## MATH 61-02: WORKSHEET 4 (§4.1-§4.3)

(W1) Everyone in your 43-person Discrete Math class travels back in time to 2016 and goes to either the Republican or the Democratic presidential primary. In the Republican presidential primary, Kasich, Trump, Rubio, and Cruz are running. In the Democratic presidential primary, Clinton, Sanders, and O'Malley are running.

(a) Suppose that everyone chooses one of the party primaries, and then chooses their favorite candidate running in that party. How many ways can the choices go?

(b) Suppose that everyone is asked to rank all seven candidates in order of preference. Moon asks Chris, "How many ways can the choices go?", to which he replies, "42(7!)". Is he right, and why?

(c) Suppose that the Democratic ballot asks voters to rank the Dem candidates in order of preference, but the Republican ballot requires a choice of top two Republicans, unranked. How many ways can the class's choices go?

(d) Suppose that everyone at both primaries is asked to rank the candidates running in that party in order of preference, but also asked to pick their least favorite candidate running from the other party. How many ways can the choices go now?

Answer:

(a) Everyone can choose which primary they go to, so each person has a total of $4+3 = 7$ candidates to choose from. Each person makes their choice independently, so the total number of ways the class can vote is $7^{42}$.

(b) The number of ways to rank the candidates is 7!, so the number of ways the class can rank them is $(7!)^{42}$. Chris is grievously wrong, probably because he added 7! to itself 42 times instead of multiplying it with itself. This is an AND situation rather than an OR situation, because everyone is making a choice—that's why you multiply.

(c) Anyone going to the Democratic primary has 3! possible preference rankings, and anyone going to the Republican primary has $\binom{4}{2}$ possible ways to choose their two favourite candidates unranked. Since you only get to go to one primary, it's an OR situation, and the number of ways a single person can vote is $3! + \binom{4}{2}$. Since each person makes their choice independently, the total number of ways the class can vote is $(3! + \binom{4}{2})^{42} = 12^{42}$.

(d) Anyone going to the Democratic primary has 3! possible Democratic rankings and 4 possible least favorite Republican candidates. Anyone going to the Republican primary has 4! possible rankings and 3 possible least favorite candidates. Since each person makes their choice independently, the total number of ways the class can vote is $(3! \times 4 + 4! \times 3)^{42} = (24 + 72)^{42} = 96^{42}$.

Great, now let's all agree to go back in time and do that one over, OK?

(W2) *Totó* is a form of gambling in Hungary. On one ticket, you can try to bet on the outcomes of 13 soccer matches in order - for each match, you bet whether the home team will win, the away team will win, or whether the match will end in a draw. A sample ticket looks like this:

| Match | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Bet | H | D | A | A | A | D | A | H | A | D | D | D | H |

(a) How many different tickets can you buy?
(b) How many different tickets can you buy where no two consecutive bets are the same?
(c) How many different tickets can you buy where you bet H exactly three times?
(d) How many different tickets can you buy where you bet that the home team will win exactly three times, the away team will win exactly six times, and the teams will draw exactly four times?
(e) How many different tickets can you buy where you bet that the home team will win either seven or eight times?
(f) Consider the string of letters formed by your bet (for instance, "HDAAADAHADDDH" on the ticket above). How many different tickets can you buy where your bet string is a palindrome (i.e., reading the same forwards or backwards)?
(g) Combine parts (e) and (f): How many different tickets can you buy where the bet string is a palindrome with either 7 or 8 H's?
(h) Suppose you're given a tip-off that the home team will not win any prime-numbered game and the away team will win any game whose number is a multiple of 3. How many different tickets are consistent with this information?

Answer:

(a) For each match, we can choose to bet on Home, Draw, or Away. There are 3 choices for 13 independent matches, so there are $3^{13}$ different tickets we can buy.
(b) If no two consecutive matches will have the same outcome, we have 3 choices to bet on for match 1, and then 2 choices to bet on for matches 2-13, since we can't bet on whatever we bet on in the previous numbered match. Hence there are $3 \cdot 2^{12}$ different tickets we can buy.
(c) We can pick 3 matches out of 13 to mark H in $\binom{13}{3}$ ways. For the remaining matches, we can freely choose D or A, so there are 2 choices to bet on for each of these 10 matches. Hence there are $\binom{13}{3} \cdot 2^{10}$ different tickets we can buy.
(d) We can pick which matches to mark H in exactly $\binom{13}{3}$ ways. For the remaining 10 matches, we can pick which matches to bet A in exactly $\binom{10}{6}$ ways. This will uniquely determine the ones to mark D. Hence there are $\binom{13}{3}\binom{10}{6}$ different tickets we can buy. (Note that there are several possible ways to answer this that look different but come out the same.)
(e) First, we can pick 7 H's in $\binom{13}{7}$ ways, leaving a 2-way choice in the remaining 6 matches. Similarly, we can bet H eight times in $\binom{13}{8}$ ways, leaving a 2-way choice in the remaining 5 matches. This gives us $\binom{13}{7} \cdot 2^6 + \binom{13}{8} \cdot 2^5$ different tickets we can buy.
(f) We can bet whatever we want for matches 1–7; being a palindrome means that this determines the bets in matches 8–13. Hence there are $3^7$ different palindromic bet strings.
(g) Since this bet string needs to be a palindrome, it is determined by the bets in matches 1–7, just like in the last part. First consider how to get 7 H's. Since this is an odd number, the middle bet must be an H, and three of the first six must be H as well—then we can pick A or D freely in the remaining three. That makes $\binom{6}{3} \cdot 2^3$ possibilities. Similarly, if we bet that the home team will win eight matches, we need to bet that the home team will win four of the first six matches, and for the other two of the first six, plus match 7, we can choose freely. Hence there are $\binom{6}{3} \cdot 2^3 + \binom{6}{4} \cdot 2^3$ different tickets we can buy.
(h) For matches 3, 6, 9, and 12, we know the away team will win. For matches 2, 5, 7, 11, and 13, we have two choices: the away team will win or the teams will draw. (Notice that 3 is prime, but we already know the outcome for that match.) For the remaining matches - 1, 4, 8, and 10 - we do not know anything and still have three outcomes to choose from. Hence we have $2^5 \cdot 3^4$ different tickets we can buy.

(W3) (a) Let's say that a "diagonal" of a polygon is a straight line from vertex to vertex that is not equal to an edge of the polygon. For instance, a triangle has no diagonals and a square has two. How many diagonals does a regular $n$-gon have (where $n \geq 3$)? Prove your answer using:

(i) counting. (Hint: each vertex is connected to how many others? And how many times does this count each edge?)

(ii) induction.

(b) Suppose that $m, n \geq 2$. Consider $S = \{(p, q) \mid 1 \leq p \leq m, 1 \leq q \leq n\}$, an $m \times n$ array of lattice points. Moon asks Chris, "How many ways are there to choose four points from this array that are the corners of a rectangle with sides parallel to the $x$– and $y$–axes?" Chris answers "I count $\binom{m}{2} \cdot \binom{n}{2}$." Is he right, and why?

Answer:

(a) (i) Label the points 1 through $n$. Then vertex 1 can form a diagonal with vertices $3, 4, \ldots, n-1$, which means there are $n-3$ diagonals coming out of it. So there must be $n-3$ diagonals coming out of each vertex. But if we write $n(n-3)$, this will count each diagonal twice, so we must divide by 2 to correct the overcounting. Hence there are $\frac{n(n-3)}{2}$ diagonals in a convex $n$-gon.

(ii) Let $D_n$ be the statement that *there are* $\frac{n(n-3)}{2}$ *diagonals in a convex $n$-gon*; we want to prove $D_n \quad \forall n \geq 3$. We already discussed $n = 3$ and $n = 4$, which handles the base case.

Inductive hypothesis: Now suppose we knew that for a convex $k$-gon, for some paticular $k \geq 3$, there are $\frac{k(k-3)}{2}$ diagonals.

Inductive step: Take a regular $(k+1)$-gon, and number its vertices 1 through $k+1$. Note that the vertices 1 through $k$ form a $k$-gon; it's not regular, but that does not affect its number of diagonals. Color those diagonals red just to keep track of them. How many additional diagonals does the $(k+1)$-gon have? It's got the one between vertex 1 and vertex $k$, plus the ones that connect vertex $k+1$ to each of the vertices numbered 2 through $k-1$; that is $k-1$ new diagonals in all. But now, our $(k+1)$-gon has $\frac{k(k-3)}{2} + (k-1) = \frac{k^2-3k+2k-4+2}{2} = \frac{k^2-k-2}{2} = \frac{(k+1)(k-2)}{2}$ diagonals, so induction is complete and we have the desired result.

(b) This problem just boils down to picking two rows and picking two columns! Then you select the four vertices at the intersections of the rows and the columns. There are clearly $\binom{m}{2} \cdot \binom{n}{2}$ ways to do this, so Chris is right as rain.