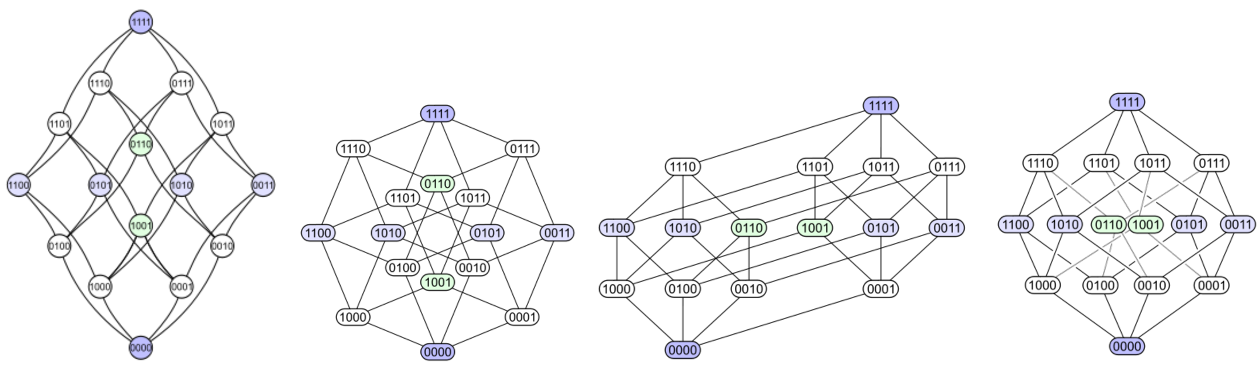
MATH 61-02: WORKSHEET 6 (§5.3-4, §6.1)

(W1) Let $(P, \leq)$ be a poset. A *chain* is a sequence of distinct elements $x_1 \leq x_2 \leq \ldots \leq x_k$, and we say that k is the *length* of the chain. An *antichain* is a subset $A \subset P$ such that $x \| y$ for all $x, y \in A$, and we say that $|A|$ is the *width* of the antichain. (In other words, a chain is a subset of P in which any two elements are comparable; an antichain is a subset of P in which no two elements are comparable.)

(a) We have seen that $(\mathcal{P}([n]), \subseteq)$ is a poset. What is the length of the longest chain in this poset?

(b) Recall that $Subs_k(S)$ is the set of all k–element subsets of S. Verify that for any $0 \leq k \leq n$, the poset contains $Subs_k([n])$ as an antichain. What is its width?

(c) The following are four Hasse diagrams of $(\mathcal{P}([4]), \subseteq)$. Which one is organized to make the chains and antichains easy to recognize? Explain.



Answer. (a) For one finite set to be properly contained in another, the second set must have at least one more element. So if I take a chain formed by starting with the empty set and adding the elements of $[n]$ one at a time, such as $\emptyset \subseteq [1] \subseteq [2] \subseteq \cdots \subseteq [n]$, this is largest-possible. The length of such a chain is $n + 1$.

(b) It is clearly impossible for two distinct elements with the same cardinality to be comparable, because neither can be a subset of the other unless they are equal. So if I consider all subsets with cardinality k, they are all mutually noncomparable and thus form an antichain. The width is $|\,Subs_k([n])| = \binom{n}{k}$, because that's the number of ways to choose k elements out of n to form a subset.

(c) Both the third and fourth are organized with the antichains $Sub_k([4])$ as the horizontal rows. (Note that the widths are $1, 4, 6, 4, 1$ as in Pascal's triangle.) But the third one groups them in a way that is irrelevant to the chain/antichain structure. In either case, chains are paths following edges from the bottom node to the top node. For instance, the chain described in the solution to part (a) is the one along the left side of the diagram.

(W2) We saw several examples of topological quotient spaces in class. For instance, $[0,1]/0 \sim 1$ is a circle.

(a) Let $\mathbb{D}$ be the unit disk $\{(x,y) : x^2+y^2 \leq 1\} \subset \mathbb{R}^2$. Define an equivalence relation by

$$(x,y) \sim (z,w) \iff (x,y) = (z,w) \text{ or } x^2+y^2 = z^2+w^2 = 1,$$

and describe the resulting quotient space $\mathbb{D}/\sim$.

(b) Come up with an equivalence relation that turns an annulus $\mathbb{A} = \{(x,y) : 1 \leq x^2+y^2 \leq 4\}$ into a torus.

Answer. (a) This glues the whole circular boundary of $\mathbb{D}$ to a single point, so I can represent $\mathbb{D}/\sim$ by a sphere with that special point as the north pole.

(b) To do this, I should glue the inner boundary to the outer boundary. I want an equivalence relation that picks out when two pairs of points are on a line of the same slope and one unit apart, such as:

$$(x,y) \sim (z,w) \iff xw = zy \text{ and } (x-z)^2 + (y-w)^2 = 1.$$

(W3) Let L be a line in the plane. Let $\mathcal{S}$ be the set of lines in the plane not parallel to L. Define a relation $\sim$ on $\mathcal{S}$ by

$$L_1 \sim L_2 \iff L_1 \cap L = L_2 \cap L$$

(so two lines are related if they intersect L in the same set).

Describe the quotient space $\mathcal{S}/\sim$.

Answer. First let's check it's an equivalence relation.

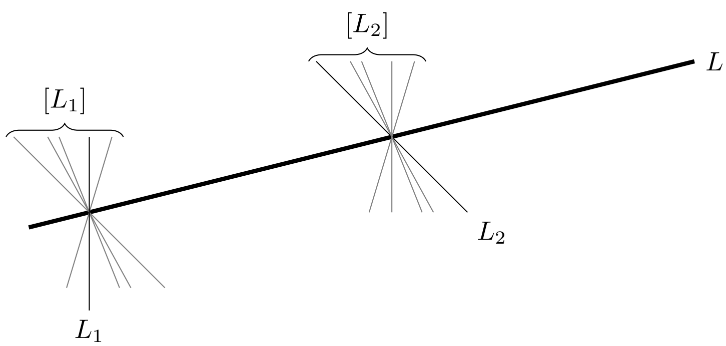
Reflexive: $L_1 \overset{?}{\sim} L_1 \iff L_1 \cap L = L_1 \cap L$ ✓

Symmetric: $L_1 \sim L_2 \iff L_1 \cap L = L_2 \cap L \iff L_2 \cap L = L_1 \cap L \iff L_2 \sim L_1$ ✓

Transitive: Suppose $L_1 \sim L_2$ and $L_2 \sim L_3$. Then $L_1 \cap L = L_2 \cap L$. This must be a single point, since it can't be infinite or the empty set because L_i are not parallel to L. Call it x. Similarly, $L_2 \cap L = L_3 \cap L$ must be single point; call it y. But now both x and y are on $L_2 \cap L$. Since two non-parallel lines can't intersect twice, $x = y$. Thus $L_1 \cap L = L_3 \cap L$, and we've shown $L_1 \sim L_3$. ✓

The equivalence class of L_1 is, by definition, $[L_1] = \{L' \in \mathcal{S} : L' \cap L = L_1 \cap L\}$. In other words, an equivalence class is characterized by all of its lines intersecting L at the same point. That means there is a bijective correspondence from points on L to equivalence classes; to each point corresponds all of the lines that go through it. So an efficient way to model the quotient space is by the points of L itself, i.e., $\mathcal{S}/\sim = L$.

(There are other possibilities too, such as $\mathcal{S}/\sim = \{\text{lines perpendicular to } L\}$, which is formed by taking the perpendicular at each point as a representative of all the lines meeting L there. But personally I find the first choice easier to think about.)



(W4) (a) Suppose $|A| = 10$ and $|B| = 8$. How many injections are there from $A \to B$?

(b) For the same A and B, how many surjections are there from $A \to B$?

(c) Give a bijection from the integers $\mathbb{Z}$ to the odd integers $2\mathbb{Z} + 1$.

Answer. (a) There are none. The pigeonhole principle guarantees that you can't put 10 inputs into 8 "output slots" without a collision.

(b) There are two possibilities. I need to hit all 8 points in the target space; either two of them will be hit twice, or one will be hit three times.
Case 1: Choose two outputs to be double-hit ($\binom{8}{2}$ ways) and choose a pair of inputs to map to each ($\binom{10}{2} \cdot \binom{8}{2}$). Then map the remaining 6 inputs to the remaining 6 outputs (6! ways).
Case 2: Choose one output to be triple-hit (8 ways) and choose three inputs to map there ($\binom{10}{3}$ ways). Then map the remaining 7 inputs to the remaining 7 outputs (7! ways).
All in all, since OR means add: $\binom{8}{2} \cdot \binom{10}{2} \cdot \binom{8}{2} \cdot 6! + 8 \cdot \binom{10}{3} \cdot 7!$, or about 183 million.
To get the probability of a surjection, divide by 8^{10} and you get just about 17%!

(c) Easy enough! As a function, I can write $f(n) = 2n + 1$, and I can see that it's injective ($2a + 1 = 2b + 1 \implies a = b$) and surjective (every odd is of the form $2n + 1$ for some integer n). Written as a relation, this is $f = \{(n, 2n + 1) : n \in \mathbb{Z}\}$.