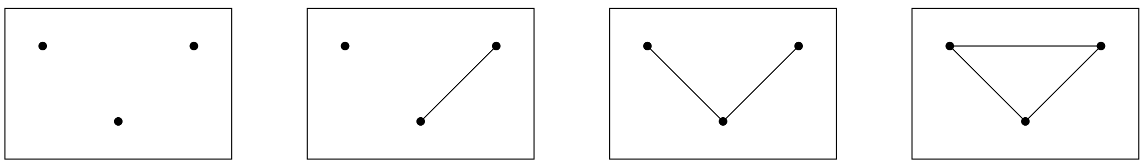
MATH 61-02: WORKSHEET 11 (GRAPH ISOMORPHISM)

Let's say that two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic*, denoted $G_1 \cong G_2$, if there is a bijection $\sigma : V_1 \to V_2$ such that for all $v, w \in V_1$,

$$\sigma(v), \sigma(w) \text{ is an edge in } E_2 \iff v, w \text{ is an edge in } E_1.$$

There are four different isomorphism classes of simple graphs with three vertices:

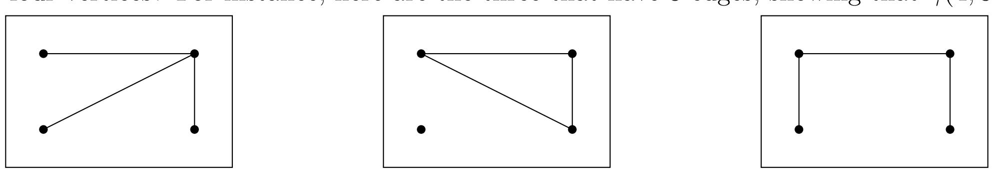


Let $\gamma(n, m)$ be the number of isomorphism types of simple graphs on n vertices with m edges, and let

$$\Gamma(n) = \sum_{m=0}^{\binom{n}{2}} \gamma(n, m)$$

be the total number of isomorphism types of graphs with n vertices. So far we've seen that $\Gamma(3) = 1 + 1 + 1 + 1 = 4$.

How about four vertices? For instance, here are the three that have 3 edges, showing that $\gamma(4, 3) = 3$.

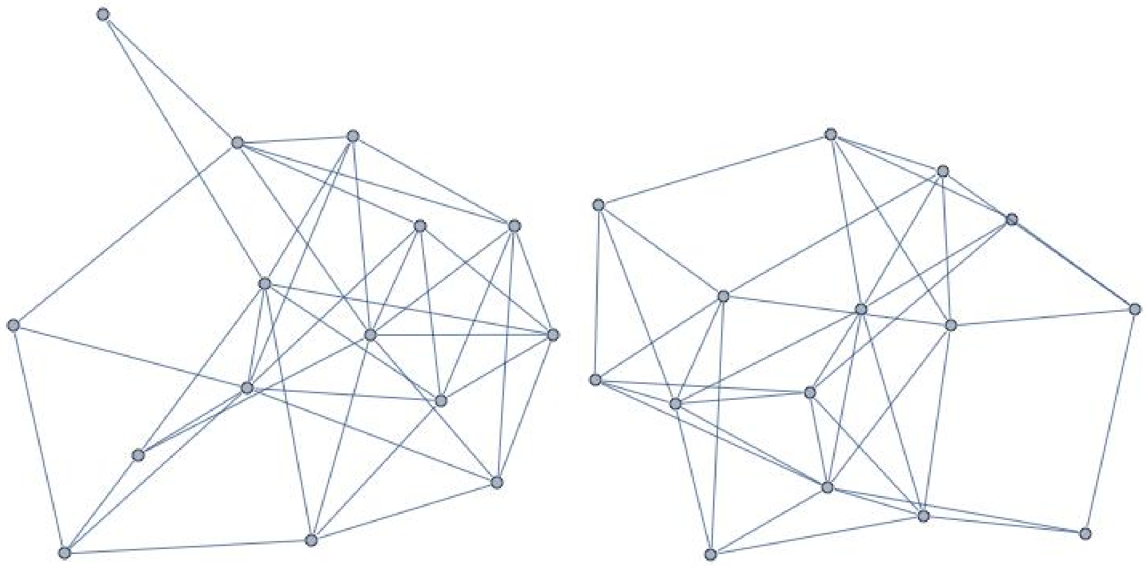


If we studied all the possibilities we would find $\Gamma(4) = 1 + 1 + 2 + 3 + 2 + 1 + 1 = 11$.

(W1) There is some symmetry there, just like for Pascal's triangle. Prove that $\gamma(n, m) = \gamma(n, \binom{n}{2} - m)$ for any $n \geq 2$ and $0 \leq m \leq \binom{n}{2}$.
(Hint: the *complement* of a graph, which we can denote G^c, has the same vertices as G but the opposite edges—that is, G^c has an edge between two vertices if and only if G does not!)

(W2) Compute $\Gamma(5)$. That is, classify all five-vertex simple graphs up to isomorphism. (Hint: the answer is between 30 and 40.)

(W3) Here are two graphs, G_1 and G_2 (15 vertices each). Let's analyze them.



The following table shows the adjacency matrices A_i for these graphs on the top row, then A_i^2 on the second row and A_i^3 on the third row.

$$\begin{pmatrix}
0&1&0&1&0&1&1&0&0&0&1&1&1&0&0\\
1&0&0&1&1&0&1&0&0&0&0&1&0&0&1\\
0&0&0&0&1&1&0&1&0&0&1&0&0&1&1\\
1&1&0&0&1&0&0&1&1&0&0&0&0&0&1\\
0&1&1&1&0&0&0&1&1&0&1&0&0&0&0\\
1&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&1&1&1&0&0&0&0&0\\
0&0&1&1&1&0&1&0&1&0&1&0&1&0&1\\
0&0&0&1&1&0&1&1&0&0&0&1&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&1&1&1&0\\
1&0&1&0&1&0&0&1&0&0&0&1&0&0&0\\
1&1&0&0&0&0&0&0&1&1&1&0&1&1&1\\
1&0&0&0&0&0&0&1&0&1&0&1&0&0&0\\
0&0&1&0&0&0&0&0&0&1&0&1&0&0&0\\
0&1&1&1&0&0&0&1&0&0&0&1&0&0&0
\end{pmatrix}
\qquad
\begin{pmatrix}
0&1&0&1&0&1&0&0&0&1&0&1&0&0&0\\
1&0&1&0&0&0&0&0&0&1&1&1&1&1&1\\
0&1&0&0&0&0&0&1&0&0&1&0&0&0&0\\
1&0&0&0&0&1&1&0&0&0&0&1&1&1&0\\
0&0&0&0&0&1&1&0&1&0&0&0&0&1&1\\
1&0&0&1&1&0&0&0&0&0&0&1&0&0&0\\
0&0&0&1&1&0&0&1&0&0&0&0&0&1&1\\
0&0&1&0&0&0&1&0&1&0&0&0&0&1&0\\
0&0&0&0&1&0&0&1&0&1&0&0&0&0&1\\
1&1&0&0&0&0&0&0&1&0&1&1&0&0&1\\
0&1&1&0&0&0&0&0&0&1&0&0&1&1&1\\
1&1&0&1&0&1&0&0&0&1&0&0&1&0&1\\
0&1&0&1&0&0&0&0&0&0&1&1&0&0&0\\
0&1&0&1&1&0&1&1&0&0&1&0&0&0&0\\
0&1&0&0&1&0&1&0&1&1&1&1&0&0&0
\end{pmatrix}$$

$$\begin{pmatrix}
7&3&2&1&3&0&1&4&3&3&1&3&1&1&3\\
3&6&2&3&1&1&1&4&4&2&3&2&2&1&2\\
2&2&6&3&2&0&1&3&2&1&2&3&1&0&1\\
1&3&3&6&3&1&4&3&2&0&3&4&2&0&2\\
3&1&2&3&6&1&3&4&2&0&2&3&1&1&4\\
0&1&0&1&1&2&1&1&0&0&2&1&1&1&1\\
1&1&1&4&3&1&5&1&1&0&2&4&3&1&2\\
4&4&3&3&4&1&1&8&3&2&2&4&0&1&2\\
3&4&2&2&2&0&1&3&5&2&3&0&2&1&3\\
3&2&1&0&0&0&0&2&2&4&1&2&1&1&1\\
1&3&2&3&2&2&2&2&3&1&5&1&3&2&3\\
3&2&3&4&3&1&4&4&0&2&1&8&2&1&1\\
1&2&1&2&1&1&3&0&2&1&3&2&4&2&2\\
1&1&0&0&1&1&1&1&1&1&2&1&2&3&2\\
3&2&1&2&4&1&2&2&3&1&3&1&2&2&5
\end{pmatrix}
\qquad
\begin{pmatrix}
5&2&1&2&1&2&1&0&1&2&2&4&3&2&3\\
2&8&1&4&2&2&2&2&2&4&5&4&2&1&3\\
1&1&3&0&0&0&1&0&1&2&1&1&2&3&2\\
2&4&0&6&3&2&1&2&0&2&2&3&1&1&2\\
1&2&0&3&5&0&2&3&1&2&2&2&0&1&2\\
2&2&0&2&0&4&2&0&1&2&0&2&2&2&2\\
1&2&1&1&2&2&5&1&3&1&2&2&1&3&1\\
0&2&0&2&3&0&1&4&0&1&2&0&0&1&2\\
1&2&1&0&1&1&3&0&4&1&2&2&0&2&2\\
2&4&2&2&2&2&1&1&1&6&2&3&3&2&4\\
2&5&1&2&2&0&2&2&2&2&6&4&1&1&2\\
4&4&1&3&2&2&2&0&2&3&4&7&2&2&2\\
3&2&2&1&0&2&1&0&0&3&1&2&4&3&3\\
2&1&3&1&1&2&3&1&2&2&1&2&3&6&4\\
3&3&2&2&2&2&1&2&2&4&2&2&3&4&7
\end{pmatrix}$$

$$\begin{pmatrix}
10&18&12&23&14&9&20&15&12&6&19&22&17&8&13\\
18&12&12&20&22&5&19&18&11&6&12&23&11&6&17\\
12&12&8&12&18&8&10&18&12&5&16&11&9&10&17\\
23&20&12&14&20&4&9&25&20&10&14&13&8&7&19\\
14&22&18&20&14&5&10&23&19&8&18&14&10&5&13\\
9&5&8&4&5&0&2&7&5&4&3&6&2&1&4\\
20&19&10&9&10&2&4&21&17&13&10&11&6&5&11\\
15&18&18&25&23&7&21&18&20&6&23&18&18&9&22\\
12&11&12&20&19&5&17&20&8&4&10&23&8&4&11\\
6&6&5&10&8&4&13&6&4&4&8&15&11&7&7\\
19&12&16&14&18&3&10&23&10&8&8&21&5&4&11\\
22&23&11&13&14&6&11&18&23&15&21&12&17&13&21\\
17&11&9&8&10&2&6&18&8&11&5&17&4&4&7\\
8&6&10&7&5&1&5&9&4&7&4&13&4&2&3\\
13&17&17&19&13&4&11&22&11&7&11&21&7&3&8
\end{pmatrix}
\qquad
\begin{pmatrix}
12&22&4&17&9&12&8&5&6&17&13&19&10&8&13\\
22&22&15&13&10&12&12&6&11&24&19&25&21&23&27\\
4&15&2&8&7&2&5&8&4&7&13&8&3&3&7\\
17&13&8&10&6&14&14&2&9&13&10&19&15&18&15\\
9&10&7&6&6&11&14&4&12&10&7&10&9&17&16\\
12&12&2&14&11&6&6&5&4&9&10&16&6&6&9\\
8&12&5&14&14&6&8&12&5&11&9&9&7&13&17\\
5&6&8&2&4&5&12&2&10&6&6&7&6&14&9\\
6&11&4&9&12&4&5&10&4&13&8&7&6&8&15\\
17&24&7&13&10&9&11&6&13&16&21&23&11&12&19\\
13&19&13&10&7&10&9&6&8&21&12&14&17&19&23\\
19&25&8&19&10&16&9&7&7&23&14&20&18&15&24\\
10&21&3&15&9&6&7&6&6&11&17&18&6&5&9\\
8&23&3&18&17&6&13&14&8&12&19&15&5&8&12\\
13&27&7&15&16&9&17&9&15&19&23&24&9&12&16
\end{pmatrix}$$

(a) Explain how to use an adjacencey matrix A to list all of the degrees of all the vertices of the graph G. Make that list for each graph on the last page.

(b) Explain how to use the matrix A^2 to get a different method of listing all of the vertex degrees. Check that you get the same list for each graph as you did in the last part.

(c) Show that the two graphs have the same total number of edges. ($|E_1| = |E_2|$)

(d) How many triangles are there in each graph?

(e) Is it possible that these graphs are isomorphic?

Extra credit: write some code (or pseudo-code) that starts with two adjacency matrices and decides whether the graphs are isomorphic. Don't just use brute force: incorporate your answers from the previous problem to make your code more efficient.