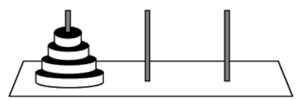
MATH 61-02: WORKSHEET 3 (§2.1)

(W1) Let S be a subset of $[9]$ such that $|S| = 6$. Show that $\exists a, b \in S$ s.t. $a + b = 10$.

Answer. Case 1: ($5 \in S$) In this case we are done, because $5 + 5 = 10$. (The setup does not require $a \neq b$.)
Case 2: ($5 \notin S$) The possible numbers are now partitioned as $\{\{4,6\},\{3,7\},\{2,8\},\{1,9\}\}$. As soon as I have both numbers from one of these pairs, I have numbers that sum to 10. Suppose I can choose numbers from $[9]$ without getting a pair that sum to ten. Then there is at most one from each of the four pairs, so at most four numbers in the set. Since S has 6 elements, this is impossible. (In the language of §2.3, this is an application of the pigeonhole principle.)

(W2) The *Towers of Hanoi* is a game consisting of 3 rods and n disks of different sizes, looking like this:



The disks start out in ascending size order on the left rod (as in the figure). The aim is to move all disks to the right rod while obeying the following rule: each move consists of taking the one top disk from one rod and moving it to another rod, while never placing a disk on top of a smaller disk. Practice with 2, 3, and 4 disks online (http://vornlocher.de/tower.html).

Prove by induction: If there are n disks, the game can be won in $2^n - 1$ moves.

(Bonus: convince yourself this is optimal.)

Answer. We prove that this is possible by induction on the number of disks. Let's let statement P_n be that *the game with n disks can be won in $2^n - 1$ moves.*

Base Case: For $n = 1$, it is clear that we can win the game in one move (by moving the one disk to the right rod on the first move). So we've proved P_1.

Inductive Hypothesis: Assume P_k for some particular k: suppose that for $n = k$ disks, we know that the game can be won in $2^k - 1$ moves,

Inductive Step: Using P_k, I will now deduce P_{k+1}. Suppose that we played with $k + 1$ disks. We can win the game as follows: move the top k disks to the center rod, however many steps that takes. Now move the bottom disk from the left to the right rod, which is one step. Now move the stack of k disks from the center to be on top of the big disk, however many steps that takes. Using the inductive hypothesis, the total number of moves we've used is at most

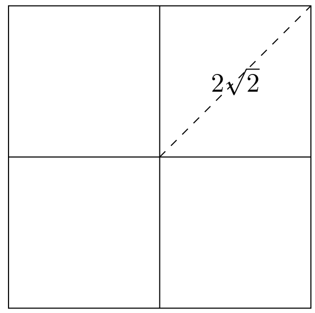
$$(2^k - 1) + 1 + (2^k - 1) = 2(2^k) - 1 = 2^{k+1} - 1,$$

which is exactly what we wanted to show.

Our induction is complete: we established P_1 and then proved $P_k \Rightarrow P_{k+1}$ for all $k \geq 1$. We knocked over the first (logical) domino, and then each domino knocks over the following one... so they all go down.

(W3) Suppose we place five points in a square of area 16. Prove that there exist two points which are a distance of at most $2\sqrt{2}$ apart. Can you come up with a similar statement for some number of points in a pentagon?

Answer. For the square, if we divide the square of area 16—hence of side length 4—into four smaller squares of side length 2, we have four regions. Each of them has diagonal $2\sqrt{2}$, so that means that any two points in the same 2×2 square will be at most $2\sqrt{2}$ apart.



But since we're dropping five points into four regions, some two points must fall in the same region by the Pigeonhole Principle! So we're done.

It is possible to come up with a variety of statements for a pentagon. One such statement is as follows: If we place six points in a regular pentagon of side length 1, there are two points which are a distance of at most 1 apart from each other. (You can prove this by dividing up the pentagon into 5 triangles!)

(W4) I went to this crazy party last weekend where they managed to cram all 5200 undergraduates at Tufts into a single house. Of course, since we didn't all know each other, some people shook hands to introduce themselves. I want to prove that there were two people at the party who shook hands with the same number of people. One possible solution starts like this: *Suppose not. Let's number the people 1 through 5200, and let $h(n)$ denote the number of people that person n shook hands with. Then the numbers $h(1), h(2), \ldots, h(5200)$ are 5200 different integers between 0 and 5199.* Finish the proof. (There are a few more steps to go.)

Answer. Suppose not. Let's number the people 1 through 5200, and let $h(n)$ denote the number of people that person n shook hands with. Then the numbers $h(1), h(2), \ldots, h(5200)$ are 5200 different integers between 0 and 5199. But there are only 5200 different integers between 0 and 5199 inclusive, so the numbers $h(1), h(2), \ldots, h(5200)$ are all different. In particular, there exist some people numbered p and q such that $h(p) = 0$ and $h(q) = 5199$. But by how we defined $h(n)$, this implies that the person numbered p shook hands with nobody, and the person numbered q shook hands with everybody except themselves, so p and q both shook hands and didn't shake hands with each other, which is a contradiction. (BOOM!)

So the numbers $h(1)$ through $h(5200)$ can't all be distinct, which means that some two people at the party shook hands with the same number of people.