MATH 61-02: WORKSHEET 8 (§6.3-6.4)

(W1) Prove that if the interval $[0, 1]$ is partitioned into nondegenerate intervals (i.e., points don't count as intervals), then the partition is countable. (That means you're trying to show the surprising result that that *you can't partition an interval into uncountably many intervals, even though it has uncountably many points.*) Explain why this implies the same result for partitions of $\mathbb{R}$.

Hint: for a given partition, first list all the intervals of length $> 1/2$...

Answer. Fix any partition $\mathcal{P}$ of $[0, 1]$. That partition must have only finitely many subintervals of length greater than $1/n$... certainly it can't have more than n of them, because the pieces of a partition are disjoint. So let's list all the intervals within $\mathcal{P}$ in order of size, from biggest to smallest. If there's a tie, I'll list them from left to right.

For instance, if $\mathcal{P} = \{\ [0, .1),\ [.1, .4],\ (.4, .5],\ (.5, .75),\ [.75, 1]\}$, then the size-ordered list would be $[.1, .4]$, $(.5, .75)$, $[.75, 1]$, $[0, .1)$, $(.4, .5]$. Now I must see that this list will eventually exhaust the whole partition in the case that the partition is infinite. Consider a subinterval I in $\mathcal{P}$. It has some positive length, so there is some n such that $1/n < |I|$. But then there are no more than $n - 1$ intervals that are longer than or equal to the length of I, so it comes no later than nth in the list! So I know everything eventually gets listed.

(W2) Suppose that you're walking on a road and every mile you go you come to another fork in the road. Suppose the roads go on forever. Consider the set P of all paths (that is, all choices of L or R at every fork; so one path is LLLLLLL... and another path is LRLRRLRRRLRRRR....). Prove that $\aleph_0 < |P| \le c$, where $\aleph_0 = |\mathbb{Z}|$ and $c = |\mathbb{R}|$. (Bonus: prove $|P| = c$.)

Answer. It is clear that $|P| \ge \aleph_0$, simply because it is infinite. Now let's show it's strictly bigger by showing that it is not countable. Suppose to the contrary that we COULD enumerate all the paths in P in a list $p_1, p_2, p_3, \ldots$ If we take the first letter of p_1, the second letter of p_2, the third letter of p_3, and so on, we can form a new string of L/R characters by changing every L to an R and vice versa. The new path differs from each p_i in the ith position, so it not on the list, which means that the list was incomplete. So we've shown $|P| > \aleph_0$.

Now let's show that $|P| \le c$, by showing that there's an injection from $P \to \mathbb{R}$. I can do this by putting a decimal point first, and then converting L/R characters to $0/1$ digits. That turns every string of L and R into a real number (for instance, it converts the sequence of all L into $.111\bar{1}$, which equals $1/9$ in decimal).

Bonus: To show that it has the *same* cardinality as the reals, I can just interpret the $0/1$ digits in binary, and I get all of $[0, 1]$, which has cardinality c. (Note that there are some details to fuss with here, because $.10000\bar{0} = .0111\bar{1}$, but the redundancy is countable and so not that serious.)

(W3) Prove that $|(0, 1)| = |[0, 1]|$, or in other words, there is a bijection between the open interval and the closed interval. (Hint: by Schroeder-Bernstein, it suffices to find injections both ways.)

Answer. First direction: Since $(0, 1)$ is a subset of $[0, 1]$, the inclusion $f(x) = x$ is an injection $(0, 1) \to [0, 1]$.

Second direction: The function $g(x) = \frac{1}{4} + \frac{1}{2}x$ is an injection $[0, 1] \to (0, 1)$ with image $[\frac{1}{4}, \frac{3}{4}]$.

Since there are injections both ways, Schroeder-Bernstein guarantees the existence of a bijection.