MATH 61-02: WORKSHEET 2 (§1.3-1.7)

(W1) In class, you have been learning the building blocks of logic. In this problem, we explore some of the *replacement rules* of propositional logic. These are steps that allow us to replace a logical sentence with an equivalent one, knowing that the truth value remains intact in the course of the replacement.[1]

(a) One replacement rule says that we can replace $P \wedge (Q \vee R)$ with $(P \wedge Q) \vee (P \wedge R)$. Verify that these two statements are equivalent using a truth table.

(b) Another replacement rule says that $P \Leftrightarrow Q$ can be replaced by $(P \wedge Q) \vee (\sim P \wedge \sim Q)$. Verify that this is true through a truth table, and explain in words why this would be true.

Answer. (a)

P	**Q**	**R**	$P \wedge (Q \vee R)$	$(P \wedge Q) \vee (P \wedge R)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

(b) The truth table has four rows, and the two expressions only come up true when P and Q are both true or both false. This replacement rule makes sense since the only combinations of truth values where $P \Leftrightarrow Q$ is true are those where P and Q have the same truth value. If both true, then $P \wedge Q$ is true. If both false, then $\sim P \wedge \sim Q$ is true.

(W2) Calculus is about limits. However, you've probably never seen a formal definition. In this problem, we look at a mathematically quantified definition of the limit of a sequence.

(a) Let $(x_n) = x_1, x_2, \dots$ be a sequence of real numbers. We call x the *limit* of the sequence (x_n) if the following condition holds:

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \quad \text{s.t.} \quad \forall n > N, \quad |x_n - x| < \epsilon.$$

Parse this statement in English and give some kind of intuitive explanation as to what this definition means.

(b) Using quantifiers, give the negation of the above statement.

(c) This definition of a limit can be expanded to define the limit of a function: Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then the precise mathematical meaning of $\lim_{x \to c} f(x) = L$ is

$$\forall \epsilon > 0, \quad \exists \delta > 0 \quad \text{s.t.} \quad |x - c| < \delta \implies |f(x) - L| < \epsilon.$$

Can you write down a similar expression for $\lim_{x \to \infty} f(x) = L$?

(d) Use quantifiers to express Fermat's Last Theorem:
No integers a, b, c satisfy the equation $a^n + b^n = c^n$ for any integer exponent n greater than 2.

[1]FYI, the rules discussed here are called *distribution*, *material equivalence*, *material implication*, and *exportation*, respectively.

Answer. (a) It says: for any positive ϵ, no matter how small, there's some threshold N such that for indices n bigger than the threshold, the difference between x_n and x is less than the tiny little ϵ. The idea is that I can force the x_n values to be as close as I want to the limit x just by waiting long enough in the sequence.

(b) $\exists \epsilon > 0 \quad \text{s.t.} \quad \forall N \in \mathbb{N}, \quad \exists n > N \quad \text{with} \quad |x_n - x| \geq \epsilon.$

(c) $\forall \epsilon > 0, \quad \exists M \in \mathbb{R} \quad \text{s.t.} \quad \forall x > M, \quad |f(x) - L| < \epsilon.$

(d) There are lots of ways to do this.
For instance: $\forall n \in \mathbb{N} \quad \text{s.t.} \quad n > 2, \quad \nexists a, b, c \in \mathbb{Z} \quad \text{s.t.} \quad a^n + b^n = c^n.$
Or: $\quad \forall a, b, c \in \mathbb{Z}, \quad \forall n \in \mathbb{N} \quad \text{s.t.} \quad n > 2, \quad a^n + b^n \neq c^n.$

(W3) In class, you saw the Sheffer stroke, a logical operator ("NAND") defined as follows:

P	**Q**	**P ↑ Q**
T	T	F
T	F	T
F	T	T
F	F	T

Recall that $P \uparrow P = \sim P$. Work out expressions for both $P \wedge Q$ and $P \vee Q$ using only $P, Q, \uparrow$, and parentheses.

Answer. One can verify that

$$P \wedge Q \quad \equiv \quad (P \uparrow Q) \uparrow (P \uparrow Q)$$

and

$$P \vee Q \quad \equiv \quad (P \uparrow P) \uparrow (Q \uparrow Q).$$