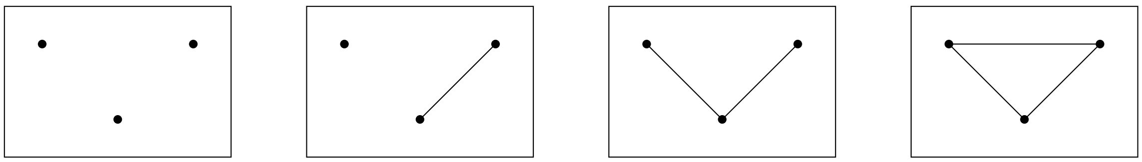
MATH 61-02: WORKSHEET 11 (GRAPH ISOMORPHISM)

Let's say that two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic*, denoted $G_1 \cong G_2$, if there is a bijection $\sigma : V_1 \to V_2$ such that for all $v, w \in V_1$,

$$\sigma(v), \sigma(w) \text{ is an edge in } E_2 \iff v, w \text{ is an edge in } E_1.$$

There are four different isomorphism classes of simple graphs with three vertices:

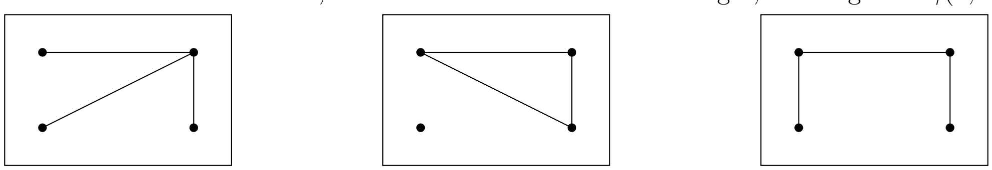


Let $\gamma(n, m)$ be the number of isomorphism types of simple graphs on n vertices with m edges, and let

$$\Gamma(n) = \sum_{m=0}^{\binom{n}{2}} \gamma(n, m)$$

be the total number of isomorphism types of graphs with n vertices. So far we've seen that $\Gamma(3) = 1 + 1 + 1 + 1 = 4$.

How about four vertices? For instance, here are the three that have 3 edges, showing that $\gamma(4, 3) = 3$.



If we studied all the possibilities we would find $\Gamma(4) = 1 + 1 + 2 + 3 + 2 + 1 + 1 = 11$.

(W1) There is some symmetry there, just like for Pascal's triangle. Prove that $\gamma(n, m) = \gamma(n, \binom{n}{2} - m)$ for any $n \geq 2$ and $0 \leq m \leq \binom{n}{2}$.

(Hint: the *complement* of a graph, which we can denote G^c, has the same vertices as G but the opposite edges—that is, G^c has an edge between two vertices if and only if G does not!)

Answer. First quick observation: if there are n vertices then there are $\binom{n}{2}$ possible edges, so if G has m edges, then G^c has the opposite edges, which is $\binom{n}{2-m}$ of them.

The key to this is just to check that $G_1 \cong G_2 \iff G_1^c \cong G_2^c$. From this it will follow that if two graphs are in the same class, then their complements are too, so the groupings are the same.

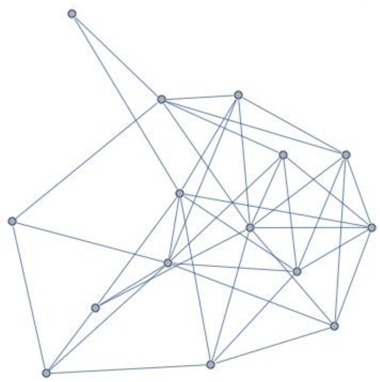
So suppose that f is a bijection carrying V_1 to V_2 such that for $v, w \in V_1$, there is an edge of G_1 between v, w if and only if there is an edge of G_2 between $f(v), f(w)$. There must be such a bijection because that's what it means for $G_1 \cong G_2$ in the first place!

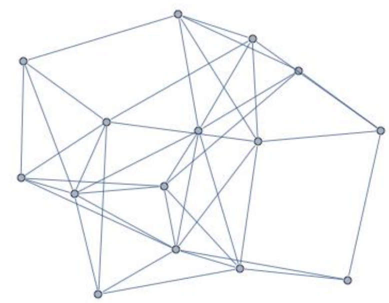
Now we need to show that the complements are isomorphic. So take two vertices $v, w \in V_1$. There's an edge between them in G_1^c iff there is *no* edge between them in G_1 iff there is no edge between $f(v), f(w)$ in G_2 iff there *is* an edge between $f(v), f(w)$ in G_2^c. This establishes the isomorphism.

(W2) Compute $\Gamma(5)$. That is, classify all five-vertex simple graphs up to isomorphism. (Hint: the answer is between 30 and 40.)

Answer. There are 34 of them, but it would take a long time to draw them here!

(W3) Here are two graphs, G_1 and G_2 (15 vertices each). Let's analyze them.





The following table shows the adjacency matrices A_i for these graphs on the top row, then A_i^2 on the second row and A_i^3 on the third row.

$$\begin{pmatrix}
0&1&0&1&0&1&1&0&0&0&1&1&1&0&0\\
1&0&0&1&1&0&1&0&0&0&0&1&0&0&1\\
0&0&0&0&1&1&0&1&0&0&1&0&0&1&1\\
1&1&0&0&1&0&0&1&1&0&0&0&0&0&1\\
0&1&1&1&0&0&0&1&1&0&1&0&0&0&0\\
1&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&1&1&1&0&0&0&0&0\\
0&0&1&1&1&0&1&0&1&0&1&0&1&0&1\\
0&0&0&1&1&0&1&1&0&0&0&1&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&1&1&1&0\\
1&0&1&0&1&0&0&1&0&0&0&1&0&0&0\\
1&1&0&0&0&0&0&0&1&1&1&0&1&1&1\\
1&0&0&0&0&0&0&1&0&1&0&1&0&0&0\\
0&0&1&0&0&0&0&0&0&1&0&1&0&0&0\\
0&1&1&1&0&0&0&1&0&0&0&1&0&0&0
\end{pmatrix}
\qquad
\begin{pmatrix}
0&1&0&1&0&1&0&0&0&1&0&1&0&0&0\\
1&0&1&0&0&0&0&0&1&1&1&1&1&1&1\\
0&1&0&0&0&0&1&0&0&1&0&0&0&0&0\\
1&0&0&0&0&1&1&0&0&0&0&1&1&1&0\\
0&0&0&0&0&1&1&0&1&0&0&0&0&1&1\\
1&0&0&1&1&0&0&0&0&0&0&1&0&0&0\\
0&0&0&1&1&0&0&1&0&0&0&0&0&1&1\\
0&0&1&0&0&0&1&0&1&0&0&0&0&1&0\\
0&0&0&0&1&0&0&1&0&1&0&0&0&0&1\\
1&1&0&0&0&0&0&0&1&0&1&1&0&0&1\\
0&1&1&0&0&0&0&0&0&1&0&0&1&1&1\\
1&1&0&1&0&1&0&0&0&1&0&0&1&0&1\\
0&1&0&1&0&0&0&0&0&0&1&1&0&0&0\\
0&1&0&1&1&0&1&1&0&0&1&0&0&0&0\\
0&1&0&0&1&0&1&0&1&1&1&1&0&0&0
\end{pmatrix}$$

$$\begin{pmatrix}
7&3&2&1&3&0&1&4&3&3&1&3&1&1&3\\
3&6&2&3&1&1&1&4&4&2&3&2&2&1&2\\
2&2&6&3&2&0&1&3&2&1&2&3&1&0&1\\
1&3&3&6&3&1&4&3&2&0&3&4&2&0&2\\
3&1&2&3&6&1&3&4&2&0&2&3&1&1&4\\
0&1&0&1&1&2&1&1&0&0&2&1&1&1&1\\
1&1&1&4&3&1&5&1&1&0&2&4&3&1&2\\
4&4&3&3&4&1&1&8&3&2&2&4&0&1&2\\
3&4&2&2&2&0&1&3&5&2&3&0&2&1&3\\
3&2&1&0&0&0&0&2&2&4&1&2&1&1&1\\
1&3&2&3&2&2&2&2&3&1&5&1&3&2&3\\
3&2&3&4&3&1&4&4&0&2&1&8&2&1&1\\
1&2&1&2&1&1&3&0&2&1&3&2&4&2&2\\
1&1&0&0&1&1&1&1&1&1&2&1&2&3&2\\
3&2&1&2&4&1&2&2&3&1&3&1&2&2&5
\end{pmatrix}
\qquad
\begin{pmatrix}
5&2&1&2&1&2&1&0&1&2&2&4&3&2&3\\
2&8&1&4&2&2&2&2&2&4&5&4&2&1&3\\
1&1&3&0&0&0&1&0&1&2&1&1&2&3&2\\
2&4&0&6&3&2&1&2&0&2&2&3&1&1&2\\
1&2&0&3&5&0&2&3&1&2&2&2&0&1&2\\
2&2&0&2&0&4&2&0&1&2&0&2&2&2&2\\
1&2&1&1&2&2&5&1&3&1&2&2&1&3&1\\
0&2&0&2&3&0&1&4&0&1&2&0&0&1&2\\
1&2&1&0&1&1&3&0&4&1&2&2&0&2&2\\
2&4&2&2&2&2&1&1&1&6&2&3&3&2&4\\
2&5&1&2&2&0&2&2&2&2&6&4&1&1&2\\
4&4&1&3&2&2&2&0&2&3&4&7&2&2&2\\
3&2&2&1&0&2&1&0&0&3&1&2&4&3&3\\
2&1&3&1&1&2&3&1&2&2&1&2&3&6&4\\
3&3&2&2&2&2&1&2&2&4&2&2&3&4&7
\end{pmatrix}$$

$$\begin{pmatrix}
10&18&12&23&14&9&20&15&12&6&19&22&17&8&13\\
18&12&12&20&22&5&19&18&11&6&12&23&11&6&17\\
12&12&8&12&18&8&10&18&12&5&16&11&9&10&17\\
23&20&12&14&20&4&9&25&20&10&14&13&8&7&19\\
14&22&18&20&14&5&10&23&19&8&18&14&10&5&13\\
9&5&8&4&5&0&2&7&5&4&3&6&2&1&4\\
20&19&10&9&10&2&4&21&17&13&10&11&6&5&11\\
15&18&18&25&23&7&21&18&20&6&23&18&18&9&22\\
12&11&12&20&19&5&17&20&8&4&10&23&8&4&11\\
6&6&5&10&8&4&13&6&4&4&8&15&11&7&7\\
19&12&16&14&18&3&10&23&10&8&8&21&5&4&11\\
22&23&11&13&14&6&11&18&23&15&21&12&17&13&21\\
17&11&9&8&10&2&6&18&8&11&5&17&4&4&7\\
8&6&10&7&5&1&5&9&4&7&4&13&4&2&3\\
13&17&17&19&13&4&11&22&11&7&11&21&7&3&8
\end{pmatrix}
\qquad
\begin{pmatrix}
12&22&4&17&9&12&8&5&6&17&13&19&10&8&13\\
22&22&15&13&10&12&12&6&11&24&19&25&21&23&27\\
4&15&2&8&7&2&5&8&4&7&13&8&3&3&7\\
17&13&8&10&6&14&14&2&9&13&10&19&15&18&15\\
9&10&7&6&6&11&14&4&12&10&7&10&9&17&16\\
12&12&2&14&11&6&6&5&4&9&10&16&6&6&9\\
8&12&5&14&14&6&8&12&5&11&9&9&7&13&17\\
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17&24&7&13&10&9&11&6&13&16&21&23&11&12&19\\
13&19&13&10&7&10&9&6&8&21&12&14&17&19&23\\
19&25&8&19&10&16&9&7&7&23&14&20&18&15&24\\
10&21&3&15&9&6&7&6&6&11&17&18&6&5&9\\
8&23&3&18&17&6&13&14&8&12&19&15&5&8&12\\
13&27&7&15&16&9&17&9&15&19&23&24&9&12&16
\end{pmatrix}$$

(a) Explain how to use an adjacency matrix A to list all of the degrees of all the vertices of the graph G. Make that list for each graph on the last page.

(b) Explain how to use the matrix A^2 to get a different method of listing all of the vertex degrees. Check that you get the same list for each graph as you did in the last part.

(c) Show that the two graphs have the same total number of edges. ($|E_1| = |E_2|$)
(d) How many triangles are there in each graph?
(e) Is it possible that these graphs are isomorphic?

Answer.

(a) You can read the degrees of vertices off of A by adding up the numbers in the rows. That's because the entries of row i count the number of edges from i to each other vertex.

(b) Every entry of A^2 counts paths of length two between some pair of vertices. So consider the diagonal of A^2. The entry (i, i) counts paths of length two from i to itself. But every such path must go out an edge and back again, so this total counts the number of edges incident to i, which is the degree.

(c) In each case, the diagonal entries of A^2 add up to 80. This double-counts the number of edges (because they are counted once for each endpoint), so each graph has 40 vertices.

(d) The sum of the diagonal entries of A^3 triple-counts the triangles.

(e) Nope! Vertex v_6 of the first graph has degree 2, but no vertex of the second graph does. Every isomorphism must preserve vertex degrees, so the graphs are not isomorphic.