MATH 61-02: WORKSHEET 7 (§6.1-6.3)

(W1) Show that $\mathbb{Z}^3$ is countable.

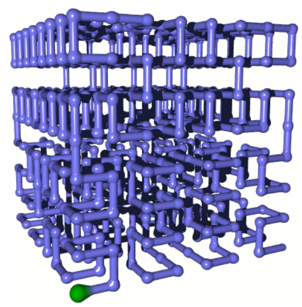
Answer. One solution is as follows: We know that $\mathbb{Z}$ is countable since we can enumerate it in a list that alternates between positive and negative integers: $0, 1, -1, 2, -2, 3, -3, \ldots$.

If we insist (and we should not), we can write this as a function $f : \mathbb{N} \to \mathbb{Z}$,

$$f(x) = \begin{cases} \frac{x-1}{2}, & x \equiv 1 \pmod 2 \\ -\frac{x}{2}, & x \equiv 0 \pmod 2 \end{cases}$$

Now we know that the Cartesian product of two countable sets is countable, and since $\mathbb{Z}$ is countable, $\mathbb{Z}\times\mathbb{Z} = \mathbb{Z}^2$ is countable. Finally, there is clearly a bijection $\mathbb{Z}^2\times\mathbb{Z} \to \mathbb{Z}^3$ given by $((a, b), c) \mapsto (a, b, c)$, so since $\mathbb{Z}^2 \times \mathbb{Z}$ is countable, $\mathbb{Z}^3$ is too.

Alternately, we can snake-enumerate the lattice points:



(W2) Show that $\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{5}}}}$ is an algebraic number.

Answer. Let $\alpha = \sqrt{1+\sqrt{2+\sqrt{3+\sqrt{5}}}}$. Then we have that

$$\alpha = \sqrt{1+\sqrt{2+\sqrt{3+\sqrt{5}}}} \implies \alpha^2 = 1+\sqrt{2+\sqrt{3+\sqrt{5}}} \implies \alpha^2 - 1 = \sqrt{2+\sqrt{3+\sqrt{5}}}$$

$$\implies (\alpha^2-1)^2 = 2+\sqrt{3+\sqrt{5}} \implies (\alpha^2-1)^2 - 2 = \sqrt{3+\sqrt{5}} \implies ((\alpha^2-1)^2-1)^2 = 3+\sqrt{5}$$
$$\implies ((\alpha^2-1)^2-1)^2 - 3 = \sqrt{5} \implies (((\alpha^2-1)^2-1)^2-3)^2 - 5 = 0,$$

and the left side gives a polynomial with integer coefficients $((((x^2-1)^2-1)^2-3)^2-5)$ with α as a solution, implying that α is an algebraic number, as desired. If you write out the polynomial, you get one of sixteenth-degree, namely $x^{16} - 8x^{14} + 24x^{12} - 32x^{10} + 10x^8 + 24x^6 - 24x^4 + 4$.

(W3) Let Q be the set of real numbers which are solutions to quadratic equations $ax^2 + bx + c = 0$ with integer coefficients (so $a, b, c \in \mathbb{Z}$).

(a) Why must $\mathbb{Q} \subset Q$? Show that Q also contains irrational numbers.

(b) Prove that Q is countable.

Answer. (a) Every rational number p/q is a root of the polynomial $qx - p = 0$, so it's in Q. But on the other hand $\sqrt{2}$ is a root of $x^2 - 2 = 0$, so Q contains some irrationals.

(b) This is just a special case of the fact that all algebraic numbers are countable, but let's spell it out here. We can make a bijective correspondence from quadratic polynomials to $\mathbb{Z}^3$ by matching up $ax^2 + bx + c$ with (a, b, c), and we just proved above that $\mathbb{Z}^3$ is countable. This means we can make an organized list of all the quadratic polynomials. But there are at most 2 real roots to any quadratic equation, so if we list out roots for all the polynomials, we still get an organized list!