Proposition 2.10, and by virtue of the fact that $\ast^m$ is a $(m-1)$-codimensional k-subalgebra of $\bigoplus_{i=1}^m k[[t_i]]/(t_i^{2\delta})$, it is easy to see that $\mathrm{ter}_{\mathcal{G}}^\delta(m)$ and $\mathrm{ter}^{\delta-m+1}(\ast^m)$ are isomorphic as schemes. The latter is connected since $\ast^m$ is local, thus so is the glued territory.

It is important to note that the Isom-Hilb components do not correspond to irreducible components of the glued territory. To see this, note that the connectedness argument above shows that the Isom-Hilb components in $\mathcal{G}_{\{1\}}$ lie in the closure of $\mathcal{G}_{\{2\}}$. Moreover, an Isom-Hilb component itself might not be irreducible.

Proposition 4.4. *For $m \geq 1$, $\mathrm{ter}_{\mathcal{G}}^m(m) \cong \mathbb{P}^{m-1}$.*

Proof. By Remark 4.3, $\mathrm{ter}_{\mathcal{G}}^m(m) \cong \mathrm{ter}^1(\ast^m) \subset G(m-1, m)$. Any $(m-1)$-dimensional k-subspace of $k\{t_1, \ldots, t_m\}$ is closed under multiplication, so in fact $\mathrm{ter}^1(\ast^m) = G(m-1, m) \cong \mathbb{P}^{m-1}$. ■

Conjecture. *For $m \geq 2$, $\mathrm{ter}_{\mathcal{G}}^\delta(m)$ is irreducible if and only if $\delta = m-1$ or $\delta = m$.*

We already know $\mathrm{ter}_{\mathcal{G}}^{m-1}(m) = \{pt\}$ and $\mathrm{ter}_{\mathcal{G}}^m(m) \cong \mathbb{P}^{m-1}$, so these are clearly irreducible. For $\delta \geq m+1$, we would ideally want to argue that the glued territory $\mathrm{ter}_{\mathcal{G}}^\delta(m)$ can be expressed as the union of two groups of Isom-Hilb components which do not entirely lie in the closure of the other. This approach comes with obstructions, however, due to the unpredictable behavior of the fibers of the Isom-Hilb components discussed in Section 3. Determining if a point in one Isom-Hilb component lies in the closure of another would require some confirmation about the automorphism groups of the finite-dimensional k-algebras, and there is no guarantee as to which k-algebras will actually appear in the gluable pairs (or even if the Isom-Hilb component is nonempty).

Observe that, if the above conjecture were true, then this would also confirm that $\mathrm{ter}_{\mathcal{G}}^{m-1}(m)$ and $\mathrm{ter}_{\mathcal{G}}^m(m)$ are the only smooth glued territories. For $\mathrm{ter}_{\mathcal{G}}^\delta(m)$ with $\delta \geq m+1$, being connected and reducible would imply there is some singular locus.

4.2. **Dimension Bounds.** It is difficult to pinpoint the precise dimension of the glued territories without explicitly computing them, even in the unibranch case. For low values of δ, it appears that the dimension of $\mathrm{ter}^\delta(1)$ is $\delta - 1$ (see Examples 1.9, 1.10, 1.11). However, this pattern quickly fails for $\delta \geq 6$.

Proposition 4.5. *There exists a semigroup stratum $T_\Gamma \subset \mathrm{ter}^\delta(1)$ of dimension strictly greater than $\delta - 1$ when $\delta \geq 6$.*

Proof. For $n \geq 2 \in \mathbb{N}$, consider the numerical semigroup $\langle n, n+1 \rangle$, which has as its complement

$$\mathbb{N} \setminus \langle n, n+1 \rangle = \{1, \ldots, n-1, n+2, \ldots, 2n-1, 2n+3, \ldots, 3n-1, \ldots\}.$$

Hence the genus of $\langle n, n+1 \rangle$ is

$$\delta = (n-1) + (n-2) + \cdots + 2 + 1 = \frac{n(n-1)}{2}.$$

All but the first $n-1$ gaps are greater than n and greater than $n+1$, so by Proposition 1.6, there is a closed immersion of $T_{\langle n, n+1 \rangle}$ into $\mathbb{A}^N$, where

$$N = 2((n-2) + (n-3) + \cdots + 1) = (n-1)(n-2).$$

There are no relations between n and $n+1$ which are smaller than the conductor of $\frac{1}{2}n(n-1)$, so this closed immersion is in fact an isomorphism. Observe that $N > \delta - 1$ when $n \geq 4$, which implies $\delta \geq 6$. ■

Hence at best we can guarantee that the dimension of $\mathrm{ter}^\delta(1)$ is bounded below by $\delta - 1$. An upper bound on the dimension of $\mathrm{ter}^\delta(1)$ can be obtained using the embedding dimension of numerical semigroups. The *embedding dimension* of a numerical semigroup $\Gamma = \langle n_1, \ldots, n_s \rangle \subset \mathbb{N}$ is the quantity $e(\Gamma) = s$, i.e. the number of elements in its minimal generating set. As the name suggests, if P is a point on a curve C, and its complete local ring $\widehat{\mathscr{O}}_{C,P}$ is a k-subalgebra of $k[[t]]$ with numerical semigroup Γ, then

$$e(\Gamma) = \dim_k \mathbb{T}_{C,P} = \dim_k \mathfrak{m}_P / \mathfrak{m}_P^2.$$

The smallest generator n_1 is the *multiplicity* $m(\Gamma)$ of Γ. It is always the case that $e(\Gamma) \leq m(\Gamma)$ [GR09, Proposition 2.10]. Clearly the maximum possible multiplicity that can be obtained by a numerical semigroup of genus δ is $\delta + 1$. Hence if $g(\Gamma) = \delta$, then $e(\Gamma) \leq \delta + 1$.

From Remark 4.3, $\mathrm{ter}^\delta(1)$ is the reduced scheme structure of $\mathrm{Ter}^\delta(k[[t]]/(t^{2\delta}))$. For any geometric point $S \in \mathrm{Ter}^\delta(k[[t]]/(t^{2\delta}))$, the tangent space $\mathbb{T}_{\mathrm{Ter}^\delta(k[[t]]/(t^{2\delta})),S}$ at S is isomorphic to $\mathrm{Der}_k(S, k[[t]]/S)$ [Ish80, Proposition 1]. Hence we examine these k-derivations. Recall that, for a fixed δ, the maximum degree subalgebra refers to the k-subalgebra $k[[t^{\delta+1}, \ldots, t^{2\delta+1}]]$.

Proposition 4.6. *Take $S \in \mathrm{ter}^\delta(1)$, so that S is local with maximal ideal $\mathfrak{m}$. The space of k-derivations $\mathrm{Der}_k(S, k[[t]]/S)$ can be identified with a subspace of $\mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k[[t]]/S)$. In particular, if S is the maximum degree subalgebra, then $\mathrm{Der}_k(S, k[[t]]/S) \cong \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k[[t]]/S)$.*

Proof. If $\partial : S \to k[[t]]/S$ is a k-derivation, then $\partial(\mathbf{c}) = 0$ for all $\mathbf{c} = (c, \ldots, c) \in S \setminus \mathfrak{m}$, and the Leibniz rule must be satisfied, i.e. $\partial(fg) = f\partial(g) + g\partial(f)$ for any $f, g \in S$. Hence ∂ is entirely determined by the image of $\mathfrak{m}/\mathfrak{m}^2$.

If S is the maximum degree subalgebra, then as k-vector spaces, $S = k\{1, t^{\delta+1}, \ldots, t^{2\delta-1}\}$ and $k[[t]]/S = k\{t, t^2, \ldots, t^\delta\}$. If $f, g \in \mathfrak{m}/\mathfrak{m}^2$, then $\partial(fg) = f\partial(g) + g\partial(f)$ is trivial, as both sides equal zero. The left hand side is zero due to the fact that $fg = 0$ in S. For the right hand side, note that $\partial(g) = a_1 t + \cdots + a_\delta t^\delta$ and $f = b_1 t^{\delta+1} + \cdots + b_{\delta-1} t^{2\delta-1}$, so their product is already annihilated in $k[[t]]/S$, and the same applies to the product $g\partial(f)$. Hence the Leibniz rule does not yield any additional restrictions, so in fact all k-linear maps $\mathfrak{m}/\mathfrak{m}^2 \to k[[t]]/S$ are k-derivations. ■

Proposition 4.7. *For any δ, $\dim \mathrm{ter}^\delta(1) \leq \delta(\delta+1)$.*

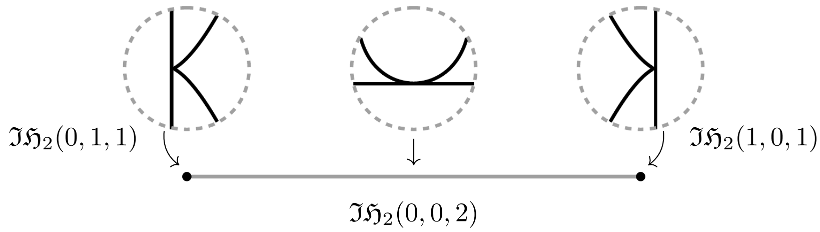

FIGURE 1. $\mathrm{ter}^2_{\mathcal{G}}(2)$

Proof. We have the following inequalities:

$$\dim \mathrm{ter}^\delta(1) \leq \max_{S\in \mathrm{ter}^\delta(1)} \mathbb{T}_{\mathrm{ter}^\delta(1),S} \leq \max_{S\in \mathrm{Ter}^\delta(k[[t]]/(t^{2\delta}))} \mathbb{T}_{\mathrm{Ter}^\delta(k[[t]]/(t^{2\delta})),S}$$

For any S with maximal ideal $\mathfrak{m}$, $\mathbb{T}_{\mathrm{Ter}^\delta(k[[t]]/(t^{2\delta})),S} \cong \mathrm{Der}_k(S, k[[t]]/S)$, which we showed is isomorphic to a subspace of $\mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k[[t]]/S)$. But $\dim_k \mathfrak{m}/\mathfrak{m}^2 = e(\Gamma_S)$, where Γ_S is the numerical semigroup of S, and the maximum possible embedding dimension of Γ_S is $\delta+1$. Hence $\dim_k \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k[[t]]/S) \leq \delta(\delta+1)$. ■

We now present some dimension bounds for the multibranch glued territories.

Proposition 4.8. *For $m \geq 2$ and $\delta \geq m$,*

$$\delta - 1 \leq \dim \mathrm{ter}^\delta_{\mathcal{G}}(m) \leq (\delta-1)(2\delta m - m - \delta + 1).$$

Proof. For the lower bound, consider the Isom-Hilb component $\mathfrak{IH}_m(m-2, \delta-m, 2)$. We know all fibers are isomorphic to $\mathbb{A}^1 \setminus \{0\}$. We also know $\mathrm{ter}^{m-2}_{\mathcal{G}}(m-1) = \{\ast^{m-1}\}$, and $\mathrm{Hilb}(\ast^{m-1}, 2)$ is parametrized by $\mathbb{P}^{m-2}$ by Example 3.9. Hence $\mathfrak{Hilb}_{m-1}(m-2, 2)$ is also parametrized by $\mathbb{P}^{m-2}$. On the other hand, in $\mathrm{ter}^{\delta-m}(1)$, the semigroup stratum $T_{\langle \delta-m+1,\ldots,2\delta-2m+1\rangle}$ consists of the unique k-subalgebra $S = k[[t^{\delta-m+1}, \ldots, t^{2\delta-2m+1}]]$, and again by Example 3.9, $\mathrm{Hilb}(S, 2)$ is parametrized by $\mathbb{P}^{\delta-m}$. So the subset $Y \subset \mathfrak{IH}_m(m-2, \delta-m, 2)$ of glued subalgebras with the Goursat data $(\ast^{m-1}, I_1, S, I_2, \varphi)$ is parametrized by $\mathbb{P}^{m-2} \times \mathbb{P}^{\delta-m} \times (\mathbb{A}^1 \setminus \{0\})$. Thus

$$\dim \mathrm{ter}^\delta_{\mathcal{G}}(m) \geq \dim \mathfrak{IH}_m(m-2, \delta-m, 2) \geq \dim Y \geq (m-2) + (\delta-m) + 1 = \delta - 1.$$

The upper bound comes from the fact that $\mathrm{ter}^\delta_{\mathcal{G}}(m)$ is a closed subscheme of the sub-Grassmannian $G(2\delta m - m - \delta + 1, 2\delta m - m)$. ■

5. EXAMPLES

Example 5.1. $\mathrm{ter}^2_{\mathcal{G}}(2)$. By Proposition 4.4, this should be isomorphic to $\mathbb{P}^1$. We will confirm this with the procedure outlined at the end of Section 3. There are three sufficient triples in $\mathrm{st}(2,2)$, namely $(0,1,1), (1,0,1), (0,2,2)$.

For $\mathfrak{IH}_2(0,1,1)$, note that $\mathrm{ter}^0(1) = \{k[[t]]\}$ and $\mathrm{ter}^1(1) = \{k[[t^2, t^3]]\}$. Since these are both local, $\mathrm{Hilb}(k[[t]], 1) = \{(t)\}$ and $\mathrm{Hilb}(k[[t^2, t^3]], 1) = \{(t^2, t^3)\}$. Hence $\mathfrak{Hilb}_1(0,1) \times \mathfrak{Hilb}_1(1,1)$ is parametrized by a point. As discussed in Example 3.11, this point clearly forms a gluable pair, and its fiber in the Isom-Hilb component is a point, so $\mathfrak{IH}_2(0,1,1)$ consists of the unique glued subalgebra given by the Goursat data $(k[[t_1]], (t_1), k[[t_2^2, t_2^3]], (t_2^2, t_2^3), \mathrm{id}_k)$. Equivalently, this is the glued subalgebra

$$k[[(t_1, 0), (0, t_2^2), (0, t_2^3)]].$$

For $\mathfrak{IH}_2(1,0,1)$, there is a similar construction to the one above, and we end up with a unique glued subalgebra with the Goursat data $(k[[t_1^2, t_1^3]], (t_1^2, t_1^3), k[[t_2]], (t_2), \mathrm{id}_k)$, or equivalently

$$k[[(t_1^2, 0), (t_1^3, 0), (0, t_2)]].$$

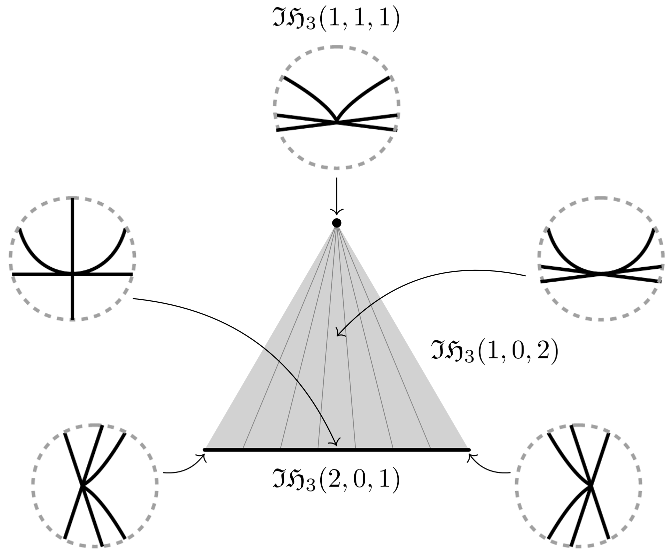

FIGURE 2. $\mathrm{ter}^3_{\mathcal{G}}(3)$

For $\mathfrak{IH}_2(0,0,2)$, note that $\mathrm{ter}^0(1) = \{k[[t]]\}$, and $\mathrm{Hilb}(k[[t]],2) = \{(t^2)\}$, so $\mathfrak{Hilb}_1(0,2) \times \mathfrak{Hilb}_1(0,2)$ is parametrized by a point. As discussed in Example 3.12, this point forms a gluable pair, and its fiber in the Isom-Hilb component is isomorphic to $\mathbb{A}^1 \setminus \{0\}$, where $c \in k^*$ corresponds to the automorphism $\varphi_c : k[x]/(x^2) \to k[x]/(x^2)$ given by $x \mapsto cx$. Thus $\mathfrak{IH}_2(0,0,2)$ is parametrized by $\mathbb{A}^1 \setminus \{0\}$, where for each $c \in k^*$, we obtain the Goursat data $(k[[t_1]],(t_1),k[[t_2]],(t_2),\varphi_c)$, or equivalently the glued subalgebra

$$k[[(t_1, ct_2),(t_1^2,0),(0,t_2^2)]].$$

Note that when $c = 0$, the glued subalgebra of $\mathfrak{IH}_2(0,1,1)$ is obtained, and when $c = \infty$, the glued subalgebra of $\mathfrak{IH}_2(1,0,1)$ is obtained.

With respect to the basis $\{(t_1,0),(t_1^2,0),(0,t_2),(0,t_2^2)\}$, it is clear that in computing Plücker coordinates for each of these k-subalgebras, we obtain $\mathbb{P}^1$. See Figure 1.

Example 5.2. $\mathrm{ter}^3_{\mathcal{G}}(3)$. Again by Proposition 4.4, this should be isomorphic to $\mathbb{P}^2$. There are three sufficient triples: $(1,1,1),(2,0,1),(1,0,2)$.

For $\mathfrak{IH}_3(1,1,1)$, note that $\mathrm{ter}^1_{\mathcal{G}}(2) = \{*^2\}$, with $\mathrm{Hilb}(*^2,1) = \{((t_1,0),(0,t_2))\}$, and $\mathrm{ter}^1(1) = \{k[[t^2,t^3]]\}$ with $\mathrm{Hilb}(k[[t^2,t^3]],1) = \{(t^2,t^3)\}$. Hence $\mathfrak{Hilb}_2(1,1) \times \mathfrak{Hilb}_1(1,1)$ is parametrized by a point. By Example 3.11, $\mathfrak{IH}_3(1,1,1)$ is parametrized by a point, and this point corresponds to the glued subalgebra

$$k[[(t_1,0,0),(0,t_2,0),(0,0,t_3^2),(0,0,t_3^3)]].$$

For $\mathfrak{IH}_3(2,0,1)$, note that $\mathrm{ter}^2_{\mathcal{G}}(2) \cong \mathbb{P}^1$ by the above example, and for each $S \in \mathrm{ter}^2_{\mathcal{G}}(2)$, $\mathrm{Hilb}(S,1) = \{\mathfrak{m}_S\}$. Hence $\mathfrak{Hilb}_2(2,1) \times \mathfrak{Hilb}_1(0,1)$ is parametrized by $\mathbb{P}^1$, with pairs given by $((S,\mathfrak{m}_S),(k[[t]],(t)))$. Again by Example 3.11, $\mathfrak{IH}_3(2,0,1)$ is parametrized by $\mathbb{P}^1$, with the corresponding glued subalgebras being

$$\left\{ \begin{array}{c} k[[(t_1,0,0),(0,t_2^2,0),(0,t_2^3,0),(0,0,t_3)]], k[[(t_1^2,0,0),(t_1^3,0,0),(0,t_2,0),(0,0,t_3)]], \\ k[[(t_1,at_2,0),(t_1^2,0,0),(0,t_2^2,0),(0,0,t_3)]] : a \in k^* \end{array} \right\}.$$

For $\mathfrak{IH}_3(1,0,2)$, note that $\mathrm{ter}^1_{\mathcal{G}}(2) = \{*^2\}$, with $\mathrm{Hilb}(*^2,2)$ parametrized by $\mathbb{P}^1$ by Example 3.9. These ideals are $((t_1,0)) + \mathfrak{m}^2$, $((0,t_2)) + \mathfrak{m}^2$, and $((t_1,bt_2)) + \mathfrak{m}^2$ for each $b \in k^*$. Hence $\mathfrak{Hilb}_2(1,2) \times \mathfrak{Hilb}_1(0,2)$ is parametrized by $\mathbb{P}^1$, and thus $\mathfrak{IH}_3(1,0,2)$ is parametrized by $\mathbb{P}^1 \times (\mathbb{A}^1 \setminus \{0\})$, with the

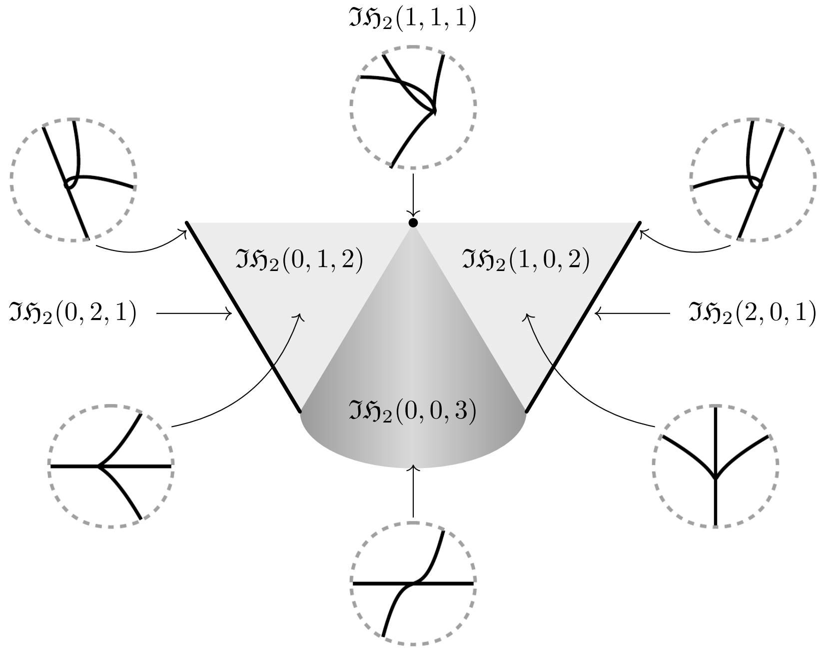

FIGURE 3. $\mathrm{ter}^3_{\mathcal{G}}(2)$

glued subalgebras being

$$\left\{\begin{array}{c} k[[(t_1,0,0),(0,t_2,ct_3),(0,t_2^2,0),(0,0,t_3^2)]], k[[(t_1,0,ct_3),(0,t_2,0),(t_1^2,0,0),(0,0,t_3^2)]], \\ k[[(t_1,bt_2,0),(t_1,0,ct_3),(t_1^2,0,0),(0,t_2^2,0),(0,0,t_3^2)]] : b,c \in k^* \end{array}\right\}.$$

Note that when $c=0$ in each of these fibers, the resulting glued subalgebra is the one in $\mathfrak{I}\mathfrak{H}_3(1,1,1)$, and when $c=\infty$, the resulting glued subalgebras lie in $\mathfrak{I}\mathfrak{H}_3(2,0,1)$. Computing Plücker coordinates, we confirm that these Isom-Hilb components form a $\mathbb{P}^2$. See Figure 2.

Example 5.3. $\mathrm{ter}^3_{\mathcal{G}}(2)$. There are six sufficient triples in $\mathrm{st}(3,2)$. We begin with the Isom-Hilb components with $\tau=1$.

A parametrization for $\mathfrak{I}\mathfrak{H}_2(0,2,1)$ is given by $\mathbb{P}^1$, with the glued subalgebras

$$\{k[[(t_1,0),(0,t_2^3),(0,t_2^4),(0,t_2^5)]], k[[(t_1,0),(0,t_2^2+at_2^3),(0,t_2^5)]] : a \in k\}$$

and similarly $\mathfrak{I}\mathfrak{H}_2(2,0,1)$ is parametrized by $\mathbb{P}^1$, with the glued subalgebras

$$\{k[[(t_1^3,0),(t_1^4,0),(t_1^5,0),(0,t_2)]], k[[(t_1^2+at_1^3,0),(t_1^5,0),(0,t_2)]] : a \in k\}.$$

A parametrization for $\mathfrak{I}\mathfrak{H}_2(1,1,1)$ is a point, with the unique glued subalgebra being

$$k[[(t_1^2,0),(t_1^3,0),(0,t_2^2),(0,t_2^3)]].$$

Next are the Isom-Hilb components with $\tau=2$. For $\mathfrak{I}\mathfrak{H}_2(0,1,2)$, we know $\mathfrak{Hilb}_1(0,2)$ is parametrized by a point, and $\mathrm{ter}^1(1)=\{k[[t^2,t^3]]\}$, with $\mathrm{Hilb}(k[[t^2,t^3]],2)$ parametrized by $\mathbb{A}^1 \sqcup \{pt\}$ by Example 3.8. Hence $\mathfrak{Hilb}_1(0,2)\times\mathfrak{Hilb}_1(1,2)$ is parametrized by $\mathbb{A}^1 \sqcup \{pt\}$, so $\mathfrak{I}\mathfrak{H}_2(0,1,2)$ is parametrized by $(\mathbb{A}^1 \sqcup \{pt\}) \times (\mathbb{A}^1 \setminus \{0\})$. The glued subalgebras are

$$\{k[[(t_1,ct_2^3),(t_1^2,0),(t_1^3,0),(0,t_2^2+at_2^3)]], k[[(t_1,ct_2^2),(0,t_2^3),(0,t_2^4)]] : a \in k, c \in k^*\}.$$

Note that when $c=0$ in each fiber, the glued subalgebras in $\mathfrak{I}\mathfrak{H}_2(0,2,1)$ are obtained, and when $c=\infty$, the glued subalgebra in $\mathfrak{I}\mathfrak{H}_2(1,1,1)$ is obtained.

A symmetric construction shows that $\mathfrak{IH}_1(1,0,2)$ is parametrized by $(\mathbb{A}^1 \sqcup \{pt\}) \times (\mathbb{A}^1 \setminus \{0\})$, with the glued subalgebras being

$$\{k[[(t_1^3, ct_2), (t_1^2 + at_1^3, 0), (0, t_2^2), (0, t_2^3)]], k[[(t_1^2, ct_2), (t_1^3, 0), (t_1^4, 0)]] : a \in k, c \in k^*\}.$$

When $c = 0$ in each fiber, the glued subalgebra in $\mathfrak{IH}_2(1,1,1)$ is obtained, and when $c = \infty$, the glued subalgebras in $\mathfrak{IH}_2(2,0,1)$ are obtained.

Lastly we compute $\mathfrak{IH}_2(0,0,3)$. Note that $\mathfrak{Hilb}_1(0,3) \times \mathfrak{Hilb}_1(0,3)$ is parametrized by a point, given by the pair $((k[[t]], (t^3)), (k[[t]], (t^3)))$. Clearly this is a gluable pair, and its fiber in the Isom-Hilb component is $\mathrm{Aut}_k(k[x]/(x^3)) \cong (\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^1$. For each $u \in k^*$ and $v \in k$, the automorphism $\varphi_{u,v}$ is given by $x \mapsto ux + vx^2$. Hence $\mathfrak{IH}_2(0,0,3)$ is parametrized by $(\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^1$, with the glued subalgebras being

$$\{k[[(t_1, ut_2 + vt_2^2), (0, t_2^3), (0, t_2^4), (0, t_2^5)]] : u \in k^*, v \in k\}.$$

We now compute the Plücker coordinates of all of these glued subalgebras when viewed as points in $G(8,10)$, with respect to the basis

$$\{(t_1, 0), (0, t_2), (t_1^2, 0), (0, t_2^2), (t_1^3, 0), (0, t_2^3), (t_1^4, 0), (0, t_2^4), (t_1^5, 0), (0, t_2^5)\}.$$

We can easily check that the Isom-Hilb components of gluing dimension 1 and 2 form two copies of $\mathbb{P}^2$ intersecting at a point, namely the unique point in $\mathfrak{IH}_2(1,1,1)$. Each glued subalgebra in $\mathfrak{IH}_2(0,0,3)$ can be viewed as the k-vector space

$$k\{(t_1, ut_2 + vt_2^2), (t_1^2, u^2t_2^2), (t_1^3, 0), (0, t_2^3), (t_1^4, 0), (0, t_2^4), (t_1^5, 0), (0, t_2^5)\}$$

so by finding the maximal minors of the matrix

$$\begin{bmatrix} 1 & u & 0 & v & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & u^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

we find that the Plücker coordinates of the points in $\mathfrak{IH}_2(0,0,3)$ form the affine patch $x_0 \neq 0$ on the projective quadric surface $Q = V(x_0x_2 - x_1^2, x_0x_3 - x_1x_2) \subset \mathbb{P}^4$. Thus

$$\mathrm{ter}_{\mathcal{G}}^3(2) \cong \mathbb{P}^2 \cup \mathbb{P}^2 \cup Q$$

where $\mathbb{P}^2 \cap Q \cong \mathbb{P}^1$ for each copy of $\mathbb{P}^2$, and the intersection of all three of these components is a point. See Figure 3.

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