Name

Math 61-02: Second Midterm

(1) [20 mins] Consider a set $S = \{x, y, z\}$. As you know, a *relation* is any $R \subseteq S \times S$. (For instance, *equality* is the relation $R_0 = \{(x, x), (y, y), (z, z)\}$, which has three elements.)

That means that the set of all relations on S is $\mathcal{R} = \mathcal{P}(S \times S)$. Note that the relations themselves form a poset, ordered by $\subseteq$.

(a) Does the poset $(\mathcal{R}, \subseteq)$ have a minimum and a maximum?

(b) How many different relations are there on S?

(c) Give an example of a symmetric relation on S with five elements.

(d) Give an example of a non-symmetric relation on S with one element.

(e) What is the definition of a *function* $f : S \to S$? What is the probability that a random relation on S is a function?

(f) Briefly verify that $R = \{(x, x), (x, y), (y, y), (y, x), (z, z)\}$ is an equivalence relation. What is the cardinality of the quotient space S/R?

(g) Prove that if R_1 is a reflexive relation and $R_1 \subseteq R_2$, then R_2 is reflexive as well.

(2) [16 mins]

(a) Let $W = \{a, b, \ldots, y, z\}$ be the western alphabet and let $G = \{\alpha, \beta, \ldots, \phi, \omega\}$ be the Greek alphabet; they have 26 and 24 letters, respectively. How many functions $f : W \to G$ are surjections with $|f^{-1}(\omega)| = 3$?

(b) Let g be a function from $\{N, S, E, W\}$ to $\{\heartsuit, \clubsuit, \diamondsuit, \spadesuit\}$. Prove that if g is injective, then g is bijective.

(c) Let $\mathcal{F}$ be the set of all finite sets. Consider the relation Ⓑ on $\mathcal{F}$ that is defined by X Ⓑ Y if and only if there exists a bijection from X to Y. It is clearly reflexive and symmetric. Briefly explain why it is transitive and describe the quotient space $\mathcal{F}/$Ⓑ .

(d) Suppose an equivalence relation on a sphere identifies all points on the equator with each other, and otherwise each point is alone in its equivalence class. Describe the quotient space, using pictures.

(3) [8 mins] Here is an incorrect proof that relations that are symmetric and transitive must be reflexive.

Suppose a relation $*$ *is symmetric and transitive. By symmetry,* $x * y$ *and* $y * x$*. Since* $*$ *is also transitive, it follows that* $x * x$*. So the relation is reflexive.*

Find all of the errors in this proof.

(4) [16 mins]

(a) How many solutions are there to $n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_7 + n_8 + n_9 + n_{10} = 100$ with all $n_i \in \mathbb{N}$?

(b) How many solutions to the same equation where all the n_i are odd natural numbers?

(c) How about if all the n_i are natural numbers and at least two of them are required to be ≥ 10?