MATH 61-02: WORKSHEET 9 (§7.1-2)

(W1) (a) For a vertex v in a graph $G = (V, E)$, let $\deg(v)$ be the *degree* of v, which is the number of times v appears as the endpoint of an edge. (So loops count double.) What is the smallest possible graph with at least one loop where every vertex has odd degree?

(b) Let K_n be the complete graph on n vertices (the simple graph where each vertex is connected by one edge to each other vertex). Sketch K_4, K_5, and K_6. For general n, what is $|E(K_n)|$ and what is the degree of each vertex? (Note: we discussed this in class.)

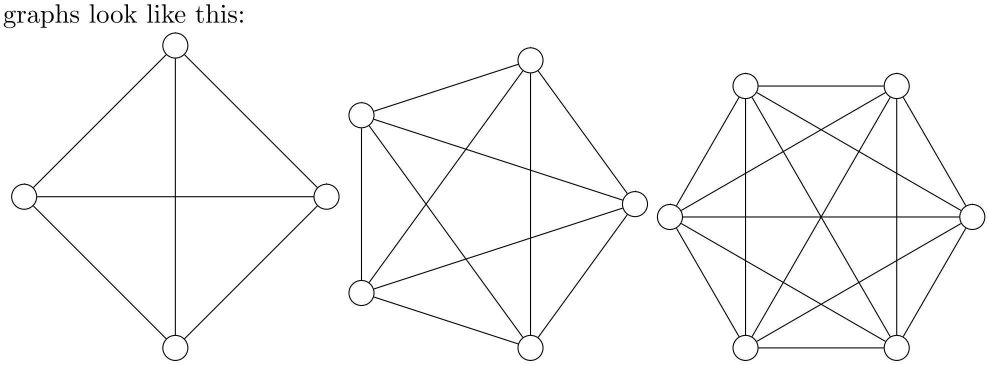
(c) Prove that $\sum_{v \in V} \deg(v) = 2|E|$ for any graph.

(d) Is it possible for a graph to have 11 vertices, all of which have degree 3?

(e) Is it possible for a graph to have 19 vertices, each of which have degree 1, 5, or 9? (Hint: consider the degree sum mod 4.)

Answer. (a) A graph on one vertex v with any number of loops must have $\deg(v)$ even, so our graph must have at least two vertices. It is easy to see that if we have two vertices v_0, v_1 with a loop at v_0 and an edge from v_0 to v_1, this satisfies the given constraint.

(b) The graphs look like this:



In general, $|E(K_n)| = \binom{n}{2}$, since an edge is uniquely defined by two distinct vertices and so it suffices to count all pairs of points in K_n. For each vertex $v \in K_n$, $\deg(v) = n - 1$, since we connect v to every point aside from itself.

(c) Suppose for a graph G we take the sum $\sum_{v \in V} \deg(v)$. This adds up the degrees of all vertices, which counts the edge-ends. Since every edge has two of those, this double-counts the edges, so adds up to $2|E|$.

(d) No: Suppose it did. Then we would have that $\sum \deg(v) = 11 \cdot 3 = 33 = 2|E|$, but it doesn't make sense to have $16\frac{1}{2}$ edges.

(e) No: Suppose it did, and let a, b, and c be the number of vertices with degree $1, 5, 9$, respectively. Then we would have that $\sum \deg(v) = 1a + 5b + 9c$, for some a, b, c satisfying $a + b + c = 19$. But let's consider everything mod 4. Since $5 \equiv 1 \pmod 4$ and $9 \equiv 1 \pmod 4$, we can reduce and obtain $a + 5b + 9c \equiv a + b + c$. Now we see that the degree sum has the same remainder as $19 \equiv 1 \pmod 4$, and that's impossible because it has to be even.

Actually, it's nice to practice your modular arithmetic and all, but there's a much simpler way to think about this: if all the degrees are odd, and you add up 19 odd numbers, you'll get an odd total. But we know that the total should be even!

(W2) The practice test for the second midterm involved the relation $R = \{(x,x),(x,y),(y,y),(y,x),(z,z)\}$ on $S = \{x,y,z\}$; you were supposed to check that it is transitive. Draw the graph associated to the relation, and use the adjacency matrix to verify transitivity.

Answer. The graph associated to this relation is a digraph consisting of three vertices x, y, z with directed loops at every vertex and directed edges going from x to y and y to x.

The adjacency matrix A of R is as follows:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We can compute that the matrix A^2 is then as follows:

$$\begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The transitivity test (Theorem 7.2.7) says that if the A^2 matrix has any nonzero entries, then the A matrix must also be nonzero in the corresponding places. There are six such positions, and the test succeeds, so we've verified transitivity.

Quick reminder of why this test works: those six nonzero entries of A^2 are telling you about paths of length 2 between particular vertices. For instance $(A^2)_{1,2} = 2$ tells you that there are two paths from v_1 to v_2 of length two. But such a path tells us that there's some (v_1, b) and also (b, v_2) in the data of the relation, and transitivity recognizes this as a "chain" and requires that (v_1, v_2) is also present.

(W3) Consider the Fig 7.7 in the book (p253), which shows airline connections between cities.

(a) Ordering the cities alphabetically, write the adjacency matrix A for the graph.
(b) Show by hand that it is possible to get from Seattle to Miami in exactly 10 flights.
(c) Using a computer (or by hand if you want), compute A^8 and A^{10}. How many ways are there to get from Seattle to Miami in 10 flights? How about Chicago to Atlanta in 8 flights? Explain the connection.

Answer. (a) The adjacency matrix A is as follows:

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) Here's one sample schedule: SEA-CHI, CHI-NYC, NYC-ATL, ATL-CHI, CHI-ATL, ATL-CHI, CHI-ATL, ATL-CHI, CHI-ATL, ATL-MIA. (I wouldn't want to fly this route, though!)

(c) We have that A^8 is the following matrix:

$$\begin{pmatrix} 2069 & 2324 & 1790 & 1382 & 606 & 1378 & 772 \\ 2324 & 2847 & 2150 & 1546 & 772 & 1622 & 850 \\ 1790 & 2150 & 1709 & 1216 & 610 & 1306 & 696 \\ 1382 & 1546 & 1216 & 931 & 412 & 940 & 528 \\ 606 & 772 & 610 & 412 & 226 & 461 & 235 \\ 1378 & 1622 & 1306 & 940 & 461 & 1004 & 543 \\ 772 & 850 & 696 & 528 & 235 & 543 & 308 \end{pmatrix}$$

and that A^{10} is the following matrix:

$$\begin{pmatrix} 19628 & 22704 & 17603 & 13166 & 6098 & 13489 & 7391 \\ 22704 & 27123 & 20880 & 15209 & 7391 & 15883 & 8492 \\ 17603 & 20880 & 16351 & 11873 & 5775 & 12492 & 6717 \\ 13166 & 15209 & 11873 & 8856 & 4114 & 9111 & 4997 \\ 6098 & 7391 & 5775 & 4114 & 2069 & 4393 & 2324 \\ 13489 & 15883 & 12492 & 9111 & 4393 & 9564 & 5171 \\ 7391 & 8492 & 6717 & 4997 & 2324 & 5171 & 2847 \end{pmatrix}$$

Noting that Chicago is represented in the second row of the matrix and Atlanta is represented in the first row of the matrix, the number of ways to get from Chicago to Atlanta in 8 flights is the number in the second row and first column of A^8 - namely, 2324. Similarly, noting the Seattle is represented in the last row of the matrix and Miami is represented in the fifth row of the matrix, the number of ways to get from Seattle to Miami in 10 flights is also 2324.
Why are these two numbers the same? Well, for any path of length 10 in this graph from Seattle to Miami, it's clear that the first edge must go from Seattle to Chicago and the last edge must go from Atlanta to Miami. But then this forms a natural bijection between walks of length ten from Seattle to Miami and walks of length eight from Chicago to Atlanta - for any walk of length ten from Seattle to Miami, we can remove the first and final edge to get a walk of length eight from Chicago to Atlanta, and for any walk of length eight from Chicago to Atlanta, we can add an edge before the first step of the walk from Seattle to Chicago and an edge after the last step of the walk from Atlanta to Miami to get a walk of length ten from Seattle to Miami.