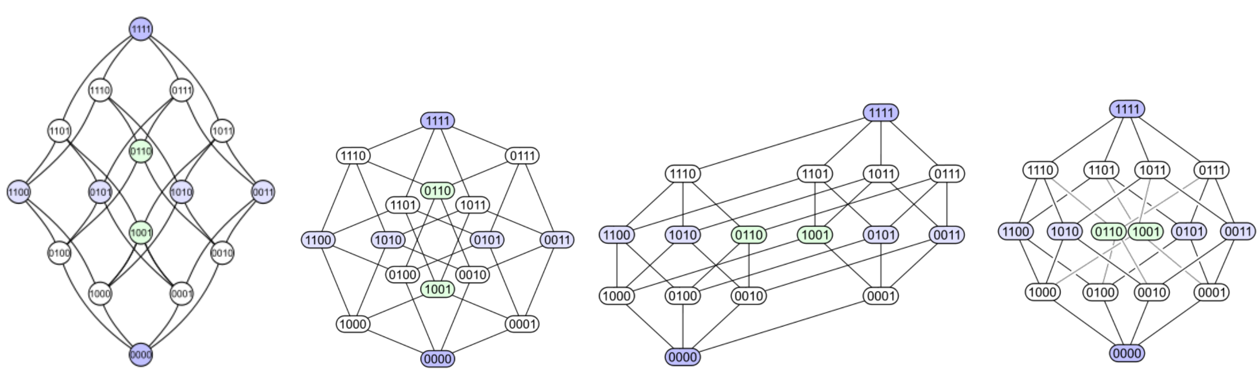
MATH 61-02: WORKSHEET 6 (§5.3-4, §6.1)

(W1) Let $(P, \leq)$ be a poset. A *chain* is a sequence of distinct elements $x_1 \leq x_2 \leq ... \leq x_k$, and we say that k is the *length* of the chain. An *antichain* is a subset $A \subset P$ such that $x \| y$ for all $x, y \in A$, and we say that $|A|$ is the *width* of the antichain. (In other words, a chain is a subset of P in which any two elements are comparable; an antichain is a subset of P in which no two elements are comparable.)

(a) We have seen that $(\mathcal{P}([n]), \subseteq)$ is a poset. What is the length of the longest chain in this poset?

(b) Recall that $Subs_k(S)$ is the set of all k–element subsets of S. Verify that for any $0 \leq k \leq n$, the poset contains $Subs_k([n])$ as an antichain. What is its width?

(c) The following are four Hasse diagrams of $(\mathcal{P}([4]), \subseteq)$. Which one is organized to make the chains and antichains easy to recognize? Explain.



(W2) We saw several examples of topological quotient spaces in class. For instance, $[0,1]/0 \sim 1$ is a circle.

(a) Let $\mathbb{D}$ be the unit disk $\{(x,y) : x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$. Define an equivalence relation by

$$(x,y) \sim (z,w) \iff (x,y) = (z,w) \text{ or } x^2 + y^2 = z^2 + w^2 = 1,$$

and describe the resulting quotient space $\mathbb{D}/\sim$.

(b) Come up with an equivalence relation that turns an annulus $\mathbb{A} = \{(x,y) : 1 \leq x^2 + y^2 \leq 4\}$ into a torus.

(W3) Let L be a line in the plane. Let $\mathcal{S}$ be the set of lines in the plane not parallel to L. Define a relation $\sim$ on $\mathcal{S}$ by

$$L_1 \sim L_2 \iff L_1 \cap L = L_2 \cap L$$

(so two lines are related if they intersect L in the same set).

Describe the quotient space $\mathcal{S}/\sim$.

(W4) (a) Suppose $|A| = 10$ and $|B| = 8$. How many injections are there from $A \to B$?

(b) For the same A and B, how many surjections are there from $A \to B$?

(c) Give a bijection from the integers $\mathbb{Z}$ to the odd integers $2\mathbb{Z} + 1$.