The following results and definitions are primarily drawn from What are zeta functions of graphs and what are they good for? by Matthew Horton, Harold Stark, and Audrey Terras and Zeta Functions and Chaos by Terras.

For all the content that follows, assume that \( G \) is a connected, finite graph with no vertices of degree less than two and that \( G \) is not a cycle. Further, we suppose that the rank of the fundamental group of \( G \) is greater than one. We’ll see that finiteness and connectedness are the key assumptions. The other assumptions keep the graphs sufficiently interesting. The condition on degree is so we may ignore graphs which have trees hanging off of them.

Beginning with a graph \( G \), we arbitrarily give orientations to each edge in \( E(G) \). We then add an additional \( |E| \) edges to \( E(G) \) such that for each \( e_i \in E(G) \), \( e_{i+|E|} = e_i^{-1} \), i.e. \( e_{i+|E|} \) has the opposite orientation as \( e_i \).

**Figure 1.** Undirected \( G \) (left), \( G \) with two directed edges for every original edge (right).

**Definition 1.** A path \( a_1a_2...a_{m-1}a_m \), where each \( a_i \in E(G) \), has a backtrack if for some \( i \), \( a_i = a_{i+1}^{-1} \). That is, the path traverses some edge \( a \) and then immediately traverses \( a^{-1} \).
Definition 2. A path $a_1a_2...a_{m-1}a_m$, where each $a_i \in E(G)$, has a tail if $a_m = a_1^{-1}$.

Definition 3. A cycle $P = a_1a_2...a_{m-1}a_m$ is prime if it is has no backtracks, no tail, and is primitive, i.e. it cannot be expressed as $C^s$ for some cycle $C$.

Definition 4. For a path $P = a_1a_2...a_{m-1}a_m$, let $l(P) = l(a_1a_2...a_{m-1}a_m) = m$. That is $l(P)$ is the length of a path $P$.

Definition 5. The equivalence class $[C]$ of a cycle $C = a_1,...,a_m$ is the set $[C] = \{a_1a_2...a_m, a_2a_3...a_ma_1,...,a_ma_1...a_{m-1}\}$.

We now have all the necessary definitions to define the Ihara Zeta Function on a graph $G$.

Definition 6. The Ihara Zeta Function of a graph $G$ is defined as the following product over all classes of primes $[P]$ in $G$:

$$\zeta(u, G) = \prod_{[P]} (1 - u^{l(P)})^{-1}$$

for $u \in \mathbb{C}$ such that $|u|$ is sufficiently small.

This zeta function was originally defined by Yasutaka Ihara in the 1960s, and was initially used to study questions about $p$-adic groups rather than graph theory. It was at the suggestion of J.P. Serre that the Ihara zeta function would be reinterpreted in the context of graph theory.

Proposition 7. As assumed, if $G$ is a finite, connected graph with $\text{rank}(\pi_1(G)) > 1$, then $G$ (after adding edges as described) has infinitely many prime cycles.
Proof. (Sketch). Suppose rank($\pi_1(G)$) = $r > 1$. Let $T$ be a maximal spanning tree of $G$. By contracting $T$ down to some vertex $v_0 \in V(G)$, we have a homotopy equivalence between $G$ and $G/T$. Since $G$ (and $G/T$) are path connected, $\pi_1(G) \cong \pi_1(G/T)$. Further, $G \sim G/T \sim \bigvee_{i=1}^{r} S^1$, the bouquet of $r$ circles meeting at $v_0$ (this result is given in detail in [1] Hatcher, Chapter 1.A).

Each of the paths in $G$ given by a copy of the circle in $G/T$ gives a prime cycle. So $G$ has a least $r$ prime cycles. In $G$, there is a path, $Q$, (possibly empty) connecting two of these prime cycles, say $P_1$ and $P_2$. Let the endpoints of $Q$ be $v_1$ and $v_2$, and say $v_1$ is on $P_1$ and $v_2$ is on $P_2$. The path $P$, starting at $v_1$, given by $P = P_1QP_2^nQ^{-1}$, where $Q$ stands for traversing $Q$ from $v_1$ to $v_2$ and $Q^{-1}$ stands for traversing $Q$ from $v_2$ to $v_1$, is prime for all $n > 0$. Hence there are infinitely many primes, as needed.

Definition 8. Let $N_m = N_m(G)$ be the number of closed paths of length $m$ in $G$ with no backtracks or tails.

Definition 9. Let $R_G$ denote the radius of convergence of $\zeta(u, G)$.
The following computation gives some insight into how one studies the radius of convergence for $\zeta(u, G)$. Many of the elementary results in this subject require computations of a similar flavor.

**Proposition 10.** Given some $G$ we can write $\log \zeta(u, G)$ as follows:

$$\log \zeta(u, G) = \sum_{m \geq 1} \frac{N_m}{m} u^m.$$ 

**Proof.** Expanding and leveraging properties of log we find that

$$\log \zeta(u, G) = \log \left( \prod_{[P]} (1 - u^{l(P)})^{-1} \right) = -\sum_{[P]} \log(1 - u^{l(P)}).$$

From here, we leverage the fact $\log(1 - x) = \sum_{k \geq 1} \frac{1}{k} x^k$ (which converges when $|x| < 1$), and change from consider path classes to considering paths:

$$\log \zeta(u, G) = \sum_{[P]} \sum_{k \geq 1} \frac{1}{k} \frac{1}{l(P)} u^{kl(P)} \quad \text{(Since } #[P] = l(P)\text{)}$$

$$= \sum_{[P]} \sum_{k \geq 1} \frac{1}{l(P^k)} u^{l(P^k)} \quad \text{(For } k \in \mathbb{N}, l(P^k) = kl(P)\text{)}$$

$$= \sum_{C} \frac{1}{l(C)} u^{l(C)}.$$

The last sum is taken over all cycles $C$ which are closed, backtrackless, and tailless. It follows from the fact that any such cycle $C$ is a power of some prime path. It remains to show that

$$\sum_{C} \frac{1}{l(C)} u^{l(C)} = \sum_{m \geq 1} \frac{N_m}{m} u^m.$$

For each cycle $C$ in the lefthand sum, $l(C) = m$ for some $m \in \mathbb{N}$, and so the term $\frac{1}{l(C)} u^{l(C)}$ becomes $\frac{1}{m} u^m$. As the left-hand sum hits all $C$ of length $m$, the sum of all those terms for cycles of length $m$ is exactly $\frac{N_m}{m} u^m$. The result follows.

Since $G$ is known, in principle one could come up with a radius of convergence for $\log \zeta(u, G)$ given this formula. Suppose $r$ is the radius of convergence for $\log \zeta(u, G)$, what can one say about $R_G$? The complex function $e^z$ is entire, i.e. it converges for all finite input $z \in \mathbb{C}$. For any $z$ such that $|z| \leq r$, we have $\log(u, G)$ converges, so $e^{\log \zeta(u, G)} = \zeta(u, G)$ must also converge. That is, $\zeta(u, G)$ converges for all $z$ such that $|z| < r$. 

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We now turn to another zeta function for the graphs under consideration, and prove a determinant formula for it.

**Definition 11.** Let the edge matrix $W$ for $G$ be a $2|E| \times 2|E|$ matrix (where $2|E|$ is the number of directed edges we obtain from a graph $G$ with undirected edges $E$) with complex variable $w_{ab}$ in entry $a, b$, corresponding to directed edges $a$ and $b$, if the head of edge $a$ is the tail of edge $b$ and 0 otherwise.

**Definition 12.** Given an oriented path $C = a_1a_2...a_n$ in $G$, let the edge norm of $C$ be $N(C) = w_{a_1a_2}w_{a_2a_3}...w_{a_{n-1}a_n}$ where each $w_{a_ia_j}$ is the $a_i, a_j$ entry in $W$.

**Definition 13.** Given a graph $G$ with edge matrix $W$, the edge Ihara zeta function is defined as

$$\zeta_E(W, G) = \prod_{[P]} (1 - N(P))^{-1}$$

such that the entries of $W$ are sufficiently small (in norm) for convergence.

**Theorem 1.** (Determinant Formula for the Edge Zeta). Where $\zeta_E(W, G)$ converges,

$$\zeta_E(W, G) = \det(I - W)^{-1}.$$  

**Lemma 14.** Assume $||W|| < 1$, then $\log \det(I - W)^{-1} = \text{Tr} \log(I - W)^{-1}$.

*Proof. (Of Lemma).* Assume $||W|| < 1$, then

$$\log(I - W) = -\sum_{k \leq 1} \frac{1}{k} W^k.$$  

Trace is both additive and continuous, so from the above it follows that

$$\text{Tr} \log(I - W) = -\sum_{k \geq 1} \frac{1}{k} \text{Tr}(W^k).$$

Since, for a matrix $M$, $\det(e^M) = e^{\text{Tr}(M)}$, so $\det(I - W)^{-1} = e^{\text{Tr} \log(I - W)^{-1}}$. Hence,

$$\log \det(I - W)^{-1} = \text{Tr} \log(I - W)^{-1}.$$  

*Proof. (Of Theorem).* In much the same way as our radius convergence calculation, we use log and Taylor series to determine

$$- \log \zeta_E(W, G) = \sum_{[P]} \sum_{j \geq 1} \frac{1}{j} N(P)^j.$$  

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Since \([P]\) has \(l(P)\) cycles in it, we may rewrite the above as

\[
\sum_{[P]} \sum_{j \geq 1} \frac{1}{j} N(P)^j = \sum_{j \geq 1} \sum_{m \geq 1 \ l(P)=m} \frac{1}{jm} N(P)^j.
\]

Any cycle \(C\) with neither backtracks nor a tail can be expressed as a power of a prime cycle, since if \(C\) has neither a tail nor backtracks, it’s either prime or not primitive. So, the above may be expressed as a sum over all backtrackless, tailless cycles \(C\), i.e.,

\[
\sum_{m \geq 1 \ l(P)=m} \sum_{j \geq 1} \frac{1}{jm} N(P)^j = \sum_{C} \frac{1}{l(C)} N(C).
\]

Now note that a power of \(W\), say \(W^m\), will have in its \(ij\)th entry, the sum of the weights (given by entries of \(W\)) of directed paths from vertex \(i\) to vertex \(j\). Thus, \(\text{Tr}(W^m)\) is the sum of the weights of the directed cycles of length \(m\). Hence,

\[
\sum_{C} \frac{1}{l(C)} N(C) = \sum_{m \geq 1} \frac{1}{m} \text{Tr}(W^m).
\]

From the previous line and the lemma,

\[
-\log \zeta_E(W, G) = \sum_{m \geq 1} \frac{1}{m} \text{Tr}(W^m) = \log \det(I - W)^{-1},
\]

and finally,

\[
\zeta_E(W, G) = \det(I - W)^{-1}.
\]

Note that by setting \(W_1\) to be \(W\) with all nonzero entries replaced by 1, we get the following equivalence:

\[
\zeta_E(uW_1, G) = \zeta(u, G).
\]

This equivalence gives us a determinant formula for \(\zeta(u, G)\). The proof follows from the equivalence above.

**Corollary 15.** Where \(\zeta(u, G)\) converges, \(\zeta(u, G) = \det(I - uW_1)^{-1}\).

We can give one last determinant formula for the Ihara zeta function, this one due to Hyman Bass. We follow the proof given in both [2] and [3]. First, we need to define some matrices:

**Definition 16.** Let \(n\) be the number of vertices in \(G\) and let \(m\) be the number of unoriented edges of \(G\). Define the matrices \(A, J, S, T,\) and \(Q\) as follows:
(i) Let $A$ be the adjacency matrix of $G$, i.e. the $n \times n$ matrix whose $ij$th entry is 1 when vertex $i$ adjacent to vertex $j$.

(ii) Let $J$ be the $2m \times 2m$ matrix $J = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$ where $I_m$ is the $m \times m$ identity matrix.

(iii) Let $S$ be the $n \times 2m$ matrix given by $(S)_{ve} = \begin{cases} 1 \text{ if } v \text{ is the starting vertex of } e \\ 0 \text{ otherwise} \end{cases}$.

(iv) Similarly, let $T$ be the $n \times 2m$ matrix given by $(T)_{ve} = \begin{cases} 1 \text{ if } v \text{ is the terminal vertex of } e \\ 0 \text{ otherwise} \end{cases}$.

(v) Let $Q + I_n$ be the $n \times n$ diagonal matrix whose $i$th diagonal entry is the degree of the $i$th vertex of $G$.

Given the above definitions, we prove the following three propositions:

**Proposition 17.** Given the definitions above, we have the following two equivalences: $SJ = T$ and $TJ = S$.

**Proof.** First, note that the $i$th row of $J$ has a 1 in the $(i + m)$-th column. That in mind, note

$$(ST)_{ve} = \sum_{i=1}^{2m} (S)_{vi} (T)_{ie}.$$ 

The right hand side picks out the $(e + m)$-th term of row $v$ of $S$, and so is 1 when $v$ is the starting vertex of edge $e + m$, hence it is the terminal vertex of edge $e$ since edge $e$ is the inverse of edge $e + m$. That is, the right hand side is exactly $(T)_{ve}$, so $SJ = T$. The fact that $TJ = S$ follows in the same way. □

**Proposition 18.** We also have the following equalities: $A = ST^t$ and $Q + I_n = SS^t = TT^t$.

**Proof.** For the first equality, consider

$$(ST^t)_{ij} = \sum_e (S)_{ie} (T^t)_{ej}.$$ 

Each term in the sum on the right is 1 when $e$ has starting vertex $i$ and terminal vertex $j$, that is there is an edge from $i$ to $j$. Thus, $(ST^t)_{ij} = (A)_{ij}$, so $ST^t = A$.

For the second equality, consider

$$(Q + I_n)_{ij} = \sum_e (S)_{ie} (S^t)_{ej}.$$
Each term in the sum gives 1 when when \( e \) has start vertex \( i \) and also start vertex \( j \), but this is possible only when \( i = j \), summing over each edge then counts the numbers edges incident to vertex \( j \), that is it gives the degree of \( j \). The third equality follows in a similar manner.

With these last three propositions, we’re ready to prove Bass’s determinant formula for the Ihara zeta function, which is as follows:

**Theorem 2.** For \( A \) and \( Q \) as above, and \( r = \text{rank}(\pi_1(G)) \),

\[
\zeta(u, G)^{-1} = (1 - u)^{-r-1} \det(I - Au - Qu^2).
\]

**Proof.** The following is the proof given in [2] and [3]. We keep the notation for \( A, S, T, Q, W_1 \) and \( J \) as defined already. Given, the previous three propositions, one can verify that

\[
\begin{pmatrix}
I_n & 0 \\
T & I_{2m}
\end{pmatrix}
\begin{pmatrix}
I_n(1 - u^2) & Su \\
0 & I_{2m} - W_1 u
\end{pmatrix}
= \begin{pmatrix}
I_n - Au + Qu^2 & Su \\
0 & I_{2m} + Ju
\end{pmatrix}
\begin{pmatrix}
I_n & 0 \\
0 & T^t - S^t u & I_{2m}
\end{pmatrix}.
\]

Taking the determinant of both sides yeilds

\[
(1 - u^2)^n \det(I - W_1 u) = \det(I_n - Au + Qu^2) \det(I_{2m} + Ju).
\]

Note that

\[
I + Ju = \begin{pmatrix}
I & Iu \\
Iu & I
\end{pmatrix},
\]

hence,

\[
\begin{pmatrix}
I & 0 \\
-Iu & I
\end{pmatrix}(I + Ju) = \begin{pmatrix}
I & 0 \\
-Iu & I
\end{pmatrix} \begin{pmatrix}
I & Iu \\
Iu & I
\end{pmatrix} = \begin{pmatrix}
I & Iu \\
0 & I(1 - u^2)
\end{pmatrix}.
\]

So \( \det(I + Ju) = (1 - u^2)^m \). From our previous proof of the infinitude of prime cycles, note that \( r = m - n + 1 \) (for a proof of this fact, see [1]). Our previously proved determinant formula for the Ihara zeta gives us that \( \zeta(u, G) = \det(I - u W_1)^{-1} \). Thus,

\[
(1 - u^2)^n \zeta(u, G)^{-1} = \det(I_n - Au + Qu^2)(1 - u^2)^m.
\]

Finally,

\[
\zeta(u, G)^{-1} = (1 - u)^{m-n} \det(I_n - Au + Qu^2)
\]

as needed.

We’ll end by stating some connections between results for the Ihara zeta function, and a number of results from number theory. Of particular interest are functional equations, an analogue of the prime number theorem, and a version of the Riemann hypothesis for the Ihara zeta function.
good reference for the number-theoretic version of things, from a more analytic viewpoint, is [4] Davenport’s *Multiplicative Number Theory*. First we introduce the the Riemann Zeta function and the Riemann hypothesis:

**Theorem 3.** (Riemann Hypothesis). Every nontrivial zero of the Riemann zeta function,

$$
\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},
$$

has real part $\frac{1}{2}$.

*Proof.* Unknown. $\square$

**Theorem 4.** The Riemann zeta function has a functional equation:

$$
\Lambda(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{1}{2}\right)\zeta(s) = \Lambda(1 - s).
$$

Set $s = \sigma + it$. The functional equation allows the behavior of $\zeta(s)$ for $\sigma < 0$ to be studied by studying the behavior of $\zeta(s)$ for $\sigma > 1$. The functional equation was proved by Riemann when he first introduced his $\zeta$ function. For some graphs $G$, the Ihara zeta function for $G$ also satisfies a functional equation.

**Theorem 5.** If $G$ is $(q + 1)$-regular on $n$ vertices, then $\zeta(u, G)$ satisfies the following functional equation:

$$
\Lambda_G(u) = (1 - u^2)^{n-1} \zeta(1 - q^2u^2)\zeta(u, G) = (-1)^n\Lambda_G\left(\frac{1}{qu}\right).
$$

The above follows by replacing $\zeta(u, G)$ with the determinant formula. Let $u = q^{-s}$ and set $\xi(s) = \zeta(q^{-s}, G)$, observe that the functional equation then relates $\xi(s)$ and $\xi(1-s)$. In particular, it says $\Lambda_G(q^{-s}) = (-1)^n\Lambda_G(q^{-(1-s)})$ giving and reflection principle about $\sigma = \frac{1}{2}$ in a similar manner as the functional equation for the Riemann zeta function.

Lamentably, this functional equation works only in the case where $G$ is $(q + 1)$-regular. There is no known functional equation for the Ihara zeta function of irregular graphs.

The Ihara zeta function also gives rise to an analogue of the Prime Number Theorem for primes in the integers, but which estimates the number of prime cycles of a given length in $G$. Recall first the prime number theorem:

**Theorem 6.** (Prime Number Theorem). Define the prime counting function as $\pi(x) = \#\{p \mid p \leq x, \text{where } p \text{ is prime}\}$. i.e. $\pi(x)$ is the number of primes less than or equal to $x$.

$$
\lim_{x \to \infty} \pi(x)/\left(\frac{x}{\log(x)}\right) = 1. \text{ That is, } \pi(x) \sim \frac{x}{\log(x)} \text{ as } m \to \infty.
$$
Before we state the Graph Prime Number Theorem, we need to fix some notation.

**Definition 19.** Define the graph prime counting function as

\[ \pi_G(n) = \#\{[P] \mid P \text{ a prime in } G \text{ of length } n}\].

It is important to note that this counts equivalence classes \([P]\) rather than primes \(P\).

**Definition 20.** Denote the greatest common divisor of path lengths in \(G\) as

\[ \Delta_G = \gcd(l(P)) \]

for each \([P]\) where \(P\) is prime.

Now we can state the prime number theorem for graphs:

**Theorem 7.** (Graph Prime Number Theorem).

Fix \(G\) as usual and let \(R_G\) be the radius of convergence for \(\zeta(u,G)\). There are two cases:

1. If \(\Delta_G = 1\), then

\[ \pi_G(m) \sim \frac{R_G^{-m}}{m}, \text{ as } m \to \infty. \]

2. If \(\Delta_G > 1\), then \(\pi_G(m) = 0\) except when \(\Delta_G \mid m\). If \(\Delta_G \mid m\), then

\[ \pi_G(m) \sim \Delta_G \frac{R_G^{-m}}{m}, \text{ as } m \to \infty. \]

We’ll close by stating the analogues of the Riemann Hypothesis. In case where \(G\) is \((q+1)\)-regular, it looks very much like the Riemann Hypothesis.

**Definition 21.** There are two cases:

1. If \(G\) is \((q+1)\)-regular, the **Riemann Hypothesis for the Ihara zeta function** says that \(\zeta(q^{-s},G)\) has no poles with \(0 < \text{Re}(s) < 1\) unless \(\text{Re}(s) = \frac{1}{2}\).

2. If \(G\) is not \((q+1)\)-regular, the Riemann Hypothesis for the Ihara zeta function can be formulated as \(\zeta(u,G)\) is pole free for \(R_G < |u| < \sqrt{R_G}\).

There are (at least) two interesting directions of investigation that follow from consideration of the graph Riemann Hypothesis which we will only mention here. First, if \(\zeta(u,G)\), for some \((q+1)\)-regular \(G\), satisfies the Riemann Hypothesis, than \(G\) is a Ramanujan graph. Ramanujan graphs are graphs where the difference of the of the largest modulus eigenvalue of \(A\) and the modulus of the smallest modulus eigenvalue of \(A\) is large. More precisely, if the eigenvalues of
$A$ are $k = \lambda_0 \geq ... \geq \lambda_{n-1}$, $\lambda(G) = \max_{\lambda_i \mid \lambda_1}$, and $\lambda(G) \leq 2\sqrt{q}$, then $G$ is Ramanujan. Ramanujan graphs are of interest because they are good expander graphs.

The second direction of study has to do with a Siegel zero. This is a conjectured zero of the Riemann Zeta function near $\sigma = 1$. Proof of its existence would disprove the Riemann Hypothesis. Stark and Terras have gone on to investigate connections between the Graph Riemann Hypothesis and Siegel zeros.
References:


