MATH 61-02: WORKSHEET 5 (§4.4,§5.1-2)

(W1) Recall that a *quadratic polynomial* in the variable x is an expression of the form $ax^2 + bx + c$. A *cubic polynomial* has degree three instead of two.

(a) What is the form of an arbitrary cubic polynomial in x? If your polynomial is called $g(x)$, evaluate $g(0)$, $g(1)$, $g(-1)$, and $g(2)$ in terms of the coefficients you used in your expression.

(b) How many cubic polynomials $f(x)$ with positive integer coefficients satisfy $f(1) = 9$?

(c) How many degree 6 polynomials $f(x)$ with positive integer coefficients satisfy $f(1) = 30$ and $f(-1) = 12$?

Answer. (a) An arbitrary cubic polynomial in x has form $a_3x^3 + a_2x^2 + a_1x^1 + a_0$. If this is $g(x)$, plugging in our values for x gives $g(0) = a_0$, $g(1) = a_3 + a_2 + a_1 + a_0$, $g(-1) = -a_3 + a_2 - a_1 + a_0$, and $g(2) = 8a_3 + 4a_2 + 2a_1 + a_0$.

(b) Let $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ be an arbitrary cubic polynomial. Then we have that $f(1) = a_3 + a_2 + a_1 + a_0 = 9$, so we can count the number of such cubic polynomials by counting the number of solutions to this equation. By stars-and-bars, there must be $\binom{8}{3}$ of these.

(c) Let $f(x) = a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ be an arbitrary degree 6 polynomial. Then we have that

$$f(1) = a_6 + a_5 + a_4 + a_3 + a_2 + a_1 + a_0 = 30,$$

and

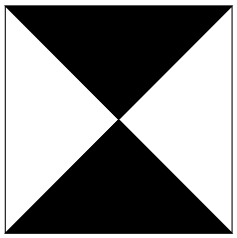
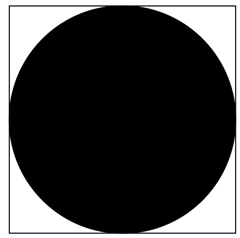
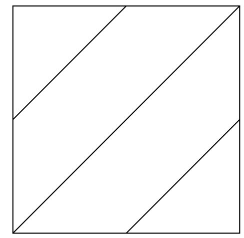
$$f(-1) = a_6 - a_5 + a_4 - a_3 + a_2 - a_1 + a_0 = 12.$$

If we add the two equations, we get that $a_6 + a_4 + a_2 + a_0 = 21$, and if we subtract the second equation from the first, we get that $a_5 + a_3 + a_1 = 9$. It is easy to see that any set of coefficients which satisfy these two equations will yield a desired degree 6 polynomial. Now, by stars-and-bars there are $\binom{20}{3}$ solutions to the first equation, and $\binom{8}{2}$ solutions to the second, so the number of combined solutions - hence solutions to the original problem - is $\binom{20}{3}\binom{8}{2}$.

(W2) Let R be the relation on the interval $[0, 1]$ defined by xRy if and only if $x^2 \leq y$. Sketch the graph of R. Decide if it is reflexive, symmetric, transitive, and/or anti-symmetric.

Answer. The graph is to the left of the parabola $y = x^2$. R is reflexive, since $x^2 \leq x$ for all $0 \leq x \leq 1$. R is not symmetric, since $0^2 \leq 1$, but $1^2 = 1$ is clearly greater than 0. R is not antisymmetric, since $0.4^2 \leq 0.6, 0.6^2 \leq 0.4$ but $0.4 \neq 0.6$. R is not transitive, since $0.6^2 \leq 0.4, 0.4^2 \leq 0.2$, but $0.6^2 = 0.36 > 0.2$.

(W3) For each of the following relations on $[0,1]$, decide whether it is reflexive, symmetric, and/or antisymmetric.
(Bonus: also decide whether each is transitive, and use that to identify which are equivalence relations and which are partial orders.)

Answer. Call the relations R, S, T corresponding to the pictures from left to right. For R, we will assume that the boundaries of the triangles given by $y = x$ and $y = 1 - x$ are entirely contained in R. Then, R is clearly reflexive. R is not symmetric, since $(0.5, 1) \in R$ but $(1, 0.5) \notin R$. R is not antisymmetric, since $(0, 1), (1, 0) \in R$ but $0 \neq 1$.

For S, we will assume that the boundary of the circle $(x - 0.5)^2 + (y - 0.5)^2 = (0.5)^2$ is entirely contained in S. S is not reflexive, since $(0, 0) \notin S$. S is symmetric, since the circle is symmetric across the line $y = x$, so if a point $(a, b) \in S$, then the corresponding point $(b, a) \in S$. S is not antisymmetric, however, since the points $(0.4, 0.6), (0.6, 0.4) \in S$, but $0.4 \neq 0.6$.

T is reflexive, since all points (a, a) for $0 \leq a \leq 1$ are clearly contained in T. T is symmetric, since the graph of T is symmetric across the line $y = x$, so if a point $(a, b) \in T$, then the corresponding point $(b, a) \in T$. T is not antisymmetric, however, since the points $(0.25, 0.75), (0.75, 0.25) \in T$, but $0.25 \neq 0.75$.

(W4) Consider the relation on A=$\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ given by $(a, b)F(c, d) \iff ad = bc$. Check that it is an equivalence relation. Give several elements of the equivalence classes $[(0, 1)]$ and $[(2, 3)]$ in the associated partition. Sketch A and show a few blocks of the partition in your picture.

Answer. Firstly, we can realise that this equivalence relation is really just that for determining when two fractions are equivalent, since $ad = bc$ iff $\frac{a}{b} = \frac{c}{d}$, so $(a, b)F(c, d) \iff \frac{a}{b} = \frac{c}{d}$. Now, this inherits all the relation properties of equality in the real numbers, so this is clearly an equivalence relation as equality is an equivalence relation.

Phrased in this light, the equivalence classes $[(0, 1)]$ and $[(2, 3)]$ consist of all pairs (a, b) such that $\frac{a}{b} = 0$ and $\frac{a}{b} = \frac{2}{3}$ respectively. In general, an equivalence class $[(c, d)]$ consists of all lattice points (a, b) lying on the line $y = \frac{d}{c}x$, and a sketch of the blocks of the partition should reflect this.

(W5) Give a relation R on $A = \{blue, red, yellow, green\}$ that is reflexive but not transitive. What is the smallest number of elements $|R|$ required to achieve this?

Answer. Such a relation is $R = \{(blue, blue), (red, red), (yellow, yellow), (green, green), (blue, red), (red, yellow)\}$. The minimum $|R|$ required to achieve this is 6: for R to be reflexive, it must contain the four elements $(blue, blue), (red, red), (yellow, yellow), (green, green)$. If R consisted of only these four elements, it is clearly transitive: The only way that two elements (a, b) and (b, c) can be in R is if $a = b = c$, in which case (a, c) is also in R. If R had five elements consisting of the previously mentioned four elements plus a fifth element (x, y), where $x, y \in A$ and $x \neq y$, then R is still transitive: The only additional ways (aside from the ones covered above) that two elements (a, b) and (b, c) can be in R is if $a = b$ and (b, c) is the fifth element we added, in which case (a, c) is the fifth element we added and so must be in R. Since we have given an R satisfying the conditions above such that $|R| = 6$, it is now clear that this is the smallest number of elements $|R|$ required to achieve this.