MATH 61-02: WORKSHEET 10 (CH 7)

(W1) Suppose G is a simple graph (no loops) with $|V(G)| = n$. Show that if the degree of every vertex in G is at least $\frac{n-1}{2}$, then G is connected. (First convince yourself this is true for $n = 2, 3, 4$.)

There are many different ways to do this problem. Here are four of them:

Proof 1. Suppose G were not connected. Then it must have at least two connected components. In each of these connected components, there must be a vertex v with degree at least $\frac{n-1}{2}$, so each of these connected components must have at least $1 + \frac{n-1}{2}$ vertices (counting v as well). But since we have at least two of these connected components, that implies that G must have at least $2 + (n-1) = n+1$ vertices. Contradiction!

Proof 2. Suppose G were not connected. Then it must have at least two connected components, so there exists a partition of $V(G)$ into two sets of $1 \leq k \leq n-1$ and $n-k$ vertices with no edges between the two sets. But since G is simple, the first set of vertices can have at most $\binom{k}{2}$ edges among them, and the second set of vertices can have at most $\binom{n-k}{2}$ edges among them. It is easy to see that this sum $\binom{k}{2} + \binom{n-k}{2}$ is maximized when $k = \lfloor \frac{n}{2} \rfloor$. Then, the resulting sum will be at most $(\frac{n}{2})(\frac{n}{2} - 1) = \frac{n^2}{4} - \frac{n}{2}$, but the degree condition we have on our vertices combined with part (c) of the last question gives us that we have at least $\frac{n^2}{4} - \frac{n}{4}$ edges, a contradiction.

Proof 3. Take two vertices x and y in G. Now, either x and y are adjacent, or they are not. If they are not, since the degree of every vertex in G is at least $\frac{n-1}{2}$, the combined degrees of x and y must be at least $n-1$, and by the Pigeonhole Principle, there must exist a vertex v such that x and y are both adjacent to v. Hence any two vertices are either adjacent or have a common neighbor, and so the graph is clearly connected.

Proof 4. I make the same claim I showed in Proof 3, but go by contradiction: Every pair of distinct vertices are either adjacent or have a common neighbor. Suppose not. Then there exist two vertices u, v that aren't adjacent and don't have a common neighbor. So that means that all the neighbors of u and v must be distinct, so as the degrees of u, v are at least $\frac{n-1}{2}$ each, they have $n-1$ distinct neighbors, and adding u and v gives us that the graph has $n+1$ vertices. Contradiction!