MATH 61-02: PRACTICE PROBLEMS FOR FINAL EXAM

(FP1) The *exclusive or* operation, denoted by $\oplus$ and sometimes known as XOR, is defined so that $P \oplus Q$ is true iff P is true or Q is true, but not both. Prove (through a truth table, or otherwise) that for any statements P, Q, R:
(a) $((P \oplus Q) \oplus R) \Leftrightarrow (P \oplus (Q \oplus R))$
(b) $(P \wedge (Q \oplus R)) \Leftrightarrow ((P \wedge Q) \oplus (P \wedge R))$
(Suggestion: first look at the expressions and analyze them to see what combination of P, Q, R is possible, then use a truth table to confirm your idea.)

Answer. One could show both sentences were true through a truth table, either through showing the entire sentence given in the problem is a tautology (so every row of the truth table is true) or showing that for each part, the sentences on either side of the $\Leftrightarrow$ have the same truth value. These truth tables should read:

P	**Q**	**R**	$((P \oplus Q) \oplus R)$	$(P \oplus (Q \oplus R))$	$(P \wedge (Q \oplus R))$	$((P \wedge Q) \oplus (P \wedge R))$
T	T	T	T	T	F	F
T	T	F	F	F	T	T
T	F	T	F	F	T	T
T	F	F	T	T	F	F
F	T	T	F	F	F	F
F	T	F	T	T	F	F
F	F	T	T	T	F	F
F	F	F	F	F	F	F

Alternately, here is an argument without truth tables:
(a) If $((P \oplus Q) \oplus R)$, we have that either (a) exactly one of P or Q is true, or (b) R is true, but not both (a) and (b). This means that either precisely one of P, Q, R must be true or all three or P, Q, R are true if $((P \oplus Q) \oplus R)$ is true, and by the same argument, $(P \oplus (Q \oplus R))$ is true if precisely one of P, Q, R is true or all three of P, Q, R are true. The desired statement follows.
(b) $(P \wedge (Q \oplus R))$ is true iff P is true and precisely one of Q and R is true; this is true iff precisely one of $P \wedge Q$ and $P \wedge R$ is true, i.e., iff $((P \wedge Q) \oplus (P \wedge R))$ is true.

(FP2) Prove the following:
(a) $1 \cdot 2 + 2 \cdot 3 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}$ for all $n \in \mathbb{N}$
(b) $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$ for $n \geq 2$
(c) If $A_1, A_2, \ldots, A_n, B$ are sets ($n \geq 1$), then $\bigcup_{i \in [n]} (A_i \setminus B) = (\bigcup_{i \in [n]} A_i) \setminus B$.
(d) Let F_n denote the nth term of the Fibonacci sequence (where $F_1 = 1, F_2 = 1$, and $F_k = F_{k-2} + F_{k-1}$ $\forall k \geq 3$). Show that $\forall n \in \mathbb{N}$, $F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$.

Answer. All of these can be proven by induction.
(a) Base case: For $n = 1$, it is easy to see that $1 \cdot 2 = \frac{1 \cdot 2 \cdot 3}{3}$. ✓ Inductive hypothesis: Now suppose that our statement is true for $n = k$, that is, $1 \cdot 2 + 2 \cdot 3 + \cdots + k \cdot (k+1) = \frac{k(k+1)(k+2)}{3}$. Then we have that

$$1 \cdot 2 + 2 \cdot 3 + \cdots + k \cdot (k+1) + (k+1) \cdot (k+2) = \frac{k(k+1)(k+2)}{3} + (k+1) \cdot (k+2)$$
$$= \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3} = \frac{(k+1)(k+2)(k+3)}{3},$$

and so if our statement is true for $n = k$, it must also be true for $n = k+1$. Hence our statement is true for all $n \in \mathbb{N}$ by induction.

(b) Base case: For $n = 2$, we can easily see that $1 + \frac{1}{4} < 2 - \frac{1}{2}$. ✓ Inductive hypothesis: Now suppose our statement is true for $n = k$, that is, $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2} < 2 - \frac{1}{k}$. Then we must have that $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$. Now, I claim that for $k \geq 2$, we must have that $\frac{1}{k} \geq \frac{1}{k+1} + \frac{1}{(k+1)^2}$: Since $1 > 0$, we can add $k^2 + 2k$ to both sides to get $k^2 + 2k + 1 > k^2 + 2k$, but then $(k+1)^2 > k(k+2)$ and so $\frac{1}{k} > \frac{k+2}{(k+1)^2} = \frac{1}{k+1} + \frac{1}{(k+1)^2}$. But then it immediately follows that $2 - \frac{1}{k} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1}$, so if our statement is true for $n = k$, it must also be true for $n = k+1$, and hence our statement is true for all $n \geq 2$ by induction.

(c) Base case: For $n = 1$, we have that $A_1 \setminus B = A_1 \setminus B$. ✓ Now suppose that for some $n = k$, $\bigcup_{i \in [k]} (A_i \setminus B) = (\bigcup_{i \in [k]} A_i) \setminus B$. Then $\bigcup_{i \in [k+1]} (A_i \setminus B) = ((\bigcup_{i \in [k]} A_i) \setminus B) \cup (A_{k+1} \setminus B)$. I claim that $((\bigcup_{i \in [k]} A_i) \setminus B) \cup (A_{k+1} \setminus B) = (\bigcup_{i \in [k+1]} A_i) \setminus B$:

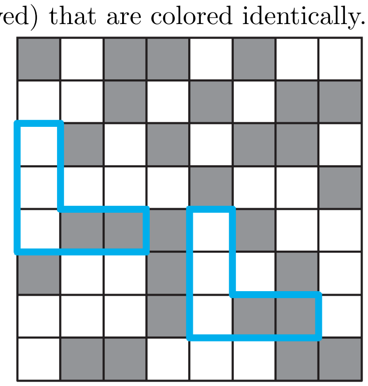
- Let $x \in ((\bigcup_{i \in [k]} A_i) \setminus B) \cup (A_{k+1} \setminus B)$. Then either $x \in (\bigcup_{i \in [k]} A_i) \setminus B$, or $x \in A_{k+1} \setminus B$. In the latter case, x would be in A_{k+1} but not B, so x must also be an element of $(\bigcup_{i \in [k+1]} A_i) \setminus B$. In the former case, x is in some A_i for $1 \leq i \leq k$, and is not in B, so x must likewise be an element of $(\bigcup_{i \in [k+1]} A_i) \setminus B$. Hence $((\bigcup_{i \in [k]} A_i) \setminus B) \cup (A_{k+1} \setminus B) \subseteq (\bigcup_{i \in [k+1]} A_i) \setminus B$.
- Let $x \in (\bigcup_{i \in [k+1]} A_i) \setminus B$. By definition of union, x is some element of A_i for some $1 \leq i \leq k+1$ and also $x \notin B$. Hence, if $x \notin (\bigcup_{i \in [k]} A_i) \setminus B$, then the only A_i that x could be an element of is A_{k+1}, and in particular it must be an element of $A_{k+1} \setminus B$. Thus $x \in ((\bigcup_{i \in [k]} A_i) \setminus B) \cup (A_{k+1} \setminus B)$ and $((\bigcup_{i \in [k]} A_i) \setminus B) \cup (A_{k+1} \setminus B) \supseteq (\bigcup_{i \in [k+1]} A_i) \setminus B$.

So after assuming the statement for $n = k$, we proved it for $n = k+1$, and hence our statement is true for all $n \in \mathbb{N}$ by induction.

(d) Base case: for $n = 1$, we have that $F_1^2 = F_1 F_2 = 1$. ✓ Now suppose that for some $n = k$ we had that $F_1^2 + F_2^2 + \cdots + F_k^2 = F_k F_{k+1}$. It follows that $F_1^2 + F_2^2 + \cdots + F_k^2 + F_{k+1}^2 = F_k F_{k+1} + F_{k+1}^2 = F_{k+1}(F_k + F_{k+1}) = F_{k+1} F_{k+2}$. Hence by assuming the statement for $n = k$, we proved it for $n = k+1$, so our statement is true for all $n \in \mathbb{N}$ by induction.

(FP3) (a) For a fixed natural number $m \geq 2$, let's write $\mathbb{Z}_m$ for the quotient space $\mathbb{Z}/\equiv_m$ of equivalence classes mod m. Consider the map $f : \mathbb{Z}^n \to (\mathbb{Z}_m)^n$ given by taking the remainder of each coordinate mod m, so for instance if $m = 4$ and $n = 3$, we have $f((4, 10, 2)) = (0, 2, 2)$. If A is a subset of $\mathbb{Z}^n$, how big must its cardinality be (in terms of m and n) in order to ensure that $|f(A)| < |A|$?

(b) The squares of an 8×8 grid are colored black or white. Let's use the term *L-region* for 5 squares arranged in an L, as shown in the picture (note orientation matters: the corner of the L must be in its lower left). Prove that no matter how we color the grid, there must be two distinct L-regions (partial overlap allowed) that are colored identically. See example below.



Answer. These problems can be solved via the Pigeonhole Principle.

(a) First note that $|f(A)| < |A|$ occurs precisely if f is *not injective*. (That's the only way that the number of outputs can be smaller than the number of inputs, by Pigeonhole.) But in this case, the target $(\mathbb{Z}_m)^n$ is itself a finite set: since $|\mathbb{Z}_m| = m$, the product $\mathbb{Z}_m \times \cdots \times \mathbb{Z}_m$ has order m^n. So as long as $|A| > m^n$, the map can't be injective.

(b) First note that since an L-region has five squares, there are $2^5 = 32$ different ways to color an L-region with black or white squares. Now, how many different L-regions are on the grid? An easy way to count that is to look at where we can place the corner square. Looking at the picture, we see that it can be positioned anywhere in the lower-left 6×6 region of the grid, so there are 36 possible positions. If there are 36 L-regions and only 32 ways to color them, it follows by the Pigeonhole Principle that there must be two distinct L-regions that are colored identically!

(FP4) Fix n, and for any function $f : [n] \to [n]$, define $N(f) := \prod_{i=1}^{n} (i - f(i))$.

(a) If $n = 5$ and f is the constant map $f(x) = 1$, compute $N(f)$.
(b) Give necessary and sufficient conditions on f for $N(f) \neq 0$.
(c) For $n = 4$, give an example of a bijection f with $N(f) > 0$.
(d) (harder) If n is odd, prove that f is a bijection $\implies$ $N(f)$ is even.

Answer. (a) Since $f(1) = 1$, we have that $1 - f(1) = 0$, and since this is a term in the product $N(f)$, we have that $N(f) = 0$ in this case.

(b) I claim that $N(f) \neq 0$ iff $f(x) \neq x$, $\forall x \in [n]$: We show the contrapositive of both directions. If $N(f) = 0$, there is some term $(i - f(i))$ in the product $N(f)$ that must be zero, so for this i we have that $f(i) = i$. If there is an $i \in [n]$ such that $f(i) = i$, the product $N(f)$ will contain the term $(i - f(i)) = 0$, so the entire product will be zero.

(c) There are multiple possible answers. One is as follows: Let $f : [4] \to [4]$ be given by $f(1) = 2, f(2) = 4, f(3) = 1, f(4) = 3$. Then we have that

$$N(f) = \prod_{i=1}^{4} (i - f(i)) = (1-2)(2-4)(3-1)(4-3) = 4 > 0.$$

(d) Suppose n is odd and f is a bijection from $[n]$ to itself. Let's suppose that $N(f)$ is odd, and seek a contradiction. Then every term $(i - f(i))$ in the product $N(f)$ must be odd (just one even term makes the whole product even). But then it follows that all $f(\text{odd})$ must be even and $f(\text{even})$ must be odd, so that i and $f(i)$ have different parity. But since there are more odd numbers than even numbers in $[n]$, so this is impossible! That's our contradiction, and we can therefore conclude that $N(f)$ is even.

(FP5) (a) I have 30 sugarcubes, and there are 10 coffee mugs lined up in a row. How many ways are there to distribute all of the sugarcubes into mugs?

(b) If I distribute the cubes randomly (making all distributions equally likely), what is the probability that some mug has at least four sugarcubes?

(c) 42 chairs are set up in a row for the Discrete Math garlic-eating contest. Only six people show up. In how may ways can the eaters be seated

(i) overall?
(ii) if they aren't allowed to sit in six consecutive seats?
(iii) if they refuse to sit next to each other and they aren't allowed to sit in the chairs on the ends of the row?
(iv) the eaters are allowed to sit anywhere they want (but still refuse to sit next to each other)?

Answer. (a) The number of ways to distribute the sugarcubes into mugs is the number of whole-number solutions to $n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_7 + n_8 + n_9 + n_{10} = 30$, which is $\binom{39}{9}$ by stars-and-bars.

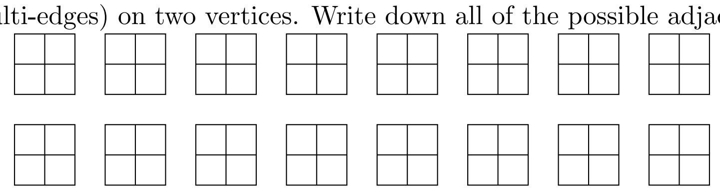
(b) Four sugarcubes is enough to kill someone. So there is only one way to avoid murder (put precisely three cubes in each mug), so the probability of murder is $\frac{\binom{39}{9}-1}{\binom{39}{9}}$, which in case you're curious is 0.99999999528. Murrrrrder most likely!

(c) (i) There are $\binom{42}{6}$ ways to pick the seats and then 6! ways to arrange people once the seats are chosen, so $\binom{42}{6} \cdot 6!$.
(ii) Let's count how many seat choices are bad. Six people in a row can sit in 37 ways (the left-most person could be in the seat numbered $1, 2, \ldots, 37$) so if we throw those out we get the new answer $\left[\binom{42}{6} - 37\right] \cdot 6!$
For part (iii), the garlic-eaters will sit in six of the chairs, leaving 36 chairs empty. There are some nonnegative number of chairs to the left of the first eater, some nonnegative number of chairs between the first and second eater, and so on, and a nonnegative number of chairs to the right of the sixth eater. (Note all of these numbers are nonnegative because the eaters aren't allowed to sit on the ends of the row.) Hence the number of ways to "distribute the empty chairs among the eaters" will be the number of solutions to $n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_7 = 36$ in the natural numbers, which is $\binom{35}{6}$ - but even if we know which six chairs the eaters are seated in we can scramble who sits where, so we have to multiply by 6! to account for this, giving a final count of $\binom{35}{6} \cdot 6!$.
For part (iv), we can think of it the same way, except that the number of ways to "distribute the empty chairs among the eaters" will be the number of solutions to $n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_7 = 36$ where n_1 and n_7 are whole numbers and the rest are natural numbers. But now let $N_1 = n_1 + 1, N_7 = n_7 + 1$. N_1, N_7 are natural numbers, so it follows that by substitution the number of solutions to $n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_7 = 36$ with the constraints above is also the number of solutions to $N_1 + n_2 + n_3 + n_4 + n_5 + n_6 + N_7 = 38$ where all variables are natural numbers. We know this is $\binom{37}{6}$, and again we need to multiply by 6! to account for the order of the eaters, so we get a final count of $\binom{37}{6} \cdot 6!$.

(FP6) Let A be a finite set with cardinality n.

(a) Explain why the number of relations on A is 2^{n^2}.

(b) For $A = \{1, 2\}$, there are sixteen relations on A. You can record them as directed graphs (with loops but no multi-edges) on two vertices. Write down all of the possible adjacency matrices.



(c) Which matrix corresponds to the relation $R = \{(1, 1), (2, 1)\}$? Draw the corresponding digraph.

(d) How many of the sixteen possible relations are symmetric? How many are anti-symmetric? How many are partial orders?

(e) Now consider the general case, $|A| = n$. What is the probability that a random relation is symmetric? Anti-symmetric?

Answer. (a) A relation on A is any subset of $A \times A$, so we can get the number of relations on A by finding the number of subsets of A, or $|\mathcal{P}(A \times A)|$. Since $|A \times A| = |A|^2 = n^2$, it follows that $|\mathcal{P}(A \times A)| = 2^{n^2}$. That's because: there are n^2 elements in $A \times A$, so we can count the number of subsets of $A \times A$ by deciding whether each element of $A \times A$ will be in a given subset - either an element will be in a subset or it won't, so we have two choices for each element of $A \times A$, and since $|A \times A| = n^2$, the number of relations on A is 2^{n^2}.

(b) For each element of the matrix, we can either fill in a 1 or a 0. All sixteen adjacency matrices are given by all possible combinations of 1's and 0's in the four elements of the matrix.

(c) The adjacency matrix corresponding to R is $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and the corresponding digraph consists of two vertices v_1, v_2 with a loop at v_1 and an edge from v_2 to v_1.

(d) Basically I need to figure out whether the associated matrix is symmetric. Well, to be symmetric, I can freely choose the entries on and below the diagonal, but then the entries above the diagonal are determined (they have to match their mirror image). That means I can freely choose 3 of the 4 entries, which gives me 2^3 choices out of 2^4 matrices, or 8 in all.
For anti-symmetric, it doesn't matter what's on the diagonal, but the other two entries can be 00, 10, or 01, but not 11. So there are $3 \cdot 2^2 = 12$ anti-symmetric relations: I choose the off-diagonals in three ways, and I choose the diagonals in four ways.
Transitivity: the only way for this to fail is if we have two pairs that "chain" together, and the only nontrivial way to do this is to have both $(1,2)$ and $(2,1)$. There are four of the sixteen relations that have those two. But if you do have those, then you must also have $(1,1)$ and $(2,2)$ in order to be transitive, so only one of the four ways to fill in the diagonal entries passes the test. That means all but three relations, or 13 out of 16, are transitive.
Notice that anything antisymmetric is automatically transitive! So to count partial orders, I just need antisymmetric plus reflexive, and there are exactly three ways to do this (choose the off-diagonals three ways, and then you're forced to take the diagonals).

(e) For the next two parts, it will be helpful to count the number of matrix positions that are, say, below the diagonal. The number is $\binom{n}{2}$, because selecting a below-diagonal entry is equivalent to picking two indices out of n. (For instance, if you picked 2 and 4 and $n = 5$, the corresponding below-diagonal entry would be $a_{4,2}$.)
Symmetric: Exactly the same as above: an $n \times n$ matrix has n diagonal entries and $\binom{n}{2}$ entries below the diagonal. I can freely choose these in a symmetric matrix and the others are then determined. There are n^2 entries in all. Hence the number of symmetric relations on A is $2^{n+\binom{n}{2}}$, and the probability of choosing a symmetric relation is $\frac{2^{n+\binom{n}{2}}}{2^{n^2}}$. Another, totally equivalent, way of counting this is that for each off-diagonal pair, symmetry says they have to agree (00 or 11 but not 10 or 01). That's 50-50 for each pair, so overall the probability of symmetry is $(1/2)^{\binom{n}{2}}$.
Anti-symmetric: For each of the diagonal elements, we can choose whether or not it is in the relation freely. Now consider an off-diagonal pair, a_{ij} and a_{ji}, with $i \neq j$. Antisymmetry says they can't both be in the relation, so in terms of the associated matrix, both entries can't be 1. For each of the pairs, there are 4 ways to choose 0 and 1 and only one of the four ways produces a failure of antisymmetry, so the probability of choosing an antisymmetric relation is $(3/4)^{\binom{n}{2}}$.

(FP7) (a) Suppose that $(X, \leq_1)$ and $(Y, \leq_2)$ are posets. Show that $(X \times Y, \leq)$ is a poset where $(a, b) \leq (c, d)$ iff $a \leq_1 c$ and $b \leq_2 d$. We can call that the *product poset.*

(b) Give an example of a pair of comparable elements and a pair of non-comparable elements in the product poset if $X = \{1, 2, 5\}$, $Y = \{3, 6\}$ and both $\leq_1$ and $\leq_2$ are the standard less-than-or-equal relation on integers.

Answer. (a) $(X \times Y, \leq)$ is reflexive: We have that $\forall (a, b) \in (X \times Y)$, $(a, b) \leq (a, b)$ since $a \leq_1 a, b \leq_1 b$ as $(X, \leq_1), (Y, \leq_2)$ are posets and hence reflexive.
$(X \times Y, \leq)$ is antisymmetric: Suppose that $(a, b) \leq (c, d)$ and $(c, d) \leq (a, b)$. Then by definition we have that $a \leq_1 c, c \leq_1 a, b \leq_2 d, d \leq_2 b$. But since $(X, \leq_1), (Y, \leq_2)$ are posets and hence antisymmetric, it follows that $a = c$ and $b = d$, so $(a, b) = (c, d)$ and hence this is antisymmetric.
$(X \times Y, \leq)$ is transitive: Suppose that $(a, b) \leq (c, d)$ and $(c, d) \leq (e, f)$. Then by definition we have that $a \leq_1 c, c \leq_1 e, b \leq_2 d, d \leq_2 f$. But since $(X, \leq_1), (Y, \leq_2)$ are posets and hence transitive, it follows that $a \leq_1 e$ and $b \leq_2 f$, so by definition we have that $(a, b) \leq (e, f)$. Hence this is transitive.
Since $\leq$ is reflexive, antisymmetric, and transitive, it follows that it is a partial order.

(b) There are multiple possible answers to this question. We'll give an example: $(1, 3), (5, 6)$ are comparable elements in $X \times Y$ - since $1 \leq 5, 3 \leq 6$ it follows that $(1, 3) \leq (5, 6)$. However, $(1, 6), (5, 3)$ are not comparable elements in $X \times Y$ - since $1 \leq 5$ but $6 \geq 3$, it is neither true that $(1, 6) \leq (5, 3)$ nor $(1, 6) \geq (5, 3)$. Hence they are non-comparable.

(FP8) Let $f : X \to Y$ be a function, and let $S, T \subseteq X$ and $A, B \subseteq Y$. Furthermore, for $C \subseteq Y$ recall that $f^{-1}(C) = \{x \in X \mid f(x) \in C\}$. Prove that:
(a) $f(S \cup T) = f(S) \cup f(T)$.
(b) $f(S \cap T) \subseteq f(S) \cap f(T)$.
(c) If f is injective, then $f(S \cap T) = f(S) \cap f(T)$.
(d) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.
(e) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

Answer. (a) $x \in f(S \cup T)$ iff $x = f(a)$ for some $a \in S \cup T$ iff $a \in S$ or $a \in T$ iff $f(a) = x$ is either in $f(S)$ or $f(T)$ iff $x \in f(S) \cup f(T)$.
(b) If $x \in f(S \cap T)$, then $x = f(a)$ for some $a \in S \cap T$. So $a \in S$ and $a \in T$, so $f(a) = x$ satisfies that $x \in f(S)$ and $x \in f(T)$, so $x \in f(S) \cap f(T)$.
(c) One direction was shown in part (b) for functions in general. Suppose that f is injective, and let $x \in f(S) \cap f(T)$. Then $x \in f(S)$ and $x \in f(T)$, but since f is injective, there is a unique a such that $a \in S$, $a \in T$, and $f(a) = x$. Then $a \in S \cap T$, so $x = f(S \cap T)$, as desired.
(d) We have that $x \in f^{-1}(A \cup B)$ iff $x \in X$ and $f(x) \in A \cup B$ iff $x \in X$ and $f(x) \in A$ or $f(x) \in B$ iff $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$ iff $x \in f^{-1}(A) \cup f^{-1}(B)$.
(e) We have that $x \in f^{-1}(A \cap B)$ iff $x \in X$ and $f(x) \in A \cap B$ iff $x \in X$ and $f(x) \in A$ and $f(x) \in B$ iff $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$ iff $x \in f^{-1}A \cap f^{-1}(B)$.

(FP9) (a) Write out the quantified definitions. (For example, $A \subseteq B$ iff $\forall x \in A,\ x \in B$.)
(i) A relation R on X is reflexive/symmetric/transitive/antisymmetric iff...
(ii) A relation f from X to Y is a function iff... (Note that the notation for "*there exists a unique*" is "$\exists!$".)
(iii) A function $f : X \to Y$ is injective/surjective iff...
(b) How would you prove a function is injective? How would you prove a function is surjective?
(c) If $f : X \to Y$ and $g : Y \to X$ satisfy $g \circ f = Id_X$, show that f is injective and g is surjective.
(d) Suppose you are given a bijection $f : X \to Y$. Give an explicit bijection $g : \mathcal{P}(X) \to \mathcal{P}(Y)$. (In other words, for a subset $A \subseteq X$, you should be able to write down $g(A)$ in set-builder notation, using f.)

Answer. (a) (i) A relation R on X is reflexive iff $\forall x \in X$, $(x, x) \in R$.
A relation R on X is symmetric iff $\forall x, y \in X$, $(x, y) \in R \Rightarrow (y, x) \in R$.
A relation R on X is transitive iff $\forall x, y, z \in X$, $((x, y) \in R \wedge (y, z) \in R) \Rightarrow (x, z) \in R$.
A relation R on X is antisymmetric iff $\forall x, y \in X$, $((x, y) \in R \wedge (y, x) \in R) \Rightarrow x = y$.
(ii) A relation f from X to Y is a function iff $\forall x \in X, \exists! y \in Y$ s.t. $(x, y) \in f$.
(iii) A function $f : X \to Y$ is injective iff $\forall x, y \in X,\ f(x) = f(y) \Rightarrow x = y$.
A function $f : X \to Y$ is surjective iff $\forall y \in Y,\ \exists x \in X$ s.t. $f(x) = y$.
(b) We can prove a function is injective by showing that if we assume $f(x) = f(y)$, we can prove that $x = y$. Likewise, we can prove a function is surjective by showing that if we take a point in the target $y \in Y$, we can deduce the existence of an $x \in X$ that satisfies $f(x) = y$.
(c) Firstly, f is injective: Suppose $f(x) = f(y)$ for some $x, y \in X$. But then this satisfies that $g(f(x)) = g(f(y))$, and since $g \circ f = Id_X$, we have that $x = y$. Now, g is surjective: Let $x \in X$. I claim that $f(x) \in Y$ has an image of x under g; this is true because $g(f(x)) = x$, as $g \circ f = Id_X$.
(d) An explicit bijection is as follows: $g : \mathcal{P}(X) \to \mathcal{P}(Y)$, $A \mapsto \{f(x) \mid x \in A\}$. Now, I claim this is indeed a bijection:
- g is injective: Suppose $g(A) = g(B)$ for $A, B \subseteq X$. But then by definition, $\{f(x) \mid x \in A\} = \{f(x) \mid x \in B\}$. But since f is itself a bijection, an inverse f^{-1} exists and so $f^{-1}\{f(x) \mid x \in A\} = f^{-1}\{f(x) \mid x \in B\}$. Now, I claim $f^{-1}\{f(x) \mid x \in A\} = A$: If $a \in A$, then $a = f^{-1}f(a)$, so $a \in f^{-1}\{f(x) \mid x \in A\}$. Likewise, if $a \in f^{-1}\{f(x) \mid x \in A\}$, then if $a \notin A$, $f^{-1}f(a) = a$ would not be in $f^{-1}\{f(x) \mid x \in A\}$, which is a contradiction, so $a \in A$. Likewise, $f^{-1}\{f(x) \mid x \in B\} = B$. Hence $A = B$ and g is injective.

- g is surjective. Let $S \subseteq Y$. Since f is a bijection, it is invertible and has an inverse f^{-1}. Then the set $T = \{f^{-1}(x) \mid x \in S\} \subseteq X$ has an image S under g, since this set maps to $\{f(x) \mid x \in T\}$ and $f(f^{-1}(x)) = x$ for all $x \in Y$.

(FP10) (a) Show that the relation $\sim$ on $\mathbb{N}$ by $a \sim b$ iff $\exists n \in \mathbb{N}$ such that $ab = n^2$ is an equivalence relation.
(b) Prove that for $m \in \mathbb{N}$, we have $m \in [6]$ iff, when we break down m into its prime decomposition, the exponent of 2 and the exponent of 3 are odd, while all the other exponents are even.
(c) Describe [16]. Can you describe in general what the equivalence class $[N]$ looks like for an arbitrary $N \in \mathbb{N}$?
(d) Let $A = \mathbb{N}/\sim$ be the set of equivalence classes. Show that A is countably infinite.

Answer. (a) Reflexivity: $a \sim a$ because $aa = a^2$. ✓ Symmetry: if $ab = n^2$, then $ba = n^2$ for the same n. ✓ Transitivity: suppose $ab = n^2$ and $bc = m^2$. Then $(ab)(bc) = n^2m^2$, so $(ac)b^2 = n^2m^2 \implies ac = (\frac{nm}{b})^2$. But we know that ac is a whole number, and a whole number can't be the square of a rational number that isn't an integer, so ac is the square of an integer, which means $a \sim c$. ✓
(b) Suppose $m = 2^r \cdot 3^s \cdot M$, where M is not divisible by 2 or 3. Note that a perfect square has all even exponents in its prime decomposition (because exponents double when you square a number). Then $6 \sim m \iff 6m$ is a perfect square, which means that $2^{r+1} \cdot 3^{s+1} \cdot M$ is a perfect square, which happens exactly if r is odd, s is odd, and all of the other exponents are even.
(c) Since $16 = 2^4$, multiplying by 16 doesn't change the parity of any of the exponents in the prime decomposition. So $16 \sim m$ if and only if m is already a perfect square. In general, a number N has some (finite) collection of primes that appear with odd exponent in its prime decomposition. For $N = 6$, that set is $\{2, 3\}$; for $N = 16$, that set is $\emptyset$. The equivalence class is those values of N that have this "oddset" in common.
(d) As described in the last part, an equivalence class is characterized by a finite subset of $\mathbb{N}$. There are certainly infinitely many of these, on one hand, so let's check that it's a countable infinity. I'll list all finite subsets as follows: organize them by their largest number. There are only finitely many sets of numbers whose largest number is 100. (To be precise, there are 2^{99} of them.) So I'll form a list of all finite subsets of $\mathbb{N}$ by listing all the ones whose largest number is 1 followed by all the ones whose largest number is 2, and so on. This process is clearly exhaustive (it eventually hits any finite subset of $\mathbb{N}$), so we've verified that this quotient space is countable.
Did you think that was too complicated? Here's an easier argument: the quotient space is formed by grouping elements from $\mathbb{N}$, so there's a surjection to it from $\mathbb{N}$, so it can't be any bigger than $\mathbb{N}$ is, so it's countable.

(FP11) The diagonalization proof that $|\mathcal{P}(X)| > |X|$ goes like this: suppose $f : X \to \mathcal{P}(X)$ is a bijection. Consider $S = \{x \in X : x \notin f(x)\}$. This is an element of $\mathcal{P}(X)$. Since f is a bijection, there must be some $s \in S$ such that $f(s) = S$. But then $s \in S$ and $s \notin S$ both lead to contradictions.

Study this construction as follows: let $X = \{a, b, c\}$ and give two examples of functions from X to $\mathcal{P}(X)$. (They won't be bijections, of course, since the power set has eight elements.) For each of your functions f, compute the set S defined above and verify that it is not in the image of f.

Answer. First example: $f(a) = \emptyset, \quad f(b) = \{a, c\}, \quad f(c) = \{c\}$. Then only c is in its corresponding set, so $S = \{a, b\}$. This is not one of the sets in the image of f!
Second example: $f(a) = \{a, b, c\}, \quad f(b) = \{a, b, c\}, \quad f(c) = \{a\}$. So a, b are in their corresponding sets, so $S = \{c\}$, which sure enough is not hit by f.

(FP12) A graph G is called *bipartite* if $V(G)$ can be partitioned into two sets S and T such that every edge of G is incident to one vertex in S and one vertex in T. A *complete bipartite graph* is a bipartite graph where every possible edge is present.

(a) For what values of n are K_n, C_n, W_n bipartite? (Recall that K_n is the complete graph on n vertices, C_n is the cycle on n vertices, and we'll write W_n for the "wheel" graph with $n+1$ vertices, formed by connecting a central vertex to every vertex of a C_n.)
(b) Must bipartite graphs be simple?
(c) A complete bipartite graph where $|S| = m, |T| = n$ is denoted $K_{m,n}$. Draw a $K_{3,3}$ and a $K_{4,2}$. What are $|V(K_{m,n})|$ and $|E(K_{m,n})|$ for general m, n?
(d) A *tree* is a connected graph with no cycles. Prove that trees are bipartite. (Begin with an example of a tree and figure out what S and T should be in your example.
(e) (harder) Prove that a graph is bipartite if and only if it does not contain any odd cycles.
(f) Suppose G is a graph with $|V(G)| = 18$. Explain steps to check if it is isomorphic to $K_{6,12}$.

Answer. (a) Well, note that a graph is bipartite if and only if you can two-color its vertices (say red and blue) so that every edge goes from red to blue. Therefore K_n is bipartite only for $n = 1, 2$ (because for $n = 3$ and up, there's a triangle subgraph, and that can't be 2-colored). C_n is bipartite for $n = 1$ and all even n, because the parity determines whether the red-blue alternating coloring will work. Finally, W_n is only bipartite for $n = 1$, because after that it contains a triangle.
(b) Nope. "Simple" means two things: no loops and no multi-edges. It's true that bipartite graphs can't have loops, because both endpoints will have the same "color." However, nothing prevents multi-edges!
(c) In general, a $K_{m,n}$ will have $m + n$ vertices (m vertices in one set of vertices, n vertices in the other) and mn edges connecting every one of the m vertices to every one of the n vertices.
(d) Pick any vertex v, and color it red. Now we can take all vertices adjacent to v and color them blue, then simply alternate red/blue as we walk out from there along any path. This eventually hits all vertices because trees are connected, so there's some path from v to every other vertex. And this never causes a contradiction, because if there were ever two different paths from v to w, there would be a cycle, which can't exist in a tree. So we've 2-colored the tree.
(e) ($\Rightarrow$) This direction is kinda more obvious: if you have a cycle of odd length, there's no way to 2-color it without getting two vertices of the same color next to each other.
($\Leftarrow$) Actually this direction is much like the 2-coloring process for trees described above. We do the same thing: start with any vertex v, color it red, and alternate colors from there on out. So a vertex w will be red if the length of a path $v \to w$ is even and blue if the length of a path $v \to w$ is odd. Why can't you get a contradiction? Well, if there are two paths $v \to w$ and one has even length and the other has odd length, then if you surger them together you get a loop of odd length! So a failure of 2-coloring must imply the existence of a loop of odd length. (To be precise, this argument can be applied to each connected component of G separately to produce a coloring of the whole graph.)
(f) For G to be isomorphic to $K_{6,12}$, it would have to have six vertices of degree 12 and 12 vertices of degree 6. I can check that by looking at row sums in its adjacency matrix. If it passes this test, then let S be the set of the degree-12 vertices and T be the set of degree-6 vertices; check that each row from an S vertex has 1s in the T positions and vice versa. This confirms the isomorphism to $K_{6,12}$.

(FP13) (a) Prove by induction that a connected graph with n vertices must have at least $n - 1$ edges.
(b) Prove that a tree with at least one edge must contain at least 2 vertices of degree 1. (Such a vertex is called a *leaf*.)

Answer. (a) Let G be a connected graph, and let $|V(G)| = n$. Base case: consider $n = 0$ (the empty graph) which is technically connected because there are no two points you can't find a path between! It has no edges, and $0 \geq -1$ ✓ Second and less trivial base case: $n = 1$, which is connected with 0 edges and $0 \leq 0$ ✓ Inductive hypothesis: Now suppose that every connected graph with $n = k$ vertices has at least $k - 1$ edges. For the inductive step we'd like to use this to study a connected graph G with $k + 1$ vertices. If we can show it has at least k edges, we win. I'd like to find a subgraph G' with one vertex less that is still connected, so that I can apply the inductive hypothesis to G'. What would be great is if I could form G' by deleting a vertex of degree 1, because that can't disconnect the graph! Well, if G had all of its $k+1$ vertices of degree at least 2, we could use the known formula $2|E| = \sum \deg(v)$ to obtain $2|E| \geq 2(k+1)$ or $|E| \geq k+1$, and that is more than enough for what we need to show. Otherwise, we now have our G' by deleting the degree-1 vertex. But now the inductive hypothesis guarantees that G' has at least $k-1$ edges, and we deleted one edge to get from G to G', so G itself has at least k edges, as desired.

(b) Let T be a tree, and consider a vertex-distinct path $v_1, ..., v_n$ in T that's as long as possible. I claim v_1, v_n are leaves: If v_1 were adjacent to any other vertex not in this path, then we could form a longer path, contradiction. If v_1 were adjacent to any other vertex aside from v_2 in this path, then we would have a cycle, but trees can't have cycles. For exactly the same reason, v_n must be a leaf too.

(FP14) (a) State and explain how to use an adjacency matrix to check if a relation is transitive.

(b) For $n \geq 2$, let P_n be the path graph on n vertices (i.e., it's got the edges v_1v_2, v_2v_3,$\dots$,$v_{n-1}v_n$). If A is its adjacency matrix, which entries of A^{n-2} are zero? (Try this with $n=3$ and 4 to get started.)

Answer. (a) Let $C = A^2$. The test is this: the relation is transitive if and only if for any i, j with $c_{ij} > 0$, we have $a_{ij} > 0$ as well.

Why this works: c_{ij} counts the number of paths of length 2 from v_i to v_j. If there is such a path, say it goes through a vertex v_b. Then (i, b) is in the relation and (b, j) is too. To have transitivity, we'd have to have (i, j) in the relation as well, and this is precisely what $a_{ij} > 0$ confirms.

(b) The entries of $M = A^{n-2}$ count the paths of length $n-2$ between various pairs of vertices. In a path graph with n vertices, you can get end-to-end with $n-1$ steps, but you can be sure that vertices v_1 and v_n are not connected by a path of length $n-2$, so you definitely have $m_{1n} = m_{n1} = 0$. Any two other vertices are close enough to be connected, but there is a parity issue. For instance, vertices at distance 1 can only be connected by a path of odd length! So for the rest of the matrix, I just need to be sure that $i-j$ has the same parity as $n-2$. But note that $i-j$ has the same parity as $i+j$, and also note that $n-2$ has the same parity as n. Finally, two things have the same parity iff their sum is even. In the end, we find that $m_{ij} = 0$ if and only if $i+j+n$ is even. So the matrix has zeros in a checkerboard pattern, plus in the northeast and southwest corners!

(FP15) (a) Let $\mathcal{T}_5$ be the set of trees with five vertices, and let $\cong$ represent graph isomorphism. Fully describe the quotient space $\mathcal{T}_5/\cong$.

(b) Give a formula for the (i, j) entry of $(A \cdot B^\intercal)^\intercal$.

(c) If $n=4$, write down the elementary transposition matrix E_{12} and the elementary transposition matrix E_{24}. Verify that $E_{12}E_{24}$ is not a symmetric matrix, and that $(E_{12}E_{24})^\intercal = E_{24}E_{12}$.

(d) Draw two different-looking connected graphs on 5 vertices (with the vertices labeled cyclically clockwise) such that their adjacency matrices satisfy $E_{12}A_1E_{12} = A_2$.

Answer. (a) The quotient space has only three points! There's one graph with a vertex of degree four and everything else connected to it. There's one with a vertex of degree three and one of its neighbors having degree two. And there's a path. By considering the maximum degree of the graph, you see that this exhausts the cases 4, 3, and 2, which are the only possible ones for a connected graph of five vertices.

(b) Suppose they are $n \times n$. First, let's use the rules for transpose to simplify: $(A \cdot B^\intercal)^\intercal = B \cdot A^\intercal$. Now we use the formula for multiplication to get

$$(B \cdot A^\intercal)_{ij} = \sum_{k=1}^{n} b_{ik} a_{jk},$$

using the fact that the (k, j) entry of $A^\intercal$ is a_{jk}.

(c)

$$E_{12} = \begin{bmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix}, \quad E_{24} = \begin{bmatrix} 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \end{bmatrix}$$

$$E_{12}E_{24} = \begin{bmatrix} 0&1&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \\ 1&0&0&0 \end{bmatrix}, \quad E_{24}E_{12} = \begin{bmatrix} 0&0&0&1 \\ 1&0&0&0 \\ 0&0&1&0 \\ 0&1&0&0 \end{bmatrix}$$

It's easy to see that transposing one of these last two gives you not itself but the other one. (Rows of one are columns of the other.)

(d)