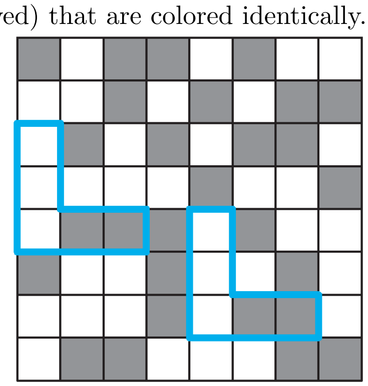
MATH 61-02: PRACTICE PROBLEMS FOR FINAL EXAM

(FP1) The *exclusive or* operation, denoted by $\oplus$ and sometimes known as XOR, is defined so that $P \oplus Q$ is true iff P is true or Q is true, but not both. Prove (through a truth table, or otherwise) that for any statements P, Q, R:
(a) $((P \oplus Q) \oplus R) \Leftrightarrow (P \oplus (Q \oplus R))$
(b) $(P \wedge (Q \oplus R)) \Leftrightarrow ((P \wedge Q) \oplus (P \wedge R))$
(Suggestion: first look at the expressions and analyze them to see what combination of P, Q, R is possible, then use a truth table to confirm your idea.)

(FP2) Prove the following:
(a) $1 \cdot 2 + 2 \cdot 3 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}$ for all $n \in \mathbb{N}$
(b) $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$ for $n \geq 2$
(c) If $A_1, A_2, \ldots, A_n, B$ are sets ($n \geq 1$), then $\bigcup_{i \in [n]} (A_i \setminus B) = (\bigcup_{i \in [n]} A_i) \setminus B$.
(d) Let F_n denote the nth term of the Fibonacci sequence (where $F_1 = 1, F_2 = 1$, and $F_k = F_{k-2} + F_{k-1}$ $\forall k \geq 3$). Show that $\forall n \in \mathbb{N}$, $F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$.

(FP3) (a) For a fixed natural number $m \geq 2$, let's write $\mathbb{Z}_m$ for the quotient space $\mathbb{Z}/\equiv_m$ of equivalence classes mod m. Consider the map $f : \mathbb{Z}^n \to (\mathbb{Z}_m)^n$ given by taking the remainder of each coordinate mod m, so for instance if $m = 4$ and $n = 3$, we have $f((4, 10, 2)) = (0, 2, 2)$. If A is a subset of $\mathbb{Z}^n$, how big must its cardinality be (in terms of m and n) in order to ensure that $|f(A)| < |A|$?
(b) The squares of an 8×8 grid are colored black or white. Let's use the term *L-region* for 5 squares arranged in an L, as shown in the picture (note orientation matters: the corner of the L must be in its lower left). Prove that no matter how we color the grid, there must be two distinct L-regions (partial overlap allowed) that are colored identically. See example below.



(FP4) Fix n, and for any function $f : [n] \to [n]$, define $N(f) := \prod_{i=1}^{n} (i - f(i))$.
(a) If $n = 5$ and f is the constant map $f(x) = 1$, compute $N(f)$.
(b) Give necessary and sufficient conditions on f for $N(f) \neq 0$.
(c) For $n = 4$, give an example of a bijection f with $N(f) > 0$.
(d) (harder) If n is odd, prove that f is a bijection $\implies$ $N(f)$ is even.

(FP5) (a) I have 30 sugarcubes, and there are 10 coffee mugs lined up in a row. How many ways are there to distribute all of the sugarcubes into mugs?

(b) If I distribute the cubes randomly (making all distributions equally likely), what is the probability that some mug has at least four sugarcubes?

(c) 42 chairs are set up in a row for the Discrete Math garlic-eating contest. Only six people show up. In how may ways can the eaters be seated

(i) overall?

(ii) if they aren't allowed to sit in six consecutive seats?

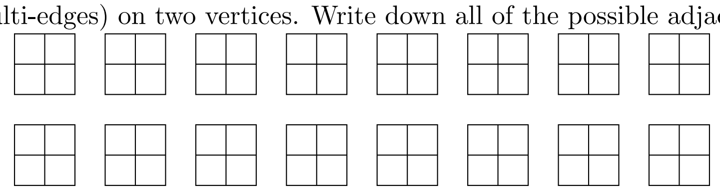
(iii) if they refuse to sit next to each other and they aren't allowed to sit in the chairs on the ends of the row?

(iv) the eaters are allowed to sit anywhere they want (but still refuse to sit next to each other)?

(FP6) Let A be a finite set with cardinality n.

(a) Explain why the number of relations on A is 2^{n^2}.

(b) For $A = \{1, 2\}$, there are sixteen relations on A. You can record them as directed graphs (with loops but no multi-edges) on two vertices. Write down all of the possible adjacency matrices.



(c) Which matrix corresponds to the relation $R = \{(1, 1), (2, 1)\}$? Draw the corresponding digraph.

(d) How many of the sixteen possible relations are symmetric? How many are anti-symmetric? How many are partial orders?

(e) Now consider the general case, $|A| = n$. What is the probability that a random relation is symmetric? Anti-symmetric?

(FP7) (a) Suppose that $(X, \leq_1)$ and $(Y, \leq_2)$ are posets. Show that $(X \times Y, \leq)$ is a poset where $(a, b) \leq (c, d)$ iff $a \leq_1 c$ and $b \leq_2 d$. We can call that the *product poset*.

(b) Give an example of a pair of comparable elements and a pair of non-comparable elements in the product poset if $X = \{1, 2, 5\}$, $Y = \{3, 6\}$ and both $\leq_1$ and $\leq_2$ are the standard less-than-or-equal relation on integers.

(FP8) Let $f : X \to Y$ be a function, and let $S, T \subseteq X$ and $A, B \subseteq Y$. Furthermore, for $C \subseteq Y$ recall that $f^{-1}(C) = \{x \in X \mid f(x) \in C\}$. Prove that:

(a) $f(S \cup T) = f(S) \cup f(T)$.

(b) $f(S \cap T) \subseteq f(S) \cap f(T)$.

(c) If f is injective, then $f(S \cap T) = f(S) \cap f(T)$.

(d) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(e) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

(FP9) (a) Write out the quantified definitions. (For example, $A \subseteq B$ iff $\forall x \in A,\ x \in B$.)

(i) A relation R on X is reflexive/symmetric/transitive/antisymmetric iff…

(ii) A relation f from X to Y is a function iff… (Note that the notation for "*there exists a unique*" is "$\exists!$".)

(iii) A function $f : X \to Y$ is injective/surjective iff…

(b) How would you prove a function is injective? How would you prove a function is surjective?

(c) If $f : X \to Y$ and $g : Y \to X$ satisfy $g \circ f = Id_X$, show that f is injective and g is surjective.

(d) Suppose you are given a bijection $f : X \to Y$. Give an explicit bijection $g : \mathcal{P}(X) \to \mathcal{P}(Y)$. (In other words, for a subset $A \subseteq X$, you should be able to write down $g(A)$ in set-builder notation, using f.)

(FP10) (a) Show that the relation $\sim$ on $\mathbb{N}$ by $a \sim b$ iff $\exists n \in \mathbb{N}$ such that $ab = n^2$ is an equivalence relation.
(b) Prove that for $m \in \mathbb{N}$, we have $m \in [6]$ iff, when we break down m into its prime decomposition, the exponent of 2 and the exponent of 3 are odd, while all the other exponents are even.
(c) Describe [16]. Can you describe in general what the equivalence class $[N]$ looks like for an arbitrary $N \in \mathbb{N}$?
(d) Let $A = \mathbb{N}/\sim$ be the set of equivalence classes. Show that A is countably infinite.

(FP11) The diagonalization proof that $|\mathcal{P}(X)| > |X|$ goes like this: suppose $f : X \to \mathcal{P}(X)$ is a bijection. Consider $S = \{x \in X : x \notin f(x)\}$. This is an element of $\mathcal{P}(X)$. Since f is a bijection, there must be some $s \in S$ such that $f(s) = S$. But then $s \in S$ and $s \notin S$ both lead to contradictions.

Study this construction as follows: let $X = \{a, b, c\}$ and give two examples of functions from X to $\mathcal{P}(X)$. (They won't be bijections, of course, since the power set has eight elements.) For each of your functions f, compute the set S defined above and verify that it is not in the image of f.

(FP12) A graph G is called *bipartite* if $V(G)$ can be partitioned into two sets S and T such that every edge of G is incident to one vertex in S and one vertex in T. A *complete bipartite graph* is a bipartite graph where every possible edge is present.
(a) For what values of n are K_n, C_n, W_n bipartite? (Recall that K_n is the complete graph on n vertices, C_n is the cycle on n vertices, and we'll write W_n for the "wheel" graph with $n+1$ vertices, formed by connecting a central vertex to every vertex of a C_n.)
(b) Must bipartite graphs be simple?
(c) A complete bipartite graph where $|S| = m, |T| = n$ is denoted $K_{m,n}$. Draw a $K_{3,3}$ and a $K_{4,2}$. What are $|V(K_{m,n})|$ and $|E(K_{m,n})|$ for general m, n?
(d) A *tree* is a connected graph with no cycles. Prove that trees are bipartite. (Begin with an example of a tree and figure out what S and T should be in your example.
(e) (harder) Prove that a graph is bipartite if and only if it does not contain any odd cycles.
(f) Suppose G is a graph with $|V(G)| = 18$. Explain steps to check if it is isomorphic to $K_{6,12}$.

(FP13) (a) Prove by induction that a connected graph with n vertices must have at least $n-1$ edges.
(b) Prove that a tree with at least one edge must contain at least 2 vertices of degree 1. (Such a vertex is called a *leaf*.)

(FP14) (a) State and explain how to use an adjacency matrix to check if a relation is transitive.
(b) For $n \geq 2$, let P_n be the path graph on n vertices (i.e., it's got the edges v_1v_2, v_2v_3,…,$v_{n-1}v_n$). If A is its adjacency matrix, which entries of A^{n-2} are zero? (Try this with $n = 3$ and 4 to get started.)

(FP15) (a) Let $\mathcal{T}_5$ be the set of trees with five vertices, and let $\cong$ represent graph isomorphism. Fully describe the quotient space $\mathcal{T}_5/\cong$.
(b) Give a formula for the (i, j) entry of $(A \cdot B^\intercal)^\intercal$.
(c) If $n = 4$, write down the elementary transposition matrix E_{12} and the elementary transposition matrix E_{24}. Verify that $E_{12}E_{24}$ is not a symmetric matrix, and that $(E_{12}E_{24})^\intercal = E_{24}E_{12}$.
(d) Draw two different-looking connected graphs on 5 vertices (with the vertices labeled cyclically clockwise) such that their adjacency matrices satisfy $E_{12}A_1E_{12} = A_2$.