

DYNAMICAL SYSTEMS WITH FAST SWITCHING AND SLOW DIFFUSION: HYPERBOLIC EQUILIBRIA AND STABLE LIMIT CYCLES

NGUYEN H. DU, ALEXANDRU HENING, DANG H. NGUYEN, AND GEORGE YIN

ABSTRACT. We study the long-term qualitative behavior of randomly perturbed dynamical systems. More specifically, we look at limit cycles of certain stochastic differential equations (SDE) with Markovian switching, in which the process switches at random times among different systems of SDEs, when the switching is fast varying and the diffusion (white noise) term is slowly changing. The system is modeled by

$$dX^{\varepsilon,\delta}(t) = f(X^{\varepsilon,\delta}(t), \alpha^\varepsilon(t))dt + \sqrt{\delta}\sigma(X^{\varepsilon,\delta}(t), \alpha^\varepsilon(t))dW(t), \quad X^\varepsilon(0) = x,$$

where $\alpha^\varepsilon(t)$ is a finite state space Markov chain with irreducible generator $Q = (q_{ij})$. The relative changing rates of the switching and the diffusion are highlighted by the two small parameters ε and δ . We associate to the system the averaged ordinary differential equation (ODE)

$$d\bar{X}(t) = \bar{f}(\bar{X}(t))dt, \quad X(0) = x,$$

where $\bar{f}(\cdot) = \sum_{i=1}^{m_0} f(\cdot, i)\nu_i$ and $(\nu_1, \dots, \nu_{m_0})$ is the unique invariant probability measure of the Markov chain with generator Q . Suppose that for each pair (ε, δ) of parameters, the process has an invariant probability measure $\mu^{\varepsilon,\delta}$, and that the averaged ODE has a limit cycle in which there is an averaged occupation measure μ^0 for the averaged equation. We are able to prove, under weak conditions, that if \bar{f} has finitely many unstable or hyperbolic fixed points, then $\mu^{\varepsilon,\delta}$ converges weakly to μ^0 as $\varepsilon \rightarrow 0$ and $\delta \rightarrow 0$. Our results generalize to the setting where the switching process α^ε is state-dependent and is given by

$$\mathbb{P}\{\alpha^\varepsilon(t + \Delta) = j \mid \alpha^\varepsilon = i, X^{\varepsilon,\delta}(s), \alpha^\varepsilon(s), s \leq t\} = q_{ij}(X^{\varepsilon,\delta}(t))\Delta + o(\Delta), \quad i \neq j$$

as long as the generator $Q(\cdot) = (q_{ij}(\cdot))$ is bounded, Lipschitz, and irreducible for all $x \in \mathbb{R}^d$.

We conclude our analysis by studying in detail an example of a stochastic predator-prey model.

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1. INTRODUCTION

Natural phenomena are almost always influenced by different types of random noise. In order to better understand the world around us, it is important to study random perturbations of dynamical systems. In the continuous dynamical systems setup, the focus then shifts from the study of the behavior of deterministic differential equations to that of differential equations with switching (piecewise deterministic Markov processes) or stochastic differential equations with switching. The long-term behavior of these systems can be analyzed by a careful study of the ergodic properties of the induced Markov processes.

Quite often, the “white noise” in the system is small compared to the deterministic component. In such cases, one is usually interested in knowing how well the deterministic system approximates the stochastic one. It is common to model continuous-time phenomena by stochastic differential equations of the type

$$(1.1) \quad dx^\delta(t) = f(x^\delta(t))dt + \sqrt{\delta}\sigma(x^\delta(t))dW(t),$$

where $f(\cdot)$ and $\sigma(\cdot)$ are sufficiently smooth functions, $W(\cdot)$ is a standard m -dimensional Brownian motion, and $\delta > 0$ is a small parameter. If we let $\delta \downarrow 0$, one would expect that the solutions of (1.1) converge, in an appropriate sense, to that of a deterministic differential equation.

Versions of this problem have been studied extensively starting with Freidlin and Wentzell [VF70, FW98], Fleming [Fle74], Kifer [Kif81] and Day [Day82].

If the process $x^\delta(t)$ has a unique ergodic probability measure μ^δ for each $\delta > 0$ and the origin of the corresponding deterministic ODE

$$(1.2) \quad dx = f(x)dt,$$

is a globally asymptotic equilibrium point, Holland established in [Hol74] asymptotic expansions of the expectation of the underlying functionals with respect to the unique ergodic probability measures μ^δ . In addition, in [Hol78], Holland considered the case when the ODE (1.2) has an asymptotically stable limit cycle and proved the weak convergence of the family $(\mu^\delta)_{\delta>0}$ to the unique stationary distribution that is concentrated on the limit cycle of the process from (1.2).

Our interest in the current problem stems from applications in ecology. Quite often, one models the dynamics of populations with continuous-time processes. This way we inherently assume that organisms can respond instantaneously to changes in the environment. However, in some cases the dynamics are better described by discrete-time models, in which demographic decisions are not made continuously. In order to model more complex systems, one has to analyze ‘hybrid’ systems where both continuous and discrete dynamics coexist. Such systems arise naturally in ecology, engineering, operations research, and physics as well as in emerging applications in wireless communications, internet traffic modeling, and financial engineering; see [YZ10] for more references.

Recently, there has been renewed interest in studying piecewise deterministic Markov processes (PDMP) [Dav84]. One may describe a PDMP by the use of a two component process. The first component is a continuous state process represented by the solution of a deterministic differential equation, whereas the second component is a discrete event process taking values in a finite set. This discrete event process is often modeled as a continuous-time Markov chain with a finite state space. At any given instance, the Markov chain takes a value (say i in the state space), and the process sojourns in state i for a random duration. During this period, the continuous state follows the flow given by a differential equation associated with i . Then at a random instance, the discrete event switches to another state $j \neq i$. The Markov chain sojourns in j for a random duration, during which, the continuous state follows another flow associated with the discrete state j .

A careful study of such processes has recently led to a better understanding of predator-prey communities where the predator evolves much faster than the prey [Cos16] and for a possible explanation of how the competitive exclusion principle from ecology, which states that multiple species competing for the same number of small resources cannot coexist, can be violated because of switching [BL16, HN18b].

It is natural to study the SDE counter-part of PDMP, that is, SDE with switching. Similarly to the piecewise deterministic Markov processes mentioned in the previous paragraph, in this setting one follows a specific system of SDE for a random time after which the discrete event switches to another state, and the process is governed by a different system of SDE. The resulting stochastic process has a discrete component (that switches among a finite number of discrete states) and a continuous component (the solution of SDE associate with each fixed discrete event state). We refer the reader to [YZ10] for an introduction to SDEs with switching. Most of the work inspired by Freidlin and Wentzell has been concerned with local phenomena that involve the exit times and exit probabilities from neighbourhoods of equilibria. One usually uses the theory of large deviations to analyze the exit problem from the domain of attraction of a stable equilibrium point. There are more complicated situations, as the one we treat in this paper, in which large deviation techniques are not sufficient, and one needs to analyze the distributional scaling limits for the exit distributions [Bak11]. There have been some previous important studies for multiscale systems with fast and slow scales [DS12, DSW12]. These previous papers have looked at large deviations in the related setting where one has a slow diffusion and the coefficients are fastly oscillating. However, in contrast to our framework, the fast oscillations come from having periodic coefficients and introducing a factor $\frac{1}{\varepsilon}$ into the periodic component of the coefficients. The way the fast oscillations are introduced in these previous papers is similar to how it is done when one does stochastic homogenization. In the present paper, the switching comes from a discrete random process α^ε .

In this paper, we consider dynamical systems represented by switching diffusions, in which the switching is rapidly varying whereas the diffusion is slowly changing. To be more precise, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. Consider the process $(X^{\varepsilon, \delta})_{t \geq 0}$ defined by

$$(1.3) \quad dX^{\varepsilon, \delta}(t) = f(X^{\varepsilon, \delta}(t), \alpha^\varepsilon(t))dt + \sqrt{\delta}\sigma(X^{\varepsilon, \delta}(t), \alpha^\varepsilon(t))dW(t), \quad X^{\varepsilon, \delta}(0) = x,$$

where $W(t)$ is an m -dimensional standard Brownian motion, $\alpha^\varepsilon(t)$ is a finite-state Markov chain that is independent of the Brownian motion and that has a state space $\mathcal{M} = \{1, \dots, m_0\}$ and generator $Q/\varepsilon = (q_{ij}/\varepsilon)_{m_0 \times m_0}$, $X^{\varepsilon, \delta}$ is an \mathbb{R}^d -valued process, $f : \mathbb{R}^d \times \mathcal{M} \rightarrow \mathbb{R}^d$, $\sigma : \mathbb{R}^d \times \mathcal{M} \rightarrow \mathbb{R}^{d \times m}$, and $\varepsilon, \delta > 0$ are two small parameters. We assume that the matrix Q is irreducible. The irreducibility of Q implies that the Markov chain associated with Q , which will be denoted by $(\tilde{\alpha}(t))_{t \geq 0}$, is ergodic thus has a unique stationary distribution $(\nu_1, \dots, \nu_{m_0})$. We denote by $X_{x,i}^{\varepsilon, \delta}(t)$ the solution of (1.3) at time $t \geq 0$ when the initial value is (x, i) and by $\alpha_i^\varepsilon(t)$ the Markov chain started at i .

Let us explore, intuitively, what happens when ε and δ are very small. In this setting, $\alpha^\varepsilon(t)$ converges very fast to its stationary distribution $(\nu_1, \dots, \nu_{m_0})$ while the diffusion is asymptotically small. As a result, on each finite time interval $[0, T]$ for $T > 0$, a solution of equation (1.3) can be approximated by the solution $\bar{X}_x(t)$ to

$$(1.4) \quad d\bar{X}(t) = \bar{f}(\bar{X}(t))dt, \quad \bar{X}(0) = x,$$

where $\bar{f}(x) = \sum_{i=1}^{m_0} f(x, i)\nu_i$.

However, if in lieu of a finite time horizon, we look at the process on the infinite time horizon $[0, \infty)$, it is not clear that $\bar{X}_x(t)$ is a good approximation. Suppose that equation (1.4) has a stable limit cycle. A natural question is whether the invariant measures $(\mu^{\varepsilon, \delta})$ of the processes (1.3) converge weakly as $\varepsilon \downarrow 0$ and $\delta \downarrow 0$, to the measure concentrated on the limit cycle. This is the main problem that we address in the current paper. In order to do this, we substantially extend the results of [Hol78] by considering the presence of both small diffusion and rapid switching. Because of the presence of both the switching and the diffusion we need to develop new mathematical techniques. In addition, even if there is no switching and we are in the SDE setting of [Hol78], our assumptions are weaker than those used in [Hol78].

Remark 1.1. *One might be interested in the following natural generalization of the setting presented above. The switching process α^ε can be state dependent, that is*

$$\mathbb{P}\{\alpha^\varepsilon(t + \Delta) = j \mid \alpha^\varepsilon = i, X^{\varepsilon, \delta}(s), \alpha^\varepsilon(s), s \leq t\} = q_{ij}(X^{\varepsilon, \delta}(t))\Delta + o(\Delta).$$

As long as the generator $Q(x) = (q_{ij}(x))$ is irreducible for each $x \in \mathbb{R}^d$ one can show that on each finite time interval $[0, T]$ one can approximate the process from (1.3) if ε, δ are small by

$$d\bar{X} = \bar{f}(\bar{X}(t))dt, \quad \bar{X}(0) = x,$$

where $\bar{f}(x) = \sum_{i=1}^{m_0} f(x, i)\nu_i(x)$ and $(\nu_1(x), \dots, \nu_{m_0}(x))$ is the stationary distribution of a Markov chain with generator $Q(x) = (q_{ij}(x))$. We will explain through a sequence of remarks that our results hold for this generalization.

The rest of the paper is organized as follows. The main assumptions and results appear in Section 1.1. In Section 2, we estimate the exit time of the solutions of (1.3) from neighborhoods around the stable manifolds of the critical points of \bar{f} . The proof of the main result is presented in Section 3. In Section 4, we apply our results to a general predator-prey model. In addition to showcasing our result in a specific setting, the proofs from Section 4 are interesting on their own right as they are quite technical and require the development

of new tools. Finally, in Section 4.1, we provide some numerical examples to illustrate our results from the predator-prey setting from Section 4.

1.1. Assumptions and main results. We denote by A' the transpose of a matrix A , by $|\cdot|$ the Euclidean norm of vectors in \mathbb{R}^d , and by $\|A\| := \sup\{|Ax| : x \in \mathbb{R}^d, |x| = 1\}$ the operator norm of a matrix $A \in \mathbb{R}^{d \times d}$. We also define $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$, and the closed ball of radius $R > 0$ centered at the origin $B_R := \{x \in \mathbb{R}^d : |x| \leq R\}$.

We recall some definitions due to Conley [Con78]. Suppose we are given a flow $(\Phi_t(\cdot))_{t \in \mathbb{R}}$. A compact invariant set K is called *isolated* if there exists a neighborhood V of K such that K is the maximal compact invariant set in V . A collection of nonempty sets $\{M_1, \dots, M_k\}$ is a *Morse decomposition* for a compact invariant set K if M_1, \dots, M_k are pairwise disjoint, compact, isolated sets for the flow Φ restricted to K and the following properties hold: 1) For each $x \in K$ there are integers $l = l(x) \leq m = m(x)$ for which the *alpha limit set* of x , $\hat{\alpha}(x) = \bigcap_{t \leq 0} \overline{\{\Phi_s(x), s \in (-\infty, t]\}}$, satisfies $\hat{\alpha}(x) \subset M_l$ and the *omega limit set* of x , $\hat{\omega}(x) := \bigcap_{t \geq 0} \overline{\{\Phi_s(x), s \in [t, \infty)\}}$, satisfies $\hat{\omega}(x) \subset M_m$ 2) If $l(x) = m(x)$ then $x \in M_l = M_m$.

Assumption 1.1. *We impose the following assumptions for the processes modeled by the systems (1.3) and (1.4).*

- (i) *For each $i \in \mathcal{M}$, $f(\cdot, i)$ and $\sigma(\cdot, i)$ are locally Lipschitz continuous.*
- (ii) *There is an $a > 0$ and a twice continuously differentiable real-valued, nonnegative function $\Phi(\cdot)$ satisfying $\liminf_{R \rightarrow \infty} \{\Phi(x) : |x| \geq R\} = \infty$ and $(\nabla \Phi)'(x)f(x, i) \leq a(\Phi(x) + 1)$, for all $(x, i) \in \mathbb{R}^d \times \mathcal{M}$.*
- (iii) *The vector field $\bar{f}(\cdot)$ has finitely many equilibrium points $\{x_1, \dots, x_{n_0-1}\}$ and a unique limit cycle Γ . The equilibrium points are either sources or hyperbolic points.*
- (iv) *There exists a Morse decomposition $\{M_1, M_2, \dots, M_{n_0}\}$ of the flow associated with \bar{f} such that $M_{n_0} = \Gamma$ is the limit cycle and for any $i < n_0$ we have $M_i = \{x_i\}$ where x_i is an equilibrium point.*
- (v) *There exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, the system (1.3) has a unique solution. Furthermore, for any $0 < \varepsilon < \varepsilon_0$, the process $(X^{\varepsilon, \delta}(t), \alpha^{\varepsilon, \delta}(t))$ has the strong Markov property and has an invariant measure $\mu^{\varepsilon, \delta}$. The family $(\mu^{\varepsilon, \delta})_{0 < \varepsilon < \varepsilon_0}$ is tight, i.e., for any $\gamma > 0$ there exists an $R = R_\gamma > 0$ such that $\mu^{\varepsilon, \delta}(B_R) > 1 - \gamma$ for all $0 < \varepsilon < \varepsilon_0$.*

Remark 1.2. *We note that using Assumption (ii), we can work in a compact state space $\mathcal{K} \subset \mathbb{R}^n$ if the diffusion term from (1.3) is zero. Assumptions (i) and (ii) are needed in order to deduce the existence and boundedness of a unique solution to equation (1.3) in the absence of the diffusion term. Assumption (iv) is used to make sure that there exist no heteroclinic cycles.*

Assumption (v) ensures that (1.3) has a unique solution that is strong Markov. Sufficient conditions that imply uniqueness and the strong Markov property exist in the literature [MY06, YZ10].

Remark 1.3. *In [Hol78] the author studied (1.1) under the assumptions that*

- (A1) $f, \sigma \in C^2(\mathbb{R}^d)$.
- (A2) *The system (1.2) has a unique limit cycle.*
- (A3) *There exists at most a finite number of critical points x^* of f . At each critical point the Jacobian matrix has only positive real parts and the matrix $\sigma' \sigma$ is positive definite.*

(A4) For any compact set B not containing critical points and any $u > 0$ there exists $T > 0$ such that if $x \in B$, then

$$d(x_x^0(t), \Gamma) < u, \text{ for } t \geq T.$$

(A5) There exists $\delta_0 > 0$ such that for $0 < \delta < \delta_0$ the stochastic differential equation has a unique ergodic measure μ_δ . Furthermore, the family $(\mu_\delta)_{0 < \delta < \delta_0}$ is tight in \mathbb{R}^d .

Our work generalizes [Hol78] significantly in the following aspects. First, we work with two types of randomness - one comes from the diffusion term and the other from the switching mechanism. Second, Assumption 1.1 (i) is weaker than (A1). Third, we can have any hyperbolic fixed points whereas assumptions (A3)-(A4) imply that all fixed points are sources and the deterministic system converges uniformly to the limit cycle. In addition, we do not need $\sigma'\sigma$ to be positive definite at the critical points.

Remark 1.4. There are several papers which look at the exit time asymptotics near hyperbolic fixed points of small perturbations of dynamical systems [Kif81, Bak08, Bak11]. We do not assume like in these paper that the noise is uniformly elliptic and we have to deal with the additional complications of a stable limit cycle as well as the switching due to α^ε .

Let $T_\Gamma > 0$ be the period of the limit cycle Γ . We can define a probability measure μ^0 , which is independent of the starting point $y \in \Gamma$, by

$$(1.5) \quad \mu^0(\cdot) = \frac{1}{T_\Gamma} \int_0^{T_\Gamma} \mathbf{1}_{\{\bar{X}_y(s) \in \cdot\}} ds,$$

where $\bar{X}_y(t)$ is the solution to equation (1.4) starting at $\bar{X}(0) = y$ and $\mathbf{1}_{\{\cdot\}}$ is the indicator function. The measure $\mu^0(\cdot)$ is the averaged occupation measure of the process \bar{X} restricted to the limit cycle Γ . Throughout the paper, we assume that δ depends on ε , i.e. $\delta = \delta(\varepsilon)$, and $\lim_{\varepsilon \downarrow 0} \delta(\varepsilon) = 0$. We will investigate the asymptotic behavior of the invariant probability measures $\mu^{\varepsilon, \delta}$ as $\varepsilon \downarrow 0$ in the following three cases:

$$(1.6) \quad \lim_{\varepsilon \downarrow 0} \frac{\delta}{\varepsilon} = \begin{cases} l \in (0, \infty), & \text{case 1} \\ 0, & \text{case 2} \\ \infty, & \text{case 3.} \end{cases}$$

The multi-scale modeling approach we use is similar to the one from [HY14].

Assumption 1.2. We impose additional conditions corresponding to the cases from (1.6).

- 1) Suppose $\lim_{\varepsilon \downarrow 0} \frac{\delta}{\varepsilon} = l \in (0, \infty)$. For any critical point x^* of \bar{f} there exists $i^* \in \mathcal{M}$ such that $\beta' f(x^*, i^*) \neq 0$ or $\beta' \sigma(x^*, i^*) \neq 0$ where β is a normal vector of the stable manifold of (1.4) at x^* .
- 2) Suppose $\lim_{\varepsilon \downarrow 0} \frac{\delta}{\varepsilon} = 0$. For any critical point x^* of \bar{f} there exists $i^* \in \mathcal{M}$ such that $\beta' f(x^*, i^*) \neq 0$ where β is a normal vector of the stable manifold of (1.4) at x^* .
- 3) Suppose $\lim_{\varepsilon \downarrow 0} \frac{\delta}{\varepsilon} = \infty$. For any critical point x^* of \bar{f} , there exists $i^* \in \mathcal{M}$ such that $\beta' \sigma(x^*, i^*) \neq 0$ where β is a normal vector of the stable manifold of (1.4) at x^* .

The intuition for the conditions of Assumption 1.2 is the following. In case 2, since δ tends to 0 much faster than ε , for sufficiently small δ , the behavior of $X^{\varepsilon, \delta}(t)$ will be close to the

process $\xi^\varepsilon(t)$ defined by

$$(1.7) \quad d\xi^\varepsilon(t) = f(\xi^\varepsilon(t), \alpha^\varepsilon(t))dt.$$

We denote from now on by $\xi_{x,i}^\varepsilon(t)$ the solution of (1.7) at time $t \geq 0$ if the initial condition is (x, i) .

If for each $i \in \mathcal{M}$, $f(x^*, i) = 0$ at a critical point x^* of \bar{f} , the Dirac mass function at x^* , δ_{x^*} , will be an invariant measure for $\xi^\varepsilon(t)$. Because of this, the sequence of invariant probability measures $(\mu^{\varepsilon,\delta})$ (or one of its subsequences) may converge to δ_{x^*} . In order to have the weak convergence of $(\mu^{\delta,\varepsilon})_{\varepsilon>0}$ to the measure μ^0 , we need to assume that there is an $i^* \in \mathcal{M}$ such that $\beta' f(x^*, i^*) \neq 0$ where β is a normal vector of the stable manifold of (1.4) at x^* . This guarantees that the process from (1.7) gets pushed away from the equilibrium x^* and away from the stable manifold (where it could get pushed back towards the equilibrium).

In case 3, the switching is very fast compared to the diffusion term, so for small ε the process will behave like

$$d\eta^\varepsilon(t) = \bar{f}(\eta^\varepsilon(t))dt + \sqrt{\delta}\sigma(\eta^\varepsilon(t), i)dW(t).$$

In order for the limit of $(\mu^{\varepsilon,\delta})$ not to put mass on the critical point x^* of \bar{f} , we need to suppose that there exists an $i^* \in \mathcal{M}$ such that $\beta'\sigma(x^*, i^*) \neq 0$ where β is a normal vector of the stable manifold of (1.4) at x^* .

For case 1, since both the switching and the diffusion are on a similar scale, we need to assume that for each critical point x^* of \bar{f} there is $i^* \in \mathcal{M}$ satisfying either $\beta'\sigma(x^*, i^*) \neq 0$ or $\beta'f(x^*, i^*) \neq 0$.

The next theorem is the main result of this paper.

Theorem 1.1. *Suppose Assumptions 1.1 and 1.2 hold. The family of invariant probability measures $(\mu^{\varepsilon,\delta})_{\varepsilon>0}$ converges weakly to the measure μ^0 given by (1.5) in the sense that for every bounded and continuous function $g : \mathbb{R}^d \times \mathcal{M} \rightarrow \mathbb{R}$,*

$$\lim_{\varepsilon \rightarrow 0} \sum_{i=1}^m \int_{\mathbb{R}^d} g(x, i) \mu^{\varepsilon,\delta}(dx, i) = \frac{1}{T_\Gamma} \int_0^{T_\Gamma} \bar{g}(\bar{X}_y(t)) dt,$$

where T_Γ is the period of the limit cycle, $y \in \Gamma$ and $\bar{g}(x) = \sum_{i \in \mathcal{M}} g(x, i) \nu_i$.

Remark 1.5. *Theorem 1.1 still holds if the switching component α^ε is state-dependent with generator $Q(x) = (q_{ij}(x))_{\mathcal{M} \times \mathcal{M}}$, $x \in \mathbb{R}^d$ as long as Q is bounded and satisfies the following conditions:*

- For all i, j the functions $q_{ii}(\cdot)$ and $\frac{q_{ij}(\cdot)}{q_{ii}(x)}$ are Lipschitz continuous.
- If $q_{ij}(x) > 0$ for some $x \in \mathbb{R}^d$ then $\inf_{x \in \mathbb{R}^d} \frac{q_{ij}(x)}{|q_{ii}(x)|} > 0$.
- For all i we have $\inf_{x \in \mathbb{R}^d} |q_{ii}(x)| > 0$.
- There exists $k \in \mathbb{N}$ such that we have $\inf_{x \in \mathbb{R}^d} \hat{q}_{ij}^{(k)}(x) > 0$ where $\hat{Q}(x) = (0 \vee q_{ij}(x))_{\mathcal{M} \times \mathcal{M}}$, and $\hat{Q}^k(x) = (\hat{q}_{ij}^{(k)}(x))$.

We explain how one can do this in Remark 2.1.

1.2. An application of Theorem 1.1. We will exhibit an example where the result of Theorem 1.1 applies. Recently there has been renewed interest in stochastic population dynamics [HN18a, BL16, Ben18, HN18b]. Suppose we have a predator-prey system of the form

$$(1.8) \quad \begin{cases} \frac{d}{dt}x(t) = x(t) [a - bx(t) - y(t)h(x(t), y(t))] \\ \frac{d}{dt}y(t) = y(t) [-c - dy(t) + x(t)fh(x(t), y(t))] \end{cases}.$$

Here $x(t), y(t)$ denote the densities of prey and predator at time $t \geq 0$, respectively; $a, b, c, d, f > 0$ describe the per-capita birth/death and competition rates, and $xh(x, y), yh(x, y)$ are the functional responses of the predator and the prey. For instance, if $h(x, y)$ is constant, the model is the classical Lotka-Volterra one [Lot25, Vol28, GH79]. If

$$h(x, y) = \frac{m_1}{m_2(i) + m_3x + m_4y},$$

the functional response is of Beddington-DeAngelis type [CC01]. The setting of (1.8) is very general and encompasses many of the models used in the ecological literature.

We explore what happens in the fast-switching slow-noise limit for the following noisy extension of (1.8)

$$(1.9) \quad \begin{cases} dX^{\varepsilon, \delta}(t) = X^{\varepsilon, \delta}(t)\varphi(X^{\varepsilon, \delta}(t), Y^{\varepsilon, \delta}(t), \alpha^\varepsilon(t))dt + \sqrt{\delta}\lambda(\alpha^\varepsilon(t))X^{\varepsilon, \delta}(t)dW_1(t) \\ dY^{\varepsilon, \delta}(t) = Y^{\varepsilon, \delta}(t)\psi(X^{\varepsilon, \delta}(t), Y^{\varepsilon, \delta}(t), \alpha^\varepsilon(t))dt + \sqrt{\delta}\rho(\alpha^\varepsilon(t))Y^{\varepsilon, \delta}(t)dW_2(t). \end{cases}$$

Here

$$\begin{aligned} \varphi(x, y, i) &= a(i) - b(i)x - yh(x, y, i) \quad \text{and} \\ \psi(x, y, i) &= -c(i) - d(i)y + f(i)xh(x, y, i), \end{aligned}$$

where $a(\cdot), b(\cdot), c(\cdot), d(\cdot), f(\cdot), \lambda(\cdot), \rho(\cdot)$ are positive functions defined on \mathcal{M} , $\delta = \delta(\varepsilon)$ depends on ε , $\lim_{\varepsilon \rightarrow 0} \delta = 0$, $W_1(t)$ and $W_2(t)$ are independent Brownian motions, and α^ε is an independent Markov chain with generator Q/ε . As before, the generator Q is assumed to be irreducible so that the Markov chain has a unique stationary distribution given by $(\nu_1, \dots, \nu_{n_0})$. The function $h(\cdot, \cdot, \cdot)$ is assumed to be positive, bounded, and continuous on $\mathcal{M} \times \mathbb{R}_+^2$.

For $g(\cdot) = a(\cdot), b(\cdot), c(\cdot), d(\cdot), f(\cdot), \varphi(\cdot), \psi(\cdot)$, define the *averaged quantities* $\bar{g} := \sum g(i)\nu_i$, $g_m = \min\{g(i) : i \in \mathcal{M}\}$, $g_M = \max\{g(i) : i \in \mathcal{M}\}$. Set $h_1(x, y) := \sum h(x, y, i)\nu_i$ and $h_2(x, y) := \sum f(i)h(x, y, i)\nu_i$. The existence and uniqueness of a global positive solution to (1.9) can be proved in the same manner as in [JJ11] or [JJNS11] and is therefore omitted. We denote by $Z_{z,i}^{\varepsilon, \delta}(t) = (X_{z,i}^{\varepsilon, \delta}(t), Y_{z,i}^{\varepsilon, \delta}(t))$ the solution to (1.9) with initial value $\alpha^\varepsilon(0) = i \in \mathcal{M}$, $Z_{z,i}^{\varepsilon, \delta}(0) = z \in \mathbb{R}_+^2$. Consider the averaged equation

$$(1.10) \quad \begin{cases} \frac{d}{dt}X(t) = X(t)\bar{\varphi}(X(t), Y(t)) = X(t) [\bar{a} - \bar{b}X(t) - Y(t)h_1(X(t), Y(t))] \\ \frac{d}{dt}Y(t) = Y(t)\bar{\psi}(X(t), Y(t)) = Y(t) [-\bar{c} - \bar{d}Y(t) + X(t)h_2(X(t), Y(t))] \end{cases}.$$

We denote by $\bar{Z}_z(t) = (\bar{X}_z(t), \bar{Y}_z(t))$, the solution to (1.10) with initial value $\bar{Z}_z(0) = z$.

Assumption 1.3. *The following properties hold.*

- (1) The system (1.10) has a finite number of positive equilibria and a unique stable limit cycle Γ . In addition, any positive solution not starting at an equilibrium converges to the stable limit cycle.
- (2) The inequality

$$\frac{\bar{a}}{\bar{b}} h_2 \left(\frac{\bar{a}}{\bar{b}}, 0 \right) > \bar{c}$$

is satisfied.

Remark 1.6. Note that the Jacobian of $\left(x\bar{\varphi}(x, y), y\bar{\psi}(x, y) \right)^\top$ at $\left(\frac{\bar{a}}{\bar{b}}, 0 \right)$ has two eigenvalues: $-\bar{c} + \frac{\bar{a}}{\bar{b}} h_2 \left(\frac{\bar{a}}{\bar{b}}, 0 \right)$ and $-\frac{\bar{b}^2}{\bar{a}} < 0$. If $-\bar{c} + \frac{\bar{a}}{\bar{b}} h_2 \left(\frac{\bar{a}}{\bar{b}}, 0 \right) < 0$, then $\left(\frac{\bar{a}}{\bar{b}}, 0 \right)$ is a stable equilibrium of (1.10), which violates condition (i) of Assumption 1.3. This shows that condition (ii) is often contained in condition (i).

We note that the model (1.10) is quite general and as such conditions on the parameters for the existence and uniqueness of a limit cycle are in general complicated.

We can apply Theorem 1.1 to this model if we can verify part (v) of Assumption 1.1 since the other conditions are clearly satisfied. Since the process $\alpha^\varepsilon(t)$ is ergodic and the diffusion is nondegenerate, an invariant probability measure of the solution $Z^{\varepsilon, \delta}(t)$ is unique if it exists. It is unlikely that one could find a Lyapunov-type function satisfying the hypothesis of [YZ10, Theorem 3.26] in order to prove the existence of an invariant probability measure. In addition, the tightness of the family of invariant probability measures $(\mu^{\varepsilon, \delta})_{\varepsilon > 0}$ cannot be proved using the methods from [DDT11, DNY16].

These difficulties can be overcome with the help of a new technical tool. We partition the domain $(0, \infty)^2$ into several parts and then construct a truncated Lyapunov-type function. We then estimate the average probability that the solution belongs to a specific part of our partition. This then allows us to prove that the family of invariant probability measures $(\mu^{\varepsilon, \delta})_{\varepsilon > 0}$ is tight on the interior of \mathbb{R}_+^2 , i.e. for any $\eta > 0$, there are $0 < \varepsilon_0, \delta_0 < 1 < L$ such that for all $\varepsilon < \varepsilon_0, \delta < \delta_0$, the unique invariant measure $\mu^{\varepsilon, \delta}$ of $(Z^{\varepsilon, \delta}(t), \alpha^\varepsilon(t))$ satisfies

$$\mu^{\varepsilon, \delta}([L^{-1}, L]^2) \geq 1 - \eta.$$

We are able to prove the following result.

Theorem 1.2. Suppose Assumption 1.3 holds. For sufficiently small δ and ε , the process given by (1.9) has a unique invariant probability measure $\mu^{\varepsilon, \delta}$ with support in $\mathbb{R}_+^{2, \circ}$ (where $\mathbb{R}_+^{2, \circ}$ denotes the interior of \mathbb{R}_+^2). In addition:

- a) If $\lim_{\varepsilon \rightarrow 0} \frac{\delta}{\varepsilon} = l \in (0, \infty]$, the family of invariant measures $(\mu^{\varepsilon, \delta})_{\varepsilon > 0}$ converges weakly to μ^0 , the occupation measure of the limit cycle of (1.10), as $\varepsilon \rightarrow 0$ (in the sense of Theorem 1.1).
- b) If $\lim_{\varepsilon \rightarrow 0} \frac{\delta}{\varepsilon} = 0$ and at each critical point (x^*, y^*) of $(\bar{\varphi}(x, y), \bar{\psi}(x, y))$, there is $i^* \in \mathcal{M}$ such that either $\varphi(x^*, y^*, i^*) \neq 0$ or $\psi(x^*, y^*, i^*) \neq 0$, then the family of invariant measures $(\mu^{\varepsilon, \delta})_{\varepsilon > 0}$ converges weakly to μ^0 , the occupation measure of the limit cycle of (1.10), as $\varepsilon \rightarrow 0$.

Remark 1.7. We note that on any finite time interval $[0, T]$ the solutions to (1.9) converge to the solutions of (1.10). However, in ecology, people are interested in the long term behavior of ecosystems as $T \rightarrow \infty$. Therefore, the above result shows rigorously that (1.10) gives the correct long-term behavior.

1.3. Sketch of proof of Theorem 1.1. Because some parts of the proofs are very technical, in order to offer some intuition to the reader we present the main ideas in this subsection.

Condition (v) of Assumption 1.1 is a tightness assumption for the family of invariant measures $(\mu^{\varepsilon, \delta})_{0 < \varepsilon < \varepsilon_0}$. This implies that any weak limit of $(\mu^{\varepsilon, \delta})_{0 < \varepsilon < \varepsilon_0}$ is an invariant measure of the limit system (1.4). The main technical issue is to show that any subsequential limit of $(\mu^{\varepsilon, \delta})_{0 < \varepsilon < \varepsilon_0}$ does not assign any mass to any of the fixed points of \bar{f} . This is done by a careful analysis of the nature of the deterministic and stochastic systems near the attracting region $\chi_l := \{y : \lim_{t \rightarrow \infty} \bar{X}_y(t) = x_l\}$, of an equilibrium x_l of \bar{f} . Note that if x_l is a source then $\chi_l = \{x_l\}$ while if x_l is hyperbolic χ_l can be an unbounded set. This makes the problem hard.

In Section 2, using large deviation techniques, we establish the following uniform estimate for the probability that the processes $X_{x,i}^{\varepsilon, \delta}$ and \bar{X}_x are close on a fixed time interval: For any R, T , and $\gamma > 0$, there is a $\kappa = \kappa(R, \gamma, T) > 0$ such that

$$(1.11) \quad \mathbb{P} \left\{ \left| X_{x,i}^{\varepsilon, \delta}(t) - \bar{X}_x(t) \right| \geq \gamma \text{ for some } t \in [0, T] \right\} < \exp \left(-\frac{\kappa}{\varepsilon + \delta} \right), x \in B_R.$$

The main task is to estimate the time of exiting the attracting region, $\chi_l \cap B_R$, of an equilibrium x_l . To be precise, we show that $X_{x,i}^{\varepsilon, \delta}$ leaves small neighborhoods of $\chi_l \cap B_R$ with strictly positive probability in finite time if we start close to $\chi_l \cap B_R$. We find uniform lower bounds for these probabilities.

In fact, for any sufficiently small $\Delta > 0$ and sufficiently large $R > 0$ to include all the sets $M_i, i = 1, \dots, n_0$, we can find $\theta_1, \theta_3 > 0$, $H_l^\Delta > 0$, and $\varepsilon_l(\Delta)$ such that for $\varepsilon < \varepsilon_l(\Delta)$,

$$(1.12) \quad \mathbb{P} \left\{ \tilde{\tau}_{x,i}^{\varepsilon, \delta} \leq H_l^\Delta \right\} \geq \psi^{\Delta, \varepsilon} := \exp \left(-\frac{\Delta}{\varepsilon + \delta} \right), |x - x_l| < \theta_1,$$

where

$$\tilde{\tau}_{x,i}^{\varepsilon, \delta} := \inf \{ t \geq 0 : X_{x,i}^{\varepsilon, \delta}(t) \in B_R \text{ and } \text{dist}(X_{x,i}^{\varepsilon, \delta}(t), \chi_l) \geq \theta_3 \}.$$

We prove the estimate (1.12) in the different cases as follows:

- 1) Suppose that there is an $i^* \in \mathcal{M}$ satisfying $\beta' f(x_l, i^*) \neq 0$, where β is a normal unit vector of the stable manifold of (1.4) at x_l . Then we estimate the time $\alpha^\varepsilon(t)$ stays in i^* and consider the diffusion in this fixed state, that is

$$dZ^\delta(t) = f(Z^\delta(t), i^*)dt + \sqrt{\delta} \sigma(Z^\delta(t), i^*)dW(t).$$

Since the drift $f(x, i^*)$ is nonzero and pushes us away from the stable manifold of x^* , and the diffusion term is small, we can estimate the exit time $\tilde{\tau}_{x,i}^{\varepsilon, \delta}$.

- 2) Suppose $\lim_{\varepsilon \rightarrow 0} \frac{\delta}{\varepsilon} \in (0, \infty]$ and there is an i^* such that $\beta' \sigma(x_l, i^*) \neq 0$. If $\lim_{\varepsilon \rightarrow 0} \frac{\delta}{\varepsilon} < \infty$, suppose in addition that $\beta' f(x_l, i) = 0, i \in \mathcal{M}$. We estimate the time $\alpha^\varepsilon(t)$ to be in i^* and consider the diffusion component in the direction β in this fixed state

$$dZ^{\varepsilon, \delta} = \sqrt{\delta} \beta' \sigma(Z^{\varepsilon, \delta}, \alpha^\varepsilon) dW(t)$$

Since the diffusion coefficient does not vanish close to x_l , we can do time change so that we get a Brownian motion. Then we can estimate the probability that the exit time exceeds a given number. Ultimately, we show that $Z^{\varepsilon,\delta}$ and $\beta'X^{\varepsilon,\delta}$ are close to each other.

Comparing the rates in (1.11) with (1.12) is key to prove the main result in Section 3 (see e.g. [Hol78, Kif12]). The idea is to estimate the time of exiting the attracting region, $\chi_l \cap B_R$, of an equilibrium x_l as well as the time of coming back to this region. Then we prove that eventually, the probability of entering $\chi_l \cap B_R$ is very small compared to the probability of exiting the region.

If we start with $\bar{X}(0)$ close to $\chi_l \cap B_R$ then after a finite time \bar{X} will be close to one of the equilibrium points or the limit cycle. Using this together with (1.11) and (1.12) we get that there exist neighborhoods N_1, G_1 of $\chi_l \cap B_R$ with $N_1 \subset G_1$ such that

$$\mathbb{P}\{\tau_{x,i}^{\varepsilon,\delta} < L\} > \frac{1}{8}\psi^{\Delta,\varepsilon}, x \in N_1$$

for some constant $L > 0$ and

$$\tau_{x,i}^{\varepsilon,\delta} = \inf\{t \geq 0 : X_{x,i}^{\varepsilon,\delta}(t) \in B_R \setminus G_1\}.$$

This can be leveraged into showing that with high probability, if we start in N_1 , we will leave the region $G_1 \supset N_1$ in a finite, uniformly bounded, time:

$$(1.13) \quad \mathbb{P}\left\{\tau_{x,i}^{\varepsilon,\delta} < T_{\Delta,1}^{\varepsilon,\delta}\right\} > \frac{1}{2}, x \in N_1$$

where $T_{\Delta,1}^{\varepsilon,\delta} := C \exp\left(\frac{\Delta}{\varepsilon + \delta}\right)$. Using (1.11) we can find a constant $\hat{T} > 0$, independent of ε such that

$$(1.14) \quad \mathbb{P}\left\{X_{x,i}^{\varepsilon,\delta}(\hat{T}) \notin G_1\right\} \geq 1 - \exp\left(-\frac{\kappa}{\varepsilon + \delta}\right), x \in B_R \setminus N_1$$

and that

$$(1.15) \quad \mathbb{P}\left\{X_{x,i}^{\varepsilon,\delta}(t) \notin N_1, \text{ for all } t \in [0, \hat{T}]\right\} \geq 1 - \exp\left(-\frac{\kappa}{\varepsilon + \delta}\right), x \in B_R \setminus G_1.$$

Note that $T_{\Delta,1}^{\varepsilon,\delta} \rightarrow \infty$ as $\varepsilon \rightarrow 0$. However, if we pick $\Delta < \kappa/2$, we have

$$(1.16) \quad \lim_{\varepsilon \rightarrow 0} T_{\Delta,1}^{\varepsilon,\delta} \exp\left(-\frac{\kappa}{\varepsilon + \delta}\right) = \lim_{\varepsilon \rightarrow 0} \exp\left(\frac{\Delta}{\varepsilon + \delta}\right) \exp\left(-\frac{\kappa}{\varepsilon + \delta}\right) = 0.$$

The estimate (1.16) shows the exit time is not long compared to the good rate of large deviations, which will be used to show that invariant measures cannot put much mass on the equilibria. Let $\tilde{X}^{\varepsilon,\delta}(t)$ be the stationary solution, whose distribution is $\mu^{\varepsilon,\delta}$ for every time $t \geq 0$. Let $\tau^{\varepsilon,\delta}$ be the first exit time of $\tilde{X}^{\varepsilon,\delta}(t)$ from G_1 . We can show that for any $\eta > 0$ we can find $R > 0$ such that $\mu^{\varepsilon,\delta}(N_1) \leq 2\eta$ by using (1.14), (1.15), and (1.16) to find the

probabilities of the events

$$\begin{aligned} K_1^{\varepsilon,\delta} &= \left\{ \tilde{X}^{\varepsilon,\delta}(T_{\Delta,1}^{\varepsilon,\delta}) \in N_1, \tau^{\varepsilon,\delta} \geq T_{\Delta,1}^{\varepsilon,\delta}, \tilde{X}^{\varepsilon,\delta}(0) \in N_1 \right\} \\ K_2^{\varepsilon,\delta} &= \left\{ \tilde{X}^{\varepsilon,\delta}(T_{\Delta,1}^{\varepsilon,\delta}) \in N_1, \tau^{\varepsilon,\delta} < T_{\Delta,1}^{\varepsilon,\delta}, \tilde{X}^{\varepsilon,\delta}(0) \in N_1 \right\} \\ K_3^{\varepsilon,\delta} &= \left\{ \tilde{X}^{\varepsilon,\delta}(T_{\Delta,1}^{\varepsilon,\delta}) \in N_1, \tilde{X}^{\varepsilon,\delta}(0) \in B_R \setminus N_1 \right\} \\ K_4^{\varepsilon,\delta} &= \left\{ \tilde{X}^{\varepsilon,\delta}(T_{\Delta,1}^{\varepsilon,\delta}) \in N_1, \tilde{X}^{\varepsilon,\delta}(0) \notin B_R \right\}. \end{aligned}$$

Similar arguments show that for any $\eta > 0$, we can find $R > 0$ and neighborhoods N_1, \dots, N_{n_0-1} of $\chi_1 \cap B_R, \dots, \chi_{n_0-1} \cap B_R$ such that

$$\limsup_{\varepsilon \rightarrow 0} \mu^{\varepsilon,\delta}(\cup_{j=1}^{n_0-1} N_j) \leq 2^{n_0} \eta.$$

Using this fact together with Assumption 1.1 and Lemma 2.2 we can establish, by a straightforward modification of the proof of [Hol78, Theorem 1], that for any $\eta > 0$ there is neighborhood N of the limit cycle Γ such that

$$\liminf_{\varepsilon \downarrow 0} \mu^{\varepsilon,\delta}(N) > 1 - 2^{n_0} \eta.$$

2. ESTIMATES FOR THE FIRST EXIT TIMES

Define for any $i = 1, \dots, n_0$ and $\theta > 0$, the sets $\chi_i := \{y : \lim_{t \rightarrow \infty} \text{dist}(\bar{X}_y(t), M_i) = 0\}$ and $M_{i,\theta} := \{y : \text{dist}(y, M_i) < \theta\}$. Let $R_0 > 1$ be large enough such that B_{R_0-1} contains all M_i , $i = 1, \dots, n_0$. Fix $\theta_0 \in (0, 1)$ such that $\{M_{i,2\theta_0}, i = 1, \dots, n_0\}$ are mutually disjoint and $M_{i,2\theta_0} \cap \chi_j = \emptyset$ for $j < i$. For any $\eta > 0$, let $R = R_\eta > 0$ such that $\mu^{\varepsilon,\delta}(B_R) > 1 - \eta$ and $R < R_0$.

The following is a well-known exponential martingale inequality (see [Mao07, Theorem 1.7.4]).

Lemma 2.1. (*Exponential martingale inequality*) *Suppose $(g(t))$ is a real-valued \mathcal{F}_t -adapted process and $\int_0^T g^2(t) dt < \infty$ almost surely. Then for any $a, b > 0$ one has*

$$\mathbb{P} \left\{ \sup_{t \in [0, T]} \left[\int_0^t g(s) dW(s) - \frac{a}{2} \int_0^t g^2(s) ds \right] > b \right\} \leq e^{-ab}.$$

We will make use of this lemma repeatedly in the proofs to follow. The next result gives us estimates on how close the solutions to (1.3) and (1.7) are on a finite time interval if they have the same starting points. The argument of the proof is pretty standard. For completeness, it is relegated to Appendix A.

Lemma 2.2. *For any R, T , and $\gamma > 0$, there is a $\kappa = \kappa(R, \gamma, T) > 0$ such that*

$$\mathbb{P} \left\{ \left| X_{x,i}^{\varepsilon,\delta}(t) - \bar{X}_x(t) \right| \geq \gamma \text{ for some } t \in [0, T] \right\} < \exp \left(-\frac{\kappa}{\varepsilon + \delta} \right), x \in B_R.$$

Lemma 2.3. *Let N be an open set in \mathbb{R}^d and let $\tilde{\tau}_{x,i}^{\varepsilon,\delta}$ be any stopping time. Suppose that there is an $\ell > 0$ such that for all starting points $(x, i) \in N \times \mathcal{M}$ one has $\mathbb{P}\{\tilde{\tau}_{x,i}^{\varepsilon,\delta} < \ell\} \geq a^{\varepsilon,\delta} > 0$, where $\lim_{\varepsilon \rightarrow 0} a^{\varepsilon,\delta} = 0$. Then $\mathbb{P}\left\{\tilde{\tau}_{x,i}^{\varepsilon,\delta} < \frac{\ell}{a^{\varepsilon,\delta}}\right\} > 1/2$ for $(x, i) \in N \times \mathcal{M}$ if ε is sufficiently small.*

Lemma 2.4. *The following properties hold:*

- (1) *For any $\theta > 0, R > 0$, there exists $\tilde{T}_1 > 0$ such that for any $y \in B_R$, $\bar{X}_y(t) \in M_{k,\theta}$ for some $t < \tilde{T}_1$, and some $k \in \{1, \dots, n_0\}$.*
- (2) *For any $y \in B_R \setminus \chi_1$ and any $\theta > 0$, there exists $\tilde{t}_y > 0$ such that $\bar{X}_y(t_y) \in \bigcup_{k=2}^{n_0} M_{k,\theta}$.*
- (3) *For any $\theta_1 > 0, R \geq R_0$, there exists $\theta_2 > 0$ such that $\text{dist}(\bar{X}_y(t), \chi_1) > \theta_2$ for any $t > 0$ and $y \in B_R$ satisfying $\text{dist}(y, \chi_1) > \theta_1$.*
- (4) *Let β be a normal unit vector of the stable manifold of (1.4) at an equilibrium x_l . Then for any $m > 0$, we can find $\tilde{\theta}_0 > 0$ such that $\{y : |\beta'y| \geq \theta, |y| \leq m\theta\} \cap \chi_l = \emptyset$ for any $\theta \in (0, \tilde{\theta}_0]$*

The following lemmas show that the process leaves small neighborhoods around the equilibrium points with strictly positive probability in finite time if we start close to the equilibrium points. Furthermore, this probability can be bounded below uniformly for all starting points close to the equilibrium. We need this because we want to show the convergence of the process to the limit cycle Γ .

Lemma 2.5. *Consider an equilibrium x_l and suppose there exists $i^* \in \mathcal{M}$ such that $\beta'f(x_l, i^*) \neq 0$ where β is a normal unit vector of the stable manifold of (1.4) at x_l . Then for any $\Delta > 0$ that is sufficiently small and any $R > R_0$, we can find $\theta_1, \theta_3 > 0, H_l^\Delta > 0$, and $\varepsilon_l(\Delta)$ such that for $\varepsilon < \varepsilon_l(\Delta)$,*

$$\mathbb{P}\left\{\tilde{\tau}_{x,i}^{\varepsilon,\delta} \leq H_l^\Delta\right\} \geq \psi^{\Delta,\varepsilon} := \exp\left(-\frac{\Delta}{\varepsilon}\right), \quad x \in M_{l,\theta_1},$$

where

$$\tilde{\tau}_{x,i}^{\varepsilon,\delta} := \inf\{t \geq 0 : X_{x,i}^{\varepsilon,\delta}(t) \in B_R \text{ and } \text{dist}(X_{x,i}^{\varepsilon,\delta}(t), \chi_l) \geq \theta_3\}.$$

Proof. Suppose without loss of generality that $x_l = 0$. Let β be a normal vector of the stable manifold at 0 such that $|\beta| = 1$ and $\beta'f(0, i^*) > 0$. Since f is locally Lipschitz we can find $a_1 > 0$ such that

$$(2.1) \quad \beta'f(x, i^*) > a_1 > 0, \quad |x| < \theta_0.$$

Then $A_1 := \sup_{|x| < \theta_0} \left\{\frac{|f(x, i^*)|}{\beta'f(x, i^*)}\right\} < \infty$.

Since β is perpendicular to the tangent of the stable manifold at 0, we can find $\theta_2 \in \left(0, \frac{1}{2+3A_1} \left(\frac{a_1\Delta}{4|q_{i^*}|} \wedge \theta_0\right)\right)$ such that

$$(2.2) \quad \text{dist}(L_l^{\theta_2}, \chi_l) := \theta_3 > 0$$

where

$$(2.3) \quad L_l^{\theta_2} = \{x : |x| \leq (2 + 3A_1)\theta_2 \text{ and } |\beta'x| > \theta_2\}.$$

The continuous dependence of the solutions of (1.4) on the starting point and the fact that 0 is an equilibrium of (1.4) imply that \bar{X} stays close to 0 for a finite time if the starting point is close enough to 0. Using this, we can derive from Lemma 2.2 that there exist numbers $\theta_1 \in (0, \theta_2)$ and $k > 0$ such that

$$(2.4) \quad \mathbb{P} \left\{ |X_{x,i}^{\varepsilon,\delta}(t)| < \theta_2, 0 < t < 1 + \frac{1}{|q_{i^*i^*}|} \right\} > 1 - \exp \left(-\frac{k}{\varepsilon + \delta} \right) \text{ for all } x \in M_{l,\theta_1}, i \in \mathcal{M}.$$

First, we consider the case $\alpha^\varepsilon(0) = i^*$. Because of the independence of $\alpha^\varepsilon(\cdot)$ and $W(\cdot)$, if $\alpha^\varepsilon(t) = i^*$ for all $t \in \left[0, \frac{\Delta}{|q_{i^*i^*}|}\right]$, the process $X_{x,i^*}^{\varepsilon,\delta}(\cdot)$ has the same distribution on the time interval $\left[0, \frac{\Delta}{|q_{i^*i^*}|}\right]$ as that of Z_x^δ given by

$$(2.5) \quad dZ^\delta(t) = f(Z^\delta(t), i^*)dt + \sqrt{\delta}\sigma(Z^\delta(t), i^*)dW(t).$$

Define the bounded stopping time

$$\rho_x^{\varepsilon,\delta} := \frac{\Delta}{|q_{i^*i^*}|} \wedge \inf\{t > 0 : |Z_x^\delta(t)| \geq \theta_0\} \wedge \inf\{t > 0 : \beta'Z_x^\delta(t) \geq \theta_2\}.$$

We have

$$(2.6) \quad \beta'Z_x^\delta(\rho_x^{\varepsilon,\delta}) = \beta'x + \int_0^{\rho_x^{\varepsilon,\delta}} \beta'f(Z_x^\delta(s), i^*)ds + \int_0^{\rho_x^{\varepsilon,\delta}} \sqrt{\delta}\beta'\sigma(Z_x^\delta(s), i^*)dW(s), |x| \leq \theta_0.$$

By the exponential martingale inequality from Lemma 2.1, there exists a constant $m_3 > 0$ independent of δ such that

$$\mathbb{P}(\Omega_x^{\varepsilon,\delta,1}) > \frac{3}{4} \text{ and } \mathbb{P}(\Omega_{x,i}^{\varepsilon,\delta,2}) > \frac{3}{4}$$

where

$$\Omega_x^{\varepsilon,\delta,1} := \left\{ -\int_0^t \sqrt{\delta}\beta'\sigma(Z_x^\delta(s), i^*)dW(s) - \frac{1}{\sqrt{\delta}} \int_0^t \delta\beta'\sigma(Z_x^\delta(s), i^*)\sigma(Z_x^\delta(s), i^*)'\beta ds < m_3\sqrt{\delta}, t \in [0, \rho_x^{\varepsilon,\delta}] \right\}$$

and

$$\Omega_x^{\varepsilon,\delta,2} := \left\{ \left| \int_0^t \sqrt{\delta}\sigma(Z_x^\delta(s), i^*)dW(s) \right| - \frac{1}{\sqrt{\delta}} \int_0^t \delta |\sigma(Z_x^\delta(s), i^*)\sigma(Z_x^\delta(s), i^*)'| ds < m_3\sqrt{\delta}, t \in [0, \rho_x^{\varepsilon,\delta}] \right\}.$$

This implies that

$$(2.7) \quad \mathbb{P}(\Omega_x^{\varepsilon,\delta,1} \cap \Omega_{x,i}^{\varepsilon,\delta,2}) > \frac{1}{2}.$$

Using (2.1) and (2.6) we note that on the set $\Omega_x^{\varepsilon,\delta,1}$

$$\begin{aligned}
(2.8) \quad \beta' Z_x^\delta(\rho_x^{\varepsilon,\delta}) &> \beta' x + \int_0^{\rho_x^{\varepsilon,\delta}} \beta' f(Z_x^\delta(s), i^*) ds \\
&\quad - \frac{1}{\sqrt{\delta}} \int_0^{\rho_x^{\varepsilon,\delta}} \beta' \delta \sigma(Z_x^\delta(s), i^*)' \sigma(Z_x^\delta(s), i^*) \beta ds - m_3 \sqrt{\delta} \\
&\geq -\theta_2 + \int_0^{\rho_x^{\varepsilon,\delta}} a_1 ds - m_3 \sqrt{\delta}
\end{aligned}$$

Let δ be so small that $m_3 \sqrt{\delta} < \frac{a_1 \Delta}{2 |q_{i^* i^*}|}$. If $\rho_x^{\varepsilon,\delta}(\omega) = \frac{\Delta}{|q_{i^* i^*}|}$ for some $\omega \in \Omega_{x,i}^{\varepsilon,\delta,1}$, using

$$\theta_2 \leq \frac{a_1 \Delta}{4 |q_{i^* i^*}|} = \frac{a_1 \rho_x^{\varepsilon,\delta}}{4}, \text{ we get}$$

$$|\beta' Z_x^\delta(\rho_x^{\varepsilon,\delta}(\omega))| \leq \theta_2 < -\theta_2 + a_1 \rho_x^{\varepsilon,\delta} - m_3 \sqrt{\delta},$$

which contradicts (2.8). As a result, if $x \leq \theta_2, \omega \in \Omega_x^{\varepsilon,\delta,1}$ and δ is sufficiently small, we have

$$(2.9) \quad \rho_x^{\varepsilon,\delta}(\omega) < \frac{\Delta}{|q_{i^* i^*}|},$$

and by (2.6) we have

$$\begin{aligned}
(2.10) \quad \int_0^{\rho_x^{\varepsilon,\delta}} \beta' f(Z_x^\delta(s), i^*) ds &\leq |\beta' Z_x^\delta(\rho_x^{\varepsilon,\delta})| + |\beta' x| + \sqrt{\delta} \int_0^{\rho_x^{\varepsilon,\delta}} |\sigma(Z_x^\delta(s), i^*) \sigma(Z_x^\delta(s), i^*)'| ds + m_3 \sqrt{\delta} \\
&< 3\theta_2.
\end{aligned}$$

on $\Omega_x^{\varepsilon,\delta,1} \cap \Omega_x^{\varepsilon,\delta,2}$. Using (2.5) and (2.10), one sees that if δ is sufficiently small and $|x| < \theta_2$ then for $\omega \in \Omega_x^{\varepsilon,\delta,1} \cap \Omega_x^{\varepsilon,\delta,2}$,

$$\begin{aligned}
(2.11) \quad |Z_x(\rho_x^{\varepsilon,\delta})| &< |x| + \int_0^{\rho_x^{\varepsilon,\delta}} |f(Z_x(t), i^*)| dt + \sqrt{\delta} \int_0^{\rho_x^{\varepsilon,\delta}} |\sigma(Z_x^\delta(s), i^*) \sigma(Z_x^\delta(s), i^*)'| ds + m_3 \sqrt{\delta} \\
&< 2\theta_2 + A_1 \int_0^{\rho_x^{\varepsilon,\delta}} \beta' f(Z_x(t), i^*) dt \\
&< (2 + 3A_1)\theta_2 < \theta_0,
\end{aligned}$$

Combining (2.11) with the definition of $\rho_x^{\varepsilon,\delta}$ shows that $\beta' Z_x(\rho_x^{\varepsilon,\delta}) = \theta_2$ and $|Z_x(\rho_x^{\varepsilon,\delta})| < (2 + 3A_1)\theta_2$ on $\Omega_x^{\varepsilon,\delta,1} \cap \Omega_x^{\varepsilon,\delta,2}$. As a result of this and (2.7), if $|x| \leq \theta_2$,

$$\mathbb{P} \left\{ \beta' Z_x(t) \geq \theta_2, |Z_x(t)| \leq (2 + 3A_1)\theta_2 \text{ for some } t \in \left[0, \frac{\Delta}{|q_{i^* i^*}|} \right] \right\} \geq \mathbb{P} \left(\Omega_x^{\varepsilon,\delta,1} \cap \Omega_{x,i}^{\varepsilon,\delta,2} \right) > \frac{1}{2}.$$

Let

$$\zeta_{x,i}^{\varepsilon,\delta} := \inf \{ t > 0 : \beta' X_{x,i}^{\varepsilon,\delta}(t) \geq \theta_2, |X_{x,i}^{\varepsilon,\delta}| \leq (2 + 3A_1)\theta_2 \} = \inf \{ t > 0 : X_{x,i}^{\varepsilon,\delta} \in L_l^{\theta_2} \}.$$

Using the independence of α^ε , the paragraph before equation (2.5), and the last two equations, we obtain

$$(2.12) \quad \mathbb{P} \left\{ \zeta_{x,i^*}^{\varepsilon,\delta} \leq \frac{\Delta}{|q_{i^*i^*}|} \right\} > \frac{1}{2} \mathbb{P} \left\{ \alpha_{i^*}^\varepsilon(t) = i^*, \text{ for all } t \in \left[0, \frac{\Delta}{|q_{i^*i^*}|} \right] \right\} = \frac{1}{2} \exp\left(-\frac{\Delta}{\varepsilon}\right), \text{ if } |x| \leq \theta_1.$$

Since $\alpha^\varepsilon(t)$ is ergodic, for any sufficiently small ε , i.e., small enough Δ ,

$$(2.13) \quad \mathbb{P}\{\alpha_i^\varepsilon(t) = i^* \text{ for some } t \in [0, 1]\} > \frac{3}{4}, i \in \mathcal{M}.$$

By the strong Markov property, we derive from (2.4), (2.12), and (2.13) that for all $(x, i) \in M_{l,\theta_1} \times \mathcal{M}$ and for ε sufficiently small

$$(2.14) \quad \mathbb{P} \left\{ \zeta_{x,i}^{\varepsilon,\delta} < 1 + \frac{\Delta}{|q_{i^*i^*}|} \right\} \geq \frac{1}{4} \exp\left(-\frac{\Delta}{\varepsilon}\right).$$

The proof is complete by combining this estimate with (2.2). \square

Lemma 2.6. *Suppose that $\lim_{\varepsilon \rightarrow 0} \frac{\delta}{\varepsilon} = r > 0$. Assume that at the equilibrium point x_l , one has $f(x_l, i) = 0$ for all $i \in \mathcal{M}$, and there is $i^* \in \mathcal{M}$ for which $\beta' \sigma(x_l, i) \neq 0$, where β is a normal unit vector of the stable manifold of (1.4) at x_l . Then for any sufficiently small $\Delta > 0$ and any $R > R_0$, we can find $\theta_1, \theta_3 > 0$, $H_1^\Delta > 0$, and $\varepsilon_l(\Delta) > 0$ such that for $\varepsilon < \varepsilon_l(\Delta)$,*

$$\mathbb{P} \left\{ \tilde{\tau}_{x,i}^{\varepsilon,\delta} \leq H_1^\Delta \right\} \geq \psi^{\Delta,\varepsilon} := \exp\left(-\frac{\Delta}{\delta}\right), \text{ for all } (x, i) \in M_{l,\theta_1} \times \mathcal{M},$$

where

$$\tilde{\tau}_{x,i}^{\varepsilon,\delta} = \inf\{t \geq 0 : X_{x,i}^{\varepsilon,\delta}(t) \in B_R \text{ and } \text{dist}(X_{x,i}^{\varepsilon,\delta}(t), \chi_l) \geq \theta_3\}.$$

Proof. We can assume without loss of generality that $x_l = 0$ and $\lim_{\varepsilon \rightarrow 0} \frac{\delta}{\varepsilon} = 1$. Since σ is locally Lipschitz, we can find $a_2 > 0$ such that

$$(2.15) \quad a_2 < \beta'(\sigma\sigma')(y, i^*)\beta, |y| < \theta_0.$$

Let $K_l > 0$ be such that $|f(x, i)| < K_l|x|$ and $|(\sigma\sigma')(x, i)| < K_l$ if $|x| < \theta_0, i \in \mathcal{M}$. Fix $T > 0$ such that $\frac{a_2\nu_{i^*}T}{2} > 1$ and let $\theta_1 > 0$ be such that

$$(2.16) \quad (2 + K_l T)^2 e^{K_l T} \theta_1 < \theta_0$$

and $\text{dist}(L_l^{\theta_1}, \chi_l) := \theta_3 > 0$ where

$$(2.17) \quad L_l^{\theta_1} := \{x : |x| \leq (2 + K_l T)^2 e^{K_l T} \theta_1 \text{ and } |\beta'x| > \theta_1\}.$$

Define

$$\zeta_{t,x,i} := \inf \left\{ u > 0 : \int_0^u \beta'(\sigma\sigma') \left(\left(1 \wedge \frac{\theta_0}{|X_{x,i}^{\varepsilon,\delta}(s)|} \right) X_{x,i}^{\varepsilon,\delta}(s), \alpha_i^\varepsilon(s) \right) \beta ds \geq t \right\}.$$

For all $t \geq 0$, we have by (2.15) and the ergodicity of the Markov chain α_i^ε that

$$\mathbb{P}(\zeta_{t,x,i} < \infty) = 1, |x| < \theta_0.$$

As a result the process $(M(t))_{t \geq 0}$ defined by

$$M(t) = \int_0^{\zeta_{t,x,i}} \beta' \sigma \left(\left(1 \wedge \frac{\theta_0}{|X_{x,i}^{\varepsilon,\delta}(s)|} \right) X_{x,i}^{\varepsilon,\delta}(s), \alpha_i^\varepsilon(s) \right) dW(s)$$

is a Brownian motion. This follows from the fact that $M(t)$ is a continuous martingale with quadratic variation $[M, M]_t = t, t \geq 0$.

Set $\theta_2 := (2 + K_l T)\theta_1$. Since $M(1)$ has the distribution of a standard normal, for sufficiently small δ , we have the estimate

$$(2.18) \quad \mathbb{P}\{\sqrt{\delta}M(1) > \theta_2\} \geq \frac{1}{2} \exp\left(-\frac{\theta_2^2}{\delta}\right), |x| < \theta_0.$$

Using the large deviation principle (see [HYZ11]), we can find $a_3 = a_3(T) > 0$ such that

$$(2.19) \quad \mathbb{P}\left\{\frac{1}{T} \int_0^T \mathbf{1}_{\{\alpha_i^\varepsilon(s)=i^*\}} ds > \frac{\nu_{i^*}}{2}\right\} \geq 1 - \exp\left(-\frac{a_3}{\varepsilon}\right).$$

Equation (2.15), the definition of $\zeta_{t,x,i}$, and $\frac{a_2 \nu_{i^*} T}{2} > 1$ yield

$$\mathbb{P}\left\{\int_0^T \beta'(\sigma\sigma') \left(\left(1 \wedge \frac{\theta_0}{|X_{x,i}^{\varepsilon,\delta}(s)|} \right) X_{x,i}^{\varepsilon,\delta}(s), \alpha_i^\varepsilon(s) \right) \beta ds \geq \frac{a_2 \nu_{i^*} T}{2}\right\} \geq 1 - \exp\left(-\frac{a_3}{\varepsilon}\right), |x| < \theta_0,$$

which leads to

$$(2.20) \quad \mathbb{P}\{\zeta_{1,x,i} \leq T\} \geq 1 - \exp\left(-\frac{a_3}{\varepsilon}\right), |x| < \theta_0.$$

Define for $|x| < \theta_0, i \in \mathcal{M}$

$$\Omega_{x,i}^{\varepsilon,\delta,3} := \left\{ \left| \sqrt{\delta} \int_0^t \sigma \left(\left(1 \wedge \frac{\theta_0}{|X_{x,i}^{\varepsilon,\delta}(s)|} \right) X_{x,i}^{\varepsilon,\delta}(s), \alpha_i^\varepsilon(s) \right) dW(s) \right| < \frac{\theta_2}{\delta} \int_0^t \delta \left| (\sigma'\sigma) \left(\left(1 \wedge \frac{\theta_0}{|X_{x,i}^{\varepsilon,\delta}(s)|} \right) X_{x,i}^{\varepsilon,\delta}(s), \alpha_i^\varepsilon(s) \right) \right| ds + \theta_2 \leq (K_l T + 1)\theta_2, t \in [0, T] \right\}$$

and note that the last inequality holds by the definition of K_l . By Lemma 2.1

$$(2.21) \quad \mathbb{P}(\Omega_{x,i}^{\varepsilon,\delta,3}) \geq 1 - \exp\left(-\frac{2\theta_2^2}{\delta}\right), |x| < \theta_0.$$

Define the stopping time

$$\zeta_{x,i} = \inf\{t > 0 : |\beta' X_{x,i}^{\varepsilon,\delta}(t)| \geq \theta_1\} \wedge \inf\{t > 0 : |X_{x,i}^{\varepsilon,\delta}(t)| \geq (K_l + 2)\theta_2 e^{K_l T}\}.$$

If $|x| \leq \theta_1$ and $\omega \in \{\sqrt{\delta}M(1) > \theta_2\} \cap \{\zeta_{1,x,i} \leq T\} \cap \Omega_{x,i}^{\varepsilon,\delta,3}$, we claim that we must have

$$(2.22) \quad \zeta_{x,i} < T.$$

We argue by contradiction. Suppose the three events $\{\sqrt{\delta}M(1) > \theta_2\}$, $\{\zeta_{1,x,i} \leq T\}$, and $\{\zeta_{x,i} \geq \zeta_{1,x,i}\}$ happen simultaneously. Then we get the contradiction

$$\begin{aligned} \theta_2 &= (2 + K_l T)\theta_1 < \sqrt{\delta}M(1) = \sqrt{\delta} \int_0^{\zeta_{1,x,i}} \beta' \sigma \left(\left(1 \wedge \frac{\theta_0}{|X_{x,i}^{\varepsilon,\delta}(s)|} \right) X_{x,i}^{\varepsilon,\delta}(s), \alpha_i^\varepsilon(s) \right) dW(s) \\ &\leq |\beta' X_{x,i}^{\varepsilon,\delta}(\zeta_1)| + |\beta' x| + \left| \int_0^{\zeta_{1,x,i}} \beta' f(X_{x,i}^{\varepsilon,\delta}(s), \alpha_i^\varepsilon(s)) ds \right| \\ &\leq 2\theta_1 + \int_0^{\zeta_{1,x,i}} K_l |\beta' X_{x,i}^{\varepsilon,\delta}(s)| ds < (2 + K_l T)\theta_1 = \theta_2, \end{aligned}$$

where we used that $\left(1 \wedge \frac{\theta_0}{|X_{x,i}^{\varepsilon,\delta}(s)|} \right) X_{x,i}^{\varepsilon,\delta}(s) = X_{x,i}^{\varepsilon,\delta}(s)$ if $s < \zeta_{x,i}$ by the definition of $\zeta_{x,i}$ and (2.16).

For $|x| \leq \theta_1$ and $\omega \in \{\sqrt{\delta}M(1) > \theta_2\} \cap \{\zeta_{x,i} \leq T\} \cap \Omega_{x,i}^{\varepsilon,\delta,3}$, for any $0 \leq t \leq \zeta_{1,x,i} \leq T$,

$$\begin{aligned} |X_{x,i}^{\varepsilon,\delta}(t)| &\leq |x| + \sqrt{\delta} \left| \int_0^t \sigma(X_{x,i}^{\varepsilon,\delta}(s), \alpha_i^\varepsilon(s)) dW(s) \right| + \int_0^t |f(X_{x,i}^{\varepsilon,\delta}(s), \alpha_i^\varepsilon(s))| ds \\ &< (K_l T + 2)\theta_2 + K_l \int_0^t |X_{x,i}^{\varepsilon,\delta}(s)| ds. \end{aligned}$$

This together with Gronwall's inequality implies that

$$|X_{x,i}^{\varepsilon,\delta}(t)| < (K_l T + 2)\theta_2 e^{K_l T}, t \in [0, \zeta_{x,i}]$$

Thus for $|x| \leq \theta_1$ and $\omega \in \{\sqrt{\delta}M(1) > \theta_2\} \cap \{\zeta_{x,i} \leq T\} \cap \Omega_{x,i}^{\varepsilon,\delta,3}$, we have that $\zeta_{x,i} < T$ and $X_{x,i}^{\varepsilon,\delta}(\zeta_{x,i}) < (K_l T + 2)\theta_2 e^{K_l T}$ and $\beta' X_{x,i}^{\varepsilon,\delta}(\zeta_{x,i}) \geq \theta_1$.

Since $\theta_2 < a_3$ and $\lim_{\varepsilon \rightarrow 0} \frac{\delta}{\varepsilon} = 1$ we have by (2.18), (2.20), (2.21) and (2.22) that for all sufficiently small ε

$$\mathbb{P}(\{\sqrt{\delta}M(1) > \theta_2\} \cap \{\zeta_{x,i} \leq T\} \cap \Omega_{x,i}^{\varepsilon,\delta,3}) \geq \frac{1}{4} \exp\left(-\frac{\theta_2^2}{\delta}\right) \geq \frac{1}{4} \exp\left(-\frac{\Delta}{\delta}\right), |x| < \theta_1$$

if $\Delta < \theta_2^2$, which completes the proof. \square

Lemma 2.7. *Suppose that $\lim_{\varepsilon \rightarrow 0} \frac{\delta}{\varepsilon} = \infty$. Assume that at the equilibrium point x_l one can find $i^* \in \mathcal{M}$ such that $\beta' \sigma(x_l, i^*) \neq 0$ where β is a normal unit vector of the stable manifold of (1.4) at x_l . Then for any sufficiently small $\Delta > 0$ and any $R < R_0$ we can find $\theta_1, \theta_3 > 0$, $H_l^\Delta > 0$, and $\varepsilon_1(\Delta)$ such that for $\varepsilon < \varepsilon_1(\Delta)$,*

$$\mathbb{P} \left\{ \tilde{\tau}_{x,i}^{\varepsilon,\delta} \leq H_l^\Delta \right\} \geq \psi^{\Delta,\varepsilon} := \exp\left(-\frac{\Delta}{\delta}\right) \text{ for all } (x, i) \in M_{l,\theta_1} \times \mathcal{M},$$

where

$$\tilde{\tau}_{x,i}^{\varepsilon,\delta} = \inf\{t \geq 0 : X_{x,i}^{\varepsilon,\delta}(t) \in B_R \text{ and } \text{dist}(X_{x,i}^{\varepsilon,\delta}(t), \chi_l) \geq \theta_3\}.$$

Proof. Assume, as in the previous lemmas, that $x_l = 0$. Pick a number $a_2 > 0$ for which

$$a_2 < \beta'(\sigma\sigma')(y, i^*)\beta, |y| < \theta_0.$$

Let $K_l > 0$ be such that $|\bar{f}(x)| < K_l|x|$ and $|(\sigma'\sigma)(x, i)| < K_l$ whenever $|x| < \theta_0$, and fix $T > 0$ such that $\frac{a_2\nu_{i^*}T}{2} > 1$. Let $\theta_1 > 0$ be such that $(3 + K_lT)^2e^{K_lT}\theta_1 < \theta_0$ and $\text{dist}(L_l^{\theta_1}, \chi_l) := \theta_3 > 0$ where

$$(2.23) \quad L_l^{\theta_1} = \{x : |x - x_l| \leq (3 + K_lT)^2e^{K_lT}\theta_1 \text{ and } |\beta'(x - x_l)| > \theta_1\}.$$

Define $\theta_2 = (3 + K_lT)\theta_1$ and let $a_2, M(t), T, \zeta_{1,x,i}$ be as in the proof of Lemma 2.6. Arguing as in the proof of (2.20), we can find $a_3 > 0$ such that

$$\mathbb{P}\{\zeta_{1,x,i} \leq T\} \geq 1 - \exp\left(-\frac{a_3}{\varepsilon}\right), |x| < \theta_0.$$

Since $\bar{f}(0) = 0$, we can apply the large deviation principle (see [HYZ11]) to show that there is $\kappa = \kappa(\Delta) > 0$ such that

$$(2.24) \quad \mathbb{P}(A) \geq 1 - \exp\left(-\frac{\kappa}{\varepsilon}\right),$$

where $A := \{|\int_0^u f(0, \alpha_i^\varepsilon(s))ds| < \theta_1, \text{ for all } u \in [0, T]\}$. The estimates

$$\begin{aligned} M(1) &= \int_0^{\zeta_{1,x,i}} \beta' \sigma(X_{x,i}^{\varepsilon,\delta}(s), \alpha_i^\varepsilon(s))dW(s) \\ &\leq |\beta' X_{x,i}^{\varepsilon,\delta}(\zeta_{1,x,i})| + |\beta' x| + \left| \int_0^{\zeta_{1,x,i}} \beta' f(0, \alpha_i^\varepsilon(s))ds \right| \\ &\quad + \int_0^{\zeta_{1,x,i}} |\beta'(f(X_{x,i}^{\varepsilon,\delta}(s), \alpha_i^\varepsilon(s)) - f(0, \alpha_i^\varepsilon(s)))|ds. \end{aligned}$$

and

$$\begin{aligned} |X_{x,i}^{\varepsilon,\delta}(t)| &\leq |x| + \sqrt{\delta} \left| \int_0^t \sigma(X_{x,i}^{\varepsilon,\delta}(s), \alpha_i^\varepsilon(s))dW(s) \right| + \int_0^t |\bar{f}(X_{x,i}^{\varepsilon,\delta}(s))|ds \\ &\quad + \int_0^t |\bar{f}(X_{x,i}^{\varepsilon,\delta}(s)) - f(X_{x,i}^{\varepsilon,\delta}(s), \alpha_i^\varepsilon(s))|ds \end{aligned}$$

together with arguments similar to those from the proof of Lemma 2.6 show that

$$\mathbb{P}\left\{X_{x,i}^{\varepsilon,\delta}(t) \in L_l^{\theta_1} \text{ for some } t \in [0, T]\right\} \geq \frac{1}{4} \exp\left(-\frac{\Delta}{\delta}\right), (x, i) \in M_{l,\theta_1} \times \mathcal{M}$$

if δ is sufficiently small. \square

Remark 2.1. *The results in this section still hold true if one assumes the generator $Q(\cdot)$ of $\alpha(\cdot)$ is state dependent – see an explanation of the exact setting in Remark 1.5. By the large deviation principle in [BDG⁺18, Section 3] and the truncation arguments in Lemma A.1, we can obtain Lemma 2.2 for the case of state-dependent switching. It should be noted that while [BDG⁺18] only considers Case 1 of (1.6), using the variational representation, the arguments in [BDG⁺18, Section 3] can be applied to obtain Lemma 2.2 for the other cases.*

We can also infer from the large deviation principle that (2.13), (2.19) and (2.24) hold in this setting. As a result, Lemmas 2.5, 2.6 and 2.7 hold. These lemmas, in combination with the proofs from Section 3 imply that the main result, Theorem 1.1, remains unchanged if one has state-dependent switching.

3. PROOF OF THE MAIN RESULT

This section provides the proofs of the convergence of $\mu^{\varepsilon, \delta}$ for the three cases given in (1.6).

Proposition 3.1. *For every $\eta > 0$, there exists $R > R_0$ and neighborhoods N_1, \dots, N_{n_0-1} of $\chi_1 \cap B_R, \dots, \chi_{n_0-1} \cap B_R$ such that*

$$\limsup_{\varepsilon \rightarrow 0} \mu^{\varepsilon, \delta}(\cup_{j=1}^{n_0-1} N_j) \leq 2^{n_0} \eta.$$

Proof. For any $\eta > 0$, let $R > R_0$ be such that $\mu^{\varepsilon, \delta}(B_R) \geq 1 - \eta$. Define

$$S_1 = \{y \in B_R : \text{dist}(y, \chi_1 \cap B_R) < \theta_0\}$$

In view of Lemma 2.4, there exists $c_2 > 0$ such that for all $t \geq 0$

$$(3.1) \quad \text{dist}(\bar{X}_y(t), \chi_1) \geq 2c_2 \text{ for any } y \in B_R \setminus S_1.$$

Define

$$G_1 = \{y \in B_R : \text{dist}(y, \chi_1 \cap B_R) < c_2\}.$$

There exists $c_3 > 0$ such that

$$(3.2) \quad \text{dist}(\bar{X}_y(t), \chi_1) \geq 2c_3 \text{ for any } y \in B_R \setminus G_1, t \geq 0.$$

Note that we have $2c_3 \leq c_2$ and $2c_2 \leq \theta_0$. Define

$$N_1 = \{y \in B_R : \text{dist}(y, \chi_1 \cap B_R) < c_3\}$$

In view of Lemma 2.4, for any $y \notin \chi_1$, there exists \tilde{t}_y such that $\bar{X}_y(\tilde{t}_y) \in M_{i, \theta_0} \cap (B_R \setminus S_1)$ for some $i > 1$. This fact together with the continuous dependence of solutions to initial values and (3.1) implies that there exists $\hat{T} > 0$ such that

$$(3.3) \quad \text{dist}(\bar{X}_y(t), \chi_1) \geq 2c_2 \text{ for any } t \geq \hat{T}, y \in B_R \setminus N_1.$$

Let $\kappa = \kappa(R, c_3, \hat{T})$ be as in Lemma 2.2 and $\Delta < \frac{\kappa}{2}$ and θ_1 and ψ_ε^Δ be as in one of the Lemmas 2.5, 2.6 and 2.7 (depending on which case we are considering). We have

$$(3.4) \quad \mathbb{P}(\tilde{\tau}_{x,i}^{\varepsilon, \delta} < H^\Delta) \geq \psi_\varepsilon^\Delta, x \in M_{1, \theta_1}$$

where, as in Section 2, the stopping time is

$$\tilde{\tau}_{x,i}^{\varepsilon, \delta} = \inf\{t \geq 0 : X_{x,i}^{\varepsilon, \delta}(t) \in B_R \text{ and } \text{dist}(X_{x,i}^{\varepsilon, \delta}(t), \chi_1) \geq \theta_3\}.$$

Define

$$\tau_{x,i}^{\varepsilon, \delta} = \inf\{t \geq 0 : X_{x,i}^{\varepsilon, \delta}(t) \in B_R \setminus G_1\}.$$

It follows from part (1) of Lemma 2.4 that for any $x \in N_1$, there exists a $\tilde{T}_1 > 0$ such that $\bar{X}_x(t_x) \in \bigcup_{j=1}^{n_0} M_{j, \frac{\theta_1}{2}}$ for some $t_x \leq \tilde{T}_1$.

Suppose $\bar{X}_x(t_x) \in \bigcup_{j=2}^{n_0} M_{j, \frac{\theta_1}{2}}$. Note that $\bigcup_{j=2}^{n_0} M_{j, \frac{\theta_1}{2}} \cap M_{1, c_3} = \emptyset$, $\theta_1 < \theta_0$ and that by construction, $M_{1, 2\theta_0} \cap \chi_j = \emptyset, j > 1$. These facts imply that $\bigcup_{j=2}^{n_0} M_{j, \frac{\theta_1}{2}} \cap N_1 = \emptyset$. This together with Lemma 2.2 and (3.3) implies

$$(3.5) \quad \mathbb{P}\{\tau_{x,i}^{\varepsilon, \delta} < \tilde{T}_1 + \hat{T}\} > \frac{1}{2}$$

for small $\varepsilon > 0$.

When ε is sufficiently small, we have by Lemma 2.2 (applied with $\gamma = \frac{\theta_1}{2}$) that for any $x \in N_1$ satisfying $\bar{X}_x(t_x) \in M_{1, \frac{\theta_1}{2}}$ that

$$(3.6) \quad \mathbb{P}\{X_{x,i}^{\varepsilon,\delta}(t_x) \in M_{1,\theta_1}\} > \frac{1}{2}.$$

Similarly to (3.3), there exists a $\tilde{T}_2 > 0$ such that

$$\text{dist}(\bar{X}_y(t), \chi_1) \geq 2c_2 \text{ for any } t \geq \tilde{T}_2, y \in B_R, \text{dist}(y, \chi_1) \geq \theta_3,$$

which implies that by Lemma 2.2, for sufficiently small $\varepsilon > 0$,

$$(3.7) \quad \mathbb{P}\left\{\text{dist}(X_{x,i}^{\varepsilon,\delta}(\tilde{T}_2), \chi_1) \geq c_2\right\} > \frac{1}{2} \text{ for any } x \in B_R, \text{dist}(x, \chi_1) \geq \theta_3, i \in \mathcal{M}.$$

Putting together (3.4), (3.6), and (3.7) we deduce that

$$(3.8) \quad \mathbb{P}\{\tau_{x,i}^{\varepsilon,\delta} < \tilde{T}_1 + H^\Delta + \tilde{T}_2\} > \frac{1}{4}\psi_\varepsilon^\Delta.$$

for ε sufficiently small. Combining (3.5) and (3.8), we get that

$$(3.9) \quad \mathbb{P}\{\tau_{x,i}^{\varepsilon,\delta} < H^\Delta + \tilde{T}_1 + \tilde{T}_2 + \hat{T}\} > \frac{1}{8}\psi_\varepsilon^\Delta, x \in N_1.$$

Define $T_{\Delta,1}^{\varepsilon,\delta} := 4\frac{H^\Delta + \tilde{T}_1 + \tilde{T}_2 + \hat{T}}{\psi_{\varepsilon,\delta}^\Delta}$. Applying Lemma 2.3 to (3.9), we have

$$(3.10) \quad \mathbb{P}\left\{\tau_{x,i}^{\varepsilon,\delta} < T_{\Delta,1}^{\varepsilon,\delta}\right\} > \frac{1}{2}, x \in N_1.$$

We will argue by contradiction that $\limsup_{\varepsilon \rightarrow 0} \mu^{\varepsilon,\delta}(N_1) \leq 2\eta$. Assume that $\limsup_{\varepsilon \rightarrow 0} \mu^{\varepsilon,\delta}(N_1) > 2\eta > 0$. Since $\Delta < \kappa/2$, we have

$$(3.11) \quad \lim_{\varepsilon \rightarrow 0} T_{\Delta,1}^{\varepsilon,\delta} \exp\left(-\frac{\kappa}{\varepsilon + \delta}\right) = 0.$$

Let $\tilde{X}^{\varepsilon,\delta}(t)$ be the stationary solution, whose distribution is $\mu^{\varepsilon,\delta}$ for every time $t \geq 0$. Let $\tau^{\varepsilon,\delta}$ be the first exit time of $\tilde{X}^{\varepsilon,\delta}(t)$ from G_1 . Define the events

$$\begin{aligned} K_1^{\varepsilon,\delta} &= \left\{ \tilde{X}^{\varepsilon,\delta}(T_{\Delta,1}^{\varepsilon,\delta}) \in N_1, \tau^{\varepsilon,\delta} \geq T_{\Delta,1}^{\varepsilon,\delta}, \tilde{X}^{\varepsilon,\delta}(0) \in N_1 \right\} \\ K_2^{\varepsilon,\delta} &= \left\{ \tilde{X}^{\varepsilon,\delta}(T_{\Delta,1}^{\varepsilon,\delta}) \in N_1, \tau^{\varepsilon,\delta} < T_{\Delta,1}^{\varepsilon,\delta}, \tilde{X}^{\varepsilon,\delta}(0) \in N_1 \right\} \\ K_3^{\varepsilon,\delta} &= \left\{ \tilde{X}^{\varepsilon,\delta}(T_{\Delta,1}^{\varepsilon,\delta}) \in N_1, \tilde{X}^{\varepsilon,\delta}(0) \in B_R \setminus N_1 \right\} \\ K_4^{\varepsilon,\delta} &= \left\{ \tilde{X}^{\varepsilon,\delta}(T_{\Delta,1}^{\varepsilon,\delta}) \in N_1, \tilde{X}^{\varepsilon,\delta}(0) \notin B_R \right\}. \end{aligned}$$

Note that the above events are disjoint and have union N_1 . As such

$$\mu^{\varepsilon,\delta}(N_1) = \sum_{n=1}^4 \mathbb{P}\{K_n^{\varepsilon,\delta}\}.$$

Using (3.10), we get that

$$(3.12) \quad \mathbb{P}(K_1^{\varepsilon,\delta}) \leq \frac{1}{2}\mu^{\varepsilon,\delta}(N_1) \quad \text{and} \quad \mathbb{P}(K_4^{\varepsilon,\delta}) \leq 1 - \mu^{\varepsilon,\delta}(B_R) < \eta.$$

Next, we estimate $\mathbb{P}(K_3^{\varepsilon,\delta})$. It follows from Lemma 2.2, (3.2), and (3.3) that if ε is sufficiently small then

$$\mathbb{P}\left\{X_{x,i}^{\varepsilon,\delta}(\hat{T}) \notin G_1\right\} \geq 1 - \exp\left(-\frac{\kappa}{\varepsilon + \delta}\right), x \in B_R \setminus N_1$$

and

$$\mathbb{P}\left\{X_{x,i}^{\varepsilon,\delta}(t) \notin N_1, \text{ for all } t \in [0, \hat{T}]\right\} \geq 1 - \exp\left(-\frac{\kappa}{\varepsilon + \delta}\right), x \in B_R \setminus G_1.$$

Using the last two estimates together with the Markov property one sees that for any $x \in B_R \setminus G_1, i \in \mathcal{M}, s \in [0, T_{\Delta,1}^{\varepsilon,\delta}]$,

$$(3.13) \quad \begin{aligned} & \mathbb{P}\left\{X_{x,i}^{\varepsilon,\delta}(s) \in N_1\right\} \\ &= \mathbb{P}\left\{X_{x,i}^{\varepsilon,\delta}(s) \in N_1, X_{x,i}^{\varepsilon,\delta}(\hat{T}) \notin B_R \setminus G_1\right\} \\ & \quad + \sum_{n=2}^{\lfloor s/\hat{T} \rfloor} \mathbb{P}\left\{X_{x,i}^{\varepsilon,\delta}(s) \in N_1, X_{x,i}^{\varepsilon,\delta}(n\hat{T}) \notin B_R \setminus G_1, X_{x,i}^{\varepsilon,\delta}(\ell\hat{T}) \in B_R \setminus G_1, \ell = 1, \dots, n-1\right\} \\ & \quad + \mathbb{P}\left\{X_{x,i}^{\varepsilon,\delta}(s) \in N_1, X_{x,i}^{\varepsilon,\delta}(\ell\hat{T}) \in B_R \setminus G_1, \ell = 1, \dots, \lfloor s/\hat{T} \rfloor\right\} \\ & \leq \mathbb{P}\left\{X_{x,i}^{\varepsilon,\delta}(\hat{T}) \notin B_R \setminus G_1\right\} + \sum_{n=2}^{\lfloor s/\hat{T} \rfloor} \mathbb{P}\left\{X_{x,i}^{\varepsilon,\delta}(n\hat{T}) \notin B_R \setminus G_1, X_{x,i}^{\varepsilon,\delta}((n-1)\hat{T}) \in B_R \setminus G_1\right\} \\ & \quad + \mathbb{P}\left\{X_{x,i}^{\varepsilon,\delta}(t) \in N_1, \text{ for some } t \in \left[\lfloor s/\hat{T} \rfloor \hat{T}, \lfloor s/\hat{T} \rfloor \hat{T} + \hat{T}\right], X_{x,i}^{\varepsilon,\delta}(\lfloor s/\hat{T} \rfloor \hat{T}) \in B_R \setminus G_1\right\} \\ & \leq \left(\lfloor s/\hat{T} \rfloor + 1\right) \exp\left(-\frac{\kappa}{\varepsilon + \delta}\right) \\ & \leq \left(s/\hat{T} + 1\right) \exp\left(-\frac{\kappa}{\varepsilon + \delta}\right), \end{aligned}$$

where $\lfloor s/\hat{T} \rfloor$ denotes the integer part of s/\hat{T} .

Note that similar arguments show that (3.13) also holds for all $s \in [\hat{T}, T_{x,i}^{\varepsilon,\delta}]$ and $x \in B_R \setminus N_1$. It follows from this with $s = T_{\Delta,1}^{\varepsilon,\delta}$,

$$\mathbb{P}(K_3^{\varepsilon,\delta}) = \mathbb{P}\left\{\tilde{X}^{\varepsilon,\delta}(T_{\Delta,1}^{\varepsilon,\delta}) \in N_1, \tilde{X}^{\varepsilon,\delta}(0) \in B_R \setminus N_1\right\} \leq \left(T_{\Delta,1}^{\varepsilon,\delta}/\hat{T} + 1\right) \exp\left(-\frac{\kappa}{\varepsilon + \delta}\right).$$

This together with (3.11) implies that

$$(3.14) \quad \lim_{\varepsilon \downarrow 0} \mathbb{P}(K_3^{\varepsilon,\delta}) = 0.$$

Using (3.13) and the strong Markov property, we get

$$\begin{aligned}
(3.15) \quad \mathbb{P}(K_2^{\varepsilon,\delta}) &= \mathbb{P}\left\{ \tilde{X}^{\varepsilon,\delta}(T_{\Delta,1}^{\varepsilon,\delta}) \in N_1, \tau^{\varepsilon,\delta} < T_{\Delta,1}^{\varepsilon,\delta}, \tilde{X}^{\varepsilon,\delta}(0) \in N_1 \right\} \\
&= \int_0^{T_{\Delta,1}^{\varepsilon,\delta}} \mathbb{P}\{\tau^{\varepsilon,\delta} \in dt\} \left[\sum_{i \in \mathcal{M}} \int_{\partial G_1} \mathbb{P}\left\{ X_{x,i}^{\varepsilon,\delta}(T_{\Delta,1}^{\varepsilon,\delta} - t) \in N_1 \right\} \mathbb{P}\left\{ \alpha^\varepsilon(t) = i, \tilde{X}^{\varepsilon,\delta}(t) \in dx \right\} \right] \\
&\leq \left(T_{\Delta,1}^{\varepsilon,\delta} / \hat{T} + 1 \right) \exp\left(-\frac{\kappa}{\varepsilon + \delta} \right) \\
&\rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \text{ due to (3.11)}.
\end{aligned}$$

Putting together the estimates (3.12), (3.15), and (3.14), we see that

$$\limsup_{\varepsilon \rightarrow 0} \mu^{\varepsilon,\delta}(N_1) \leq \frac{1}{2} \limsup_{\varepsilon \rightarrow 0} \mu^{\varepsilon,\delta}(N_1) + 0 + 0 + \eta,$$

which contradicts the assumption that $\limsup_{\varepsilon \rightarrow 0} \mu^{\varepsilon,\delta}(N_1) > 2\eta$. We have therefore shown that

$$\lim_{\varepsilon \rightarrow 0} \mu^{\varepsilon,\delta}(N_1) \leq 2\mu.$$

Define

$$S_2 = \{y \in B_R \setminus S_1 : \text{dist}(y, \chi_2 \cap B_R \setminus S_1) < \theta_0\}.$$

There exists $c_4 > 0$ such that $\text{dist}(\bar{X}_y(t), \chi_1) \geq 2c_4$ for any $y \in B_R \setminus S_1$. Define

$$G_2 = \{y \in B_R \setminus S_1 : \text{dist}(y, \chi_2 \cap (B_R \setminus S_1)) < c_4\}$$

There exists $c_5 > 0$ such that $\text{dist}(\bar{X}_y(t), \chi_1) \geq 2c_5$ for any $y \in B_R \setminus G_1$. Define

$$N_2 = \{y \in B_R \setminus S_1 : \text{dist}(y, \chi_1 \cap B_R \setminus S_1) < c_5\}$$

Let \hat{T}_2 be such that $\bar{X}_y(\hat{T}_2) \in B_R \setminus (S_1 \cup S_2)$ given that $y \in B_R \setminus (S_1 \cup N_2)$. We can show, just as above, that there exists a $T_{\Delta,2}^{\varepsilon,\delta}$ such that $\lim_{\varepsilon \rightarrow \infty} T_{\Delta,2}^{\varepsilon,\delta} \exp\left(-\frac{\kappa}{\varepsilon + \delta}\right) = 0$ and

$$\mathbb{P}\{\tau_{x,i}^{\varepsilon,\delta} < T_{\Delta,2}^{\varepsilon,\delta}\} > \frac{1}{2}.$$

Define events

$$\begin{aligned}
K_{1,2}^{\varepsilon,\delta} &= \left\{ \tilde{X}^{\varepsilon,\delta}(T_{\Delta,2}^{\varepsilon,\delta}) \in N_2, \tau_2^{\varepsilon,\delta} \geq T_{\Delta,2}^{\varepsilon,\delta}, \tilde{X}^{\varepsilon,\delta}(0) \in N_2 \right\} \\
K_{2,2}^{\varepsilon,\delta} &= \left\{ \tilde{X}^{\varepsilon,\delta}(T_{\Delta,2}^{\varepsilon,\delta}) \in N_2, \tau_2^{\varepsilon,\delta} < T_{\Delta,2}^{\varepsilon,\delta}, \tilde{X}^{\varepsilon,\delta}(0) \in N_2 \right\} \\
K_{3,2}^{\varepsilon,\delta} &= \left\{ \tilde{X}^{\varepsilon,\delta}(T_{\Delta,2}^{\varepsilon,\delta}) \in N_2, \tilde{X}^{\varepsilon,\delta}(0) \in B_R \setminus (S_1 \cup N_2) \right\} \\
K_{4,2}^{\varepsilon,\delta} &= \left\{ \tilde{X}^{\varepsilon,\delta}(T_{\Delta,2}^{\varepsilon,\delta}) \in N_2, \tilde{X}^{\varepsilon,\delta}(0) \notin B_R \setminus S_1 \right\}.
\end{aligned}$$

Applying the same arguments as in the previous part, we can show that $\limsup_{\varepsilon \rightarrow 0} \mu^{\varepsilon,\delta}(N_2) \leq 4\eta$. Continuing this process, we can construct neighborhoods N_1, \dots, N_{n_0-1} of $\chi_1 \cap B_R, \dots, \chi_{n_0-1} \cap B_R$ such that

$$\limsup_{\varepsilon \rightarrow 0} \mu^{\varepsilon,\delta}(\cup_{j=1}^{n_0-1} N_j) \leq 2^{n_0} \eta.$$

□

Theorem 1.1. *Suppose Assumptions 1.1 and 1.2 hold. The family of invariant probability measures $(\mu^{\varepsilon,\delta})_{\varepsilon>0}$ converges weakly to the measure μ^0 given by (1.5) in the sense that for every bounded and continuous function $g : \mathbb{R}^d \times \mathcal{M} \rightarrow \mathbb{R}$,*

$$\lim_{\varepsilon \rightarrow 0} \sum_{i=1}^m \int_{\mathbb{R}^d} g(x, i) \mu^{\varepsilon,\delta}(dx, i) = \frac{1}{T_\Gamma} \int_0^{T_\Gamma} \bar{g}(\bar{X}_y(t)) dt,$$

where T_Γ is the period of the limit cycle, $y \in \Gamma$ and $\bar{g}(x) = \sum_{i \in \mathcal{M}} g(x, i) \nu_i$.

Proof. We have proved in Proposition 3.1 that for any $\eta > 0$ we can find $R > 0$ and neighborhoods N_1, \dots, N_{n_0-1} of $\chi_1 \cap B_R, \dots, \chi_{n_0-1} \cap B_R$ such that

$$\limsup_{\varepsilon \rightarrow 0} \mu^{\varepsilon,\delta}(\cup_{j=1}^{n_0-1} N_j) \leq 2^{n_0+1} \eta.$$

Using this fact together with Assumption 1.1 and Lemma 2.2, by a straightforward modification of the proof of [Hol78, Theorem 1], we can establish that for any $\vartheta > 0$ there is neighborhood N of the limit cycle Γ such that

$$\liminf_{\varepsilon \downarrow 0} \mu^{\varepsilon,\delta}(N) > 1 - \vartheta.$$

□

4. PROOF OF THEOREM 1.2

To proceed, we first need some auxiliary results.

Lemma 4.1. *There exist numbers $K_1, K_2 > 0$ such that for any $0 < \varepsilon, \delta < 1$ and any $(i_0, z_0) \in \mathcal{M} \times \mathbb{R}_+^{2^o}$, we have*

$$\frac{1}{t} \mathbb{E} \int_0^t |Z_{z_0, i_0}^{\varepsilon, \delta}(s)|^2 ds \leq K_1(1 + |z_0|), t \geq 1,$$

and

$$\limsup_{t \rightarrow \infty} \mathbb{E} |Z_{z_0, i_0}^{\varepsilon, \delta}(t)|^2 \leq K_2.$$

Proof. Let $\theta < \min\{f_M b(i), d(i) : i \in \mathcal{M}\}$. Define

$$\hat{K}_1 = \sup_{(x, y, i) \in \mathcal{M} \times \mathbb{R}_+^{2^o}} \{f_M x(a(i) - b(i)x) - y(c(i) + d(i)y) + \theta(x^2 + y^2)\} < \infty.$$

Consider $\hat{V}(x, y, i) = f_M x + y$. We can check that $\mathcal{L}^{\varepsilon, \delta} \hat{V}(x, y, i) \leq \hat{K}_1 - \theta(x^2 + y^2)$, where $\mathcal{L}^{\varepsilon, \delta}$ the generator associated with (1.9) (see [MY06, p. 48] or [YZ10] for the formula of $\mathcal{L}^{\varepsilon, \delta}$). Similarly, we can verify that there is $\hat{K}_2 > 0$ such that for all $\varepsilon < 1, \delta < 1$, $\mathcal{L}^{\varepsilon, \delta}(\hat{V}^2(x, y, i)) \leq \hat{K}_2 - \hat{V}^2(x, y, i)$. For each $k > 0$, define the stopping time $\sigma_k = \inf\{t : x(t) + y(t) > k\}$. By the generalized Itô formula for $\hat{V}(x(t), y(t), \alpha^\varepsilon(t))$

$$\begin{aligned} \mathbb{E} \hat{V}(Z_{z_0, i_0}^{\varepsilon, \delta}(t \wedge \sigma_k), \alpha^\varepsilon(t \wedge \sigma_k)) &= \hat{V}(z_0, i_0) + \mathbb{E} \int_0^{t \wedge \sigma_k} \mathcal{L}^{\varepsilon, \delta} \hat{V}(Z_{z_0, i_0}^{\varepsilon, \delta}(s), \alpha^\varepsilon(s)) ds \\ &\leq f_M x_0 + y_0 + \mathbb{E} \int_0^{t \wedge \sigma_k} [\hat{K}_1 - \theta |Z_{z_0, i_0}^{\varepsilon, \delta}(s)|^2] ds. \end{aligned} \tag{4.1}$$

Hence

$$\theta \mathbb{E} \int_0^{t \wedge \sigma_k} |Z_{z_0, i_0}^{\varepsilon, \delta}(s)|^2 ds \leq f_M x_0 + y_0 + \hat{K}_1 t.$$

Letting $k \rightarrow \infty$ and dividing both sides by θt we have

$$(4.2) \quad \frac{1}{t} \mathbb{E} \int_0^t |Z_{z_0, i_0}^{\varepsilon, \delta}(s)|^2 ds \leq \frac{f_M x_0 + y_0}{\theta t} + \frac{\hat{K}_1}{\theta}.$$

Applying the generalized Itô formula to $e^t \hat{V}^2(Z_{z_0, i_0}^{\varepsilon, \delta}(t), \alpha^\varepsilon(t))$,

$$(4.3) \quad \begin{aligned} & \mathbb{E} e^{t \wedge \sigma_k} \hat{V}^2(Z_{z_0, i_0}^{\varepsilon, \delta}(t \wedge \sigma_k), \alpha^\varepsilon(t \wedge \sigma_k)) \\ &= \hat{V}^2(z_0, i_0) + \mathbb{E} \int_0^{t \wedge \sigma_k} e^s [(\hat{V}^2(Z_{z_0, i_0}^{\varepsilon, \delta}(s), \alpha^\varepsilon(s)) + \mathcal{L}^{\varepsilon, \delta} \hat{V}^2(Z_{z_0, i_0}^{\varepsilon, \delta}(s), \alpha^\varepsilon(s)))] ds \\ &\leq (f_M x_0 + y_0)^2 + \hat{K}_2 \mathbb{E} \int_0^{t \wedge \sigma_k} e^s ds \leq (f_M x_0 + y_0)^2 + \hat{K}_2 e^t. \end{aligned}$$

Taking the limit as $k \rightarrow \infty$, and then dividing both sides by e^t , we have

$$(4.4) \quad \mathbb{E} [f_M X_{z_0, i_0}^{\varepsilon, \delta}(t) + Y_{z_0, i_0}^{\varepsilon, \delta}(t)]^2 \leq (f_M x_0 + y_0)^2 e^{-t} + \hat{K}_2.$$

The assertions of the lemma follow directly from (4.2) and (4.4). \square

Lemma 4.2. *There is a number $K_3 > 0$ such that*

$$\frac{1}{t} \mathbb{E} \int_0^t [\varphi^2(Z_{z, i}^{\varepsilon, \delta}(s), \alpha_i^\varepsilon(s)) + \psi^2(Z_{z, i}^{\varepsilon, \delta}(s), \alpha_i^\varepsilon(s))] ds \leq K_3(1 + |z|)$$

for all $\varepsilon, \delta \in (0, 1]$, $z \in \mathbb{R}_+^{2, \circ}$, $t \geq 1$.

Proof. Since the function $h(\cdot, \cdot, i)$ is bounded, we can find $C > 0$ such that

$$\varphi^2(z, i) + \psi^2(z, i) \leq C(1 + |z|^2).$$

The claim follows by an application of Lemma 4.1. \square

Recall that the two equilibria of (1.10) on the boundary are both hyperbolic. Note that the Jacobian of $(x\bar{\phi}(x, y), y\bar{\psi}(x, y))^\top$ at $(\frac{\bar{a}}{b}, 0)$ has two eigenvalues: $-\bar{c} + \frac{\bar{a}}{b} h_2(\frac{\bar{a}}{b}, 0) > 0$ and $-\frac{\bar{b}^2}{\bar{a}} < 0$. At $(0, 0)$, the two eigenvalues are $\bar{a} > 0$ and $-\bar{c} < 0$, respectively. If we consider the weighted average Lyapunov exponent, we can see that the growth rate of $\frac{2\bar{c}}{\bar{a}} \frac{d \ln X(t)}{dt} + \frac{d \ln Y(t)}{dt}$ is positive both at $(0, 0)$ and $(\frac{\bar{a}}{b}, 0)$. This suggests we should look at $\frac{2\bar{c}}{\bar{a}} \frac{d \ln X(t)}{dt} + \frac{d \ln Y(t)}{dt}$ in order to prove that the dynamics of (1.10) is pushed away from the boundary. Then we can use approximation arguments to obtain the tightness of $(Z^{\varepsilon, \delta})$ on $\mathbb{R}_+^{2, \circ}$. Define

$$\Upsilon(z, i) := \frac{2\bar{c}}{\bar{a}} \varphi(z, i) + \psi(z, i)$$

and

$$\bar{\Upsilon}(z) := \frac{2\bar{c}}{\bar{a}} \bar{\varphi}(z) + \bar{\psi}(z).$$

We have the following lemma.

Lemma 4.3. *Let $\gamma_0 = \frac{1}{2} \left(\bar{c} \wedge \left(-\bar{c} + \frac{\bar{a}}{\bar{b}} h_1 \left(\frac{\bar{a}}{\bar{b}}, 0 \right) \right) \right) > 0$. For any $H > \frac{\bar{a}}{\bar{b}} + 1$, there are numbers $T, \beta > 0$ such that for all $z \in \{(x, y) \in \mathbb{R}_+^2 \mid x \wedge y \leq \beta, x \vee y \leq H\}$*

$$(4.5) \quad \bar{X}_z(T) \vee \bar{Y}_z(T) \leq H \text{ and } \frac{1}{T} \int_0^T \bar{\Upsilon}(\bar{Z}_z(t)) dt \geq \gamma_0.$$

Proof. Since $\lim_{t \rightarrow \infty} \bar{Z}_{(0,y)}(t) \rightarrow (0, 0)$, $\forall y \in \mathbb{R}_+$ and

$$(4.6) \quad \bar{\Upsilon}(0, 0) = \frac{2\bar{c}}{\bar{a}} \bar{\varphi}(0, 0) + \bar{\psi}(0, 0) = \frac{2\bar{c}}{\bar{a}} \bar{a} - \bar{c} = \bar{c} \geq 2\gamma_0,$$

there exists $T_1 > 0$ such that

$$(4.7) \quad \frac{1}{t} \int_0^t \bar{\Upsilon}(\bar{Z}_{(0,y)}(s)) ds \geq \frac{3}{2} \gamma_0 \text{ for } t \geq T_1, y \in [0, H].$$

By (4.6) and the continuity of $\bar{\Upsilon}(\cdot)$, there exists $\beta_1 \in (0, \frac{\bar{a}}{\bar{b}})$ such that

$$(4.8) \quad \bar{\Upsilon}(x, 0) \geq \frac{7}{4} \gamma_0, \text{ if } x \leq \beta_1.$$

Since

$$\bar{\Upsilon}\left(\frac{\bar{a}}{\bar{b}}, 0\right) = \frac{2\bar{c}}{\bar{a}} \bar{\varphi}\left(\frac{\bar{a}}{\bar{b}}, 0\right) + \bar{\psi}\left(\frac{\bar{a}}{\bar{b}}, 0\right) = -\bar{c} + \frac{\bar{a}}{\bar{b}} h_1\left(\frac{\bar{a}}{\bar{b}}, 0\right) \geq 2\gamma_0$$

and

$$\lim_{t \rightarrow \infty} \bar{Z}_{(x,0)}(t) \rightarrow \left(\frac{\bar{a}}{\bar{b}}, 0\right), \forall x > 0,$$

there exists a $T_2 > 0$ such that

$$(4.9) \quad \frac{1}{t} \int_0^t \bar{\Upsilon}(\bar{Z}_{(x,0)}(s)) ds \geq \frac{7}{4} \gamma_0 \text{ for } t \geq T_2, x \in [\beta_1, H].$$

Let $\bar{M}_H = \sup_{x \in [0, H]} \{|\bar{\Upsilon}(x, 0)|\}$, $\bar{t}_x = \inf\{t \geq 0 : X_{x,0} \geq \beta_1\}$ and $T_3 = \left(4 \frac{\bar{M}_H}{\gamma_0} + 7\right) T_2$. It can be seen from the equation of $\bar{X}(t)$ that $\bar{X}_{(x,0)}(t) \in [\beta_1, H]$ if $t \geq \bar{t}_x, x \in (0, \beta_1]$. For $t \geq T_3$, we can use (4.8) and (4.9) to estimate $\frac{1}{t} \int_0^t \bar{\Upsilon}(\bar{Z}_{(x,0)}(s)) ds$ in the following three cases.

Case 1. If $t - T_2 \leq \bar{t}_x \leq t$ then

$$\begin{aligned} \int_0^t \bar{\Upsilon}(\bar{Z}_{(x,0)}(s)) ds &= \int_0^{\bar{t}_x} \bar{\Upsilon}(\bar{Z}_{(x,0)}(s)) ds + \int_{\bar{t}_x}^t \bar{\Upsilon}(\bar{Z}_{(x,0)}(s)) ds \\ &\geq \frac{7}{4} \gamma_0 (t - T_2) - T_2 \bar{M}_H \geq \frac{3}{2} \gamma_0 t, \left(\text{since } t \geq \left(4 \frac{\bar{M}_H}{\gamma_0} + 7\right) T_2 \right). \end{aligned}$$

Case 2. If $\bar{t}_x \leq t - T_2$, then

$$\begin{aligned} \int_0^t \bar{\Upsilon}(\bar{Z}_{(x,0)}(s)) ds &= \int_0^{\bar{t}_x} \bar{\Upsilon}(\bar{Z}_{(x,0)}(s)) ds + \int_{\bar{t}_x}^t \bar{\Upsilon}(\bar{Z}_{(x,0)}(s)) ds \\ &\geq \frac{7}{4} \gamma_0 (t - \bar{t}_x) + \frac{7}{4} \gamma_0 \bar{t}_x \geq \frac{3}{2} \gamma_0 t. \end{aligned}$$

Case 3. If $\bar{t}_x \geq t$, then

$$\int_0^t \bar{\Upsilon}(\bar{Z}_{(x,0)}(s)) ds = \int_0^{\bar{t}_x} \bar{\Upsilon}(\bar{Z}_{(x,0)}(s)) ds \geq \frac{7}{4} \gamma_0 \bar{t}_x \geq \frac{3}{2} \gamma_0 t.$$

As a result,

$$(4.10) \quad \frac{1}{t} \int_0^t \bar{\Upsilon}(\bar{Z}_{(x,0)}(s)) ds \geq \frac{3}{2} \gamma_0, \quad \text{if } t \geq T_3, x \in (0, H].$$

Let $T = T_1 \vee T_3$. By the continuous dependence of solutions on initial values, there is $\beta > 0$ such that

$$(4.11) \quad \bar{X}_z(T) \vee \bar{Y}_z(T) \leq H \quad \text{and} \quad \frac{1}{T} \int_0^T |\bar{\Upsilon}(\bar{Z}_{z_1}(s)) - \bar{\Upsilon}(\bar{Z}_{z_2}(s))| ds \leq \frac{1}{2} \gamma_0$$

given that $|z_1 - z_2| \leq \beta, z_1, z_2 \in [0, H]^2$. Combining (4.7), (4.10) and (4.11) we obtain the desired result. \square

Generalizing the techniques in [NY17], we divide the proof of the eventual tightness into two lemmas.

Lemma 4.4. *For any $\Delta > 0$, there exist $\varepsilon_0, \delta_0, T > 0$ and a compact set $\mathcal{K} \subset \mathbb{R}_+^{2;\circ}$ such that*

$$\liminf_{k \rightarrow \infty} \frac{1}{k} \sum_{n=0}^{k-1} \mathbb{P} \left\{ Z_{z_0, i_0}^{\varepsilon, \delta}(nT) \in \mathcal{K} \right\} \geq 1 - \frac{\Delta}{3} \quad \text{for any } \varepsilon < \varepsilon_0, \delta < \delta_0, z \in \mathbb{R}_+^{2;\circ}.$$

Proof. For any $\Delta > 0$, let $H = H(\Delta) > \frac{\bar{a}}{b} + 1$ be chosen later and define $D = \{(x, y) : 0 < x, y \leq H\}$. Let $T > 0$ and $\beta > 0$ such that (4.5) is satisfied and $D_1 = \{(x, y) : 0 < x, y \leq H, x \wedge y < \beta\} \subset D$. Define $V(x, y) = -\frac{2\bar{c}}{a} \ln x - \ln y + C$ where C is a positive constant such that $V(z) \geq 0 \forall z \in D$. In view of the generalized Itô formula,

$$\begin{aligned} V(Z_{z,i}^{\varepsilon, \delta}(t)) - V(z) &= \int_0^t \left[-\Upsilon(Z_{z,i}^{\varepsilon, \delta}(s), \alpha^\varepsilon(s)) + \frac{\delta}{2} \left(\frac{2\bar{c}}{a} \lambda^2(\alpha^\varepsilon(s)) + \rho^2(\alpha^\varepsilon(s)) \right) \right] ds \\ &\quad - \frac{2\bar{c}}{a} \int_0^t \sqrt{\delta} \lambda(\alpha^\varepsilon(s)) dW_1(s) - \int_0^t \sqrt{\delta} \rho(\alpha^\varepsilon(s)) dW_2(s). \end{aligned}$$

For $A \in \mathcal{F}$, using Holder's inequality and Itô's isometry, we have

$$\begin{aligned}
& \mathbb{E} \left(\mathbf{1}_A |V(Z_{z,i}^{\varepsilon,\delta}(T)) - V(z)| \right) \\
& \leq \left| \mathbb{E} \mathbf{1}_A \int_0^T \Upsilon(Z_{z,i}^{\varepsilon,\delta}(t), \alpha^\varepsilon(t)) dt \right| + \mathbb{E} \mathbf{1}_A \int_0^T \frac{\delta}{2} \left(\frac{2\bar{c}}{a} \lambda^2(\alpha^\varepsilon(t)) + \rho^2(\alpha^\varepsilon(t)) \right) dt \\
& \quad + \frac{2\bar{c}}{a} \mathbb{E} \mathbf{1}_A \left| \int_0^T \sqrt{\delta} \lambda(\alpha^\varepsilon(t)) dW_1(t) \right| + \mathbb{E} \mathbf{1}_A \left| \int_0^T \sqrt{\delta} \rho(\alpha^\varepsilon(t)) dW_2(t) \right| \\
(4.12) \quad & \leq T(\mathbb{E} \mathbf{1}_A)^{\frac{1}{2}} \left(\mathbb{E} \int_0^T \left[\Upsilon(Z_{z,i}^{\varepsilon,\delta}(t), \alpha^\varepsilon(t)) + \frac{\delta}{2} \left(\frac{2\bar{c}}{a} \lambda^2(\alpha^\varepsilon(t)) + \rho^2(\alpha^\varepsilon(t)) \right) \right] dt \right)^{\frac{1}{2}} \\
& \quad + \delta \sqrt{\mathbb{P}(A)} \left(\mathbb{E} \int_0^T \left(\frac{2\bar{c}}{a} \lambda^2(\alpha^\varepsilon(t)) + \rho^2(\alpha^\varepsilon(t)) \right) dt \right)^{\frac{1}{2}} \\
& \leq K_4 T(1 + |z|) \sqrt{\mathbb{P}(A)},
\end{aligned}$$

where the last inequality follows from (4.2) and the boundedness of $\rho(i)$ and $\lambda(i)$. If $A = \Omega$, we have

$$(4.13) \quad \frac{1}{T} \mathbb{E} \left(|V(Z_{z,i}^{\varepsilon,\delta}(T)) - V(z)| \right) \leq K_4(1 + |z|).$$

Let $\hat{H}_T > H$ such that $\bar{X}_z(t) \vee \bar{Y}_z(t) \leq \hat{H}_T$ for all $z \in [0, H]^2$, $0 \leq t \leq T$ and

$$\bar{d}_H = \sup \left\{ \left| \frac{\partial \bar{\Upsilon}}{\partial x}(x, y) \right|, \left| \frac{\partial \bar{\Upsilon}}{\partial y}(x, y) \right| : (x, y) \in \mathbb{R}_+^2, x \vee y \leq \hat{H}_T \right\}.$$

Let $\varsigma > 0$. Lemma 2.2 implies that there are δ_0, ε_0 such that if $\varepsilon < \varepsilon_0, \delta < \delta_0$,

$$(4.14) \quad \mathbb{P} \left\{ |\bar{X}_z(t) - X_{z,i}^{\varepsilon,\delta}(t)| + |\bar{Y}_z(t) - Y_{z,i}^{\varepsilon,\delta}(t)| < 1 \wedge \frac{\gamma_0}{2\bar{d}_H}, \text{ for all } t \in [0, T] \right\} > 1 - \frac{\varsigma}{6}, z \in \bar{D}.$$

On the other hand, if $|\bar{X}_z(t) - X_{z,i}^{\varepsilon,\delta}(t)| + |\bar{Y}_z(t) - Y_{z,i}^{\varepsilon,\delta}(t)| < 1 \wedge \frac{\gamma_0}{2\bar{d}_H}$, we have

$$\begin{aligned}
& \left| \frac{1}{T} \int_0^T \Upsilon(Z_{z,i}^{\varepsilon,\delta}(t), \alpha^\varepsilon(t)) dt - \frac{1}{T} \int_0^T \bar{\Upsilon}(\bar{Z}_{z,i}(t)) dt \right| \\
& \leq \frac{1}{T} \left| \int_0^T \left(\bar{\Upsilon}(Z_{z,i}^{\varepsilon,\delta}(t)) - \bar{\Upsilon}(\bar{Z}_{z,i}(t)) \right) dt \right| \\
(4.15) \quad & \quad + \frac{1}{T} \left| \int_0^T \left(\Upsilon(Z_{z,i}^{\varepsilon,\delta}(t), \alpha^\varepsilon(t)) - \bar{\Upsilon}(Z_{z,i}^{\varepsilon,\delta}(t)) \right) dt \right| \\
& \leq \frac{\gamma_0}{2} + \frac{F_H}{T} \int_0^T \sum_{j \in \mathcal{M}} |\mathbf{1}_{\{\alpha^\varepsilon(t)=j\}} - v_j| dt
\end{aligned}$$

where $F_H := \sup\{|\Upsilon(z, i)| : i \in \mathcal{M}, z \in [0, K_T + 1]^2\}$. In view of [HYZ11, Lemma 2.1],

$$(4.16) \quad \mathbb{E} \left| \frac{1}{T} \int_0^T \sum_{j \in \mathcal{M}} |\mathbf{1}_{\{\alpha^\varepsilon(t)=j\}} - v_j| dt \right|^2 = \mathbb{E} \left| \frac{\varepsilon}{T} \int_0^{T/\varepsilon} \sum_{j \in \mathcal{M}} |\mathbf{1}_{\{\alpha(t)=j\}} - v_j| dt \right|^2 \leq \frac{\kappa}{T} \varepsilon$$

for some constant $\kappa > 0$. On the one hand,

$$(4.17) \quad \mathbb{E} \frac{1}{T} \left| \int_0^T \left(-\frac{2\bar{c}}{a} \lambda(\alpha^\varepsilon(t)) dW_1(t) - \rho(\alpha^\varepsilon(t)) dW_2(t) \right) \right|^2 \leq \frac{4\bar{c}^2}{a^2} \lambda_M^2 + \rho_M^2.$$

Combining (4.5), (4.14), (4.15), (4.16), and (4.17), we can reselect ε_0 and δ_0 such that for $\varepsilon < \varepsilon_0, \delta < \delta_0$ we have

$$(4.18) \quad \mathbb{P} \left\{ \frac{-1}{T} \int_0^T \Upsilon(\alpha^\varepsilon(t), Z_{z,i}^{\varepsilon,\delta}(t)) dt \leq -0.5\gamma_0 \right\} \geq 1 - \frac{\varsigma}{3}, \quad z \in D_1, i \in \mathcal{M},$$

$$(4.19) \quad \mathbb{P} \left\{ X_{z,i}^{\varepsilon,\delta}(T) \vee Y_{z,i}^{\varepsilon,\delta}(T) \leq H \text{ (or equivalently } Z_{z,i}^{\varepsilon,\delta}(T) \in D) \right\} \geq 1 - \frac{\varsigma}{3}, \quad z \in D_1,$$

and

$$(4.20) \quad \mathbb{P} \left\{ \delta\vartheta + \frac{\sqrt{\delta}}{T} \left| \int_0^T \left(\frac{2\bar{c}}{a} \lambda(\alpha^\varepsilon(t)) dW_1(t) + \rho(\alpha^\varepsilon(t)) dW_2(t) \right) dt \right| < 0.25\gamma_0 \right\} > 1 - \frac{\varsigma}{3}$$

where $\vartheta = \frac{1}{2} \left(\frac{2\bar{c}}{a} \lambda_M^2 + \rho_M^2 \right)$. Consequently, for any $(z, i) \in D_1 \times \mathcal{M}$, there is a subset $\Omega_{z,i}^{\varepsilon,\delta} \subset \Omega$ with $\mathbb{P}(\Omega_{z,i}^{\varepsilon,\delta}) \geq 1 - \varsigma$ in which we have $Z_{z,i}^{\varepsilon,\delta}(T) \in D$ and

$$(4.21) \quad \begin{aligned} \frac{1}{T} (V(Z_{z,i}^{\varepsilon,\delta}(T)) - V(z)) &\leq \frac{-1}{T} \int_0^T \Upsilon(\alpha^\varepsilon(t), Z_{z,i}^{\varepsilon,\delta}(t)) dt + \delta\vartheta \\ &\quad + \frac{1}{T} \left| \int_0^T \sqrt{\delta} \left(\frac{2\bar{c}}{a} \lambda(\alpha^\varepsilon(t)) dW_1(t) + \rho(\alpha^\varepsilon(t)) dW_2(t) \right) \right| \\ &\leq -0.25\gamma_0 \end{aligned}$$

On the other hand, we deduce from (4.13) that for $z \in D$,

$$(4.22) \quad \mathbb{P} \left\{ \frac{1}{T} (V(Z_{z,i}^{\varepsilon,\delta}(T)) - V(z)) \leq \Lambda \right\} \geq 1 - \varsigma,$$

where $\Lambda := \frac{K_4(1+2H)}{\varsigma}$. Moreover, it also follows from (4.13) that for $z \in D \setminus D_1$

$$\mathbb{E} V(Z_{z,i}^{\varepsilon,\delta}(T)) \leq \sup_{z \in D \setminus D_1} (V(z) + K_4 T |z|).$$

Define

$$(4.23) \quad L_1 := \sup_{z \in D \setminus D_1} V(z) + \Lambda T, \quad L_2 := L_1 + 0.25\gamma_0,$$

as well as $D_2 := \{(x, y) \in \mathbb{R}_+^{2,\circ} : (x, y) \in D, V(x, y) > L_2\}$ and $U(z) = V(z) \vee L_1$. It is clear that

$$(4.24) \quad U(z_2) - U(z_1) \leq |V(z_2) - V(z_1)| \text{ for any } z_1, z_2 \in \mathbb{R}_+^{2,\circ}.$$

It follows from (4.12) that for any $\delta, \varepsilon < 1$, $A \in \mathcal{F}$, and $z \in D$, we have

$$(4.25) \quad \frac{1}{T} \mathbb{E} \mathbf{1}_A \left| V(Z_{z,i}^{\varepsilon,\delta}(T)) - V(z) \right| \leq K_4(2H + 1) \sqrt{\mathbb{P}(A)}.$$

Applying (4.25) and (4.24) with $A = \Omega \setminus \Omega_{z,i}^{\varepsilon,\delta}$, we get

$$(4.26) \quad \frac{1}{T} \mathbb{E} \mathbf{1}_{\Omega \setminus \Omega_{z,i}^{\varepsilon,\delta}} \left[U(Z_{z,i}^{\varepsilon,\delta}(T)) - U(z) \right] \leq K_4(2H+1)\sqrt{\varsigma}, \text{ if } z \in D_1.$$

In view of (4.21), for $z \in D_2$ we have $V(Z_{z,i}^{\varepsilon,\delta}(T)) < V(z) - 0.25\gamma_0 T$. By the definition of D_2 , we also have $L_1 \leq V(z) - 0.25\gamma_0 T$. Thus, for any $z \in D_2$ and $\omega \in \Omega_{z,i}^{\varepsilon,\delta}$

$$U(Z_{z,i}^{\varepsilon,\delta}(T)) = L_1 \vee V(Z_{z,i}^{\varepsilon,\delta}(T)) \leq V(z) - 0.25\gamma_0 T = U(z) - 0.25\gamma_0 T,$$

which implies

$$(4.27) \quad \frac{1}{T} \left[\mathbb{E} \mathbf{1}_{\Omega_{z,i}^{\varepsilon,\delta}} U(Z_{z,i}^{\varepsilon,\delta}(T)) - \mathbb{E} \mathbf{1}_{\Omega_{z,i}^{\varepsilon,\delta}} U(z) \right] \leq -0.25\gamma_0 \mathbb{P}(\Omega_{z,i}^{\varepsilon,\delta}) \leq -0.25\gamma_0(1-\varsigma).$$

Combining (4.26) with (4.27)

$$(4.28) \quad \frac{1}{T} \left[\mathbb{E} U(Z_{z,i}^{\varepsilon,\delta}(T)) - U(z) \right] \leq -0.25\gamma_0(1-\varsigma) + K_4(2H+1)\sqrt{\varsigma}, \quad \forall z \in D_2.$$

For $z \in D_1 \setminus D_2$, and $\omega \in \Omega_{z,i}^{\varepsilon,\delta}$, we have from (4.21) that $V(Z_{z,i}^{\varepsilon,\delta}(T)) \leq V(z)$. This shows that $U(Z_{z,i}^{\varepsilon,\delta}(T)) = L_1 \vee V(Z_{z,i}^{\varepsilon,\delta}(T)) \leq U(z) = V(z) \vee L_1$. Hence, for $z \in D_1 \setminus D_2$ and $\omega \in \Omega_{z,i}^{\varepsilon,\delta}$ one has

$$U(Z_{z,i}^{\varepsilon,\delta}(T)) - U(z) \leq 0.$$

This and (4.26) imply

$$(4.29) \quad \frac{1}{T} \left[\mathbb{E} U(Z_{z,i}^{\varepsilon,\delta}(T)) - U(z) \right] \leq K_4(2H+1)\sqrt{\varsigma}, \quad \forall z \in D_1 \setminus D_2.$$

If $z \in D \setminus D_1$, $U(z) = L_1$ and we have from (4.22) and (4.23) that

$$\mathbb{P}\{U(Z_{z,i}^{\varepsilon,\delta}(T)) = L_1\} = \mathbb{P}\{V(Z_{z,i}^{\varepsilon,\delta}(T)) \leq L_1\} \geq 1 - \varsigma.$$

Thus

$$\mathbb{P}\{U(Z_{z,i}^{\varepsilon,\delta}(T)) = U(z)\} \geq 1 - \varsigma.$$

Use (4.25) and (4.24) again to arrive at

$$(4.30) \quad \frac{1}{T} \left[\mathbb{E} U(Z_{z,i}^{\varepsilon,\delta}(T)) - U(z) \right] \leq K_4(2H+1)\sqrt{\varsigma}, \quad \forall z \in D \setminus D_1.$$

On the other hand, equations (4.13) and (4.24) imply

$$(4.31) \quad \frac{1}{T} \left[\mathbb{E} U(Z_{z,i}^{\varepsilon,\delta}(T)) - U(z) \right] \leq K_4(1+|z|), \quad z \in \mathbb{R}_+^{2,\circ}.$$

Pick an arbitrary $(z_0, i_0) \in \mathbb{R}_+^{2,\circ} \times \mathcal{M}$. An application of the Markov property yields

$$\begin{aligned} & \frac{1}{T} \left[\mathbb{E} U(Z_{z_0, i_0}^{\varepsilon,\delta}((n+1)T)) - \mathbb{E} U(Z_{z_0, i_0}^{\varepsilon,\delta}(nT)) \right] \\ &= \sum_{i \in \mathcal{M}} \int_{\mathbb{R}_+^{2,\circ}} \frac{1}{T} \left[\mathbb{E} U(Z_{z,i}^{\varepsilon,\delta}(T)) - U(z) \right] \mathbb{P}\{Z_{z_0, i_0}^{\varepsilon,\delta}(nT) \in dz, \alpha^\varepsilon(t) = i\}. \end{aligned}$$

Combining (4.28), (4.29), (4.30), and (4.31), we get

$$\begin{aligned}
& \frac{1}{T} \left[\mathbb{E}U(Z_{z_0, i_0}^{\varepsilon, \delta}((n+1)T)) - \mathbb{E}U(Z_{z_0, i_0}^{\varepsilon, \delta}(nT)) \right] \\
& \leq - [0.25\gamma_0(1-\varsigma) - K_4(2H+1)\sqrt{\varsigma}] \mathbb{P}\{Z_{z_0, i_0}^{\varepsilon, \delta}(nT) \in D_2\} \\
& \quad + K_4(2H+1)\sqrt{\varsigma} \mathbb{P}\{Z_{z_0, i_0}^{\varepsilon, \delta}(nT) \in D \setminus D_2\} \\
& \quad + K_4 \mathbb{E} \mathbf{1}_{\{Z_{z_0, i_0}^{\varepsilon, \delta}(nT) \notin D\}} (1 + |Z_{z_0, i_0}^{\varepsilon, \delta}(nT)|) \\
& \leq -0.25\gamma_0(1-\varsigma) \mathbb{P}\{Z_{z_0, i_0}^{\varepsilon, \delta}(nT) \in D_2\} + K_4(2H+1)\sqrt{\varsigma} \\
& \quad + K_4 \mathbb{P}\{Z_{z_0, i_0}^{\varepsilon, \delta}(nT) \notin D\} \mathbb{E}(1 + |Z_{z_0, i_0}^{\varepsilon, \delta}(nT)|).
\end{aligned}$$

Note that

$$\begin{aligned}
& \liminf_{k \rightarrow \infty} \frac{1}{k} \sum_{n=0}^{k-1} \frac{1}{T} \left[\mathbb{E}U(Z_{z_0, i_0}^{\varepsilon, \delta}((n+1)T)) - \mathbb{E}U(Z_{z_0, i_0}^{\varepsilon, \delta}(nT)) \right] \\
& = \liminf_{k \rightarrow \infty} \frac{1}{kT} \mathbb{E}U(Z_{z_0, i_0}^{\varepsilon, \delta}(kT)) \geq 0.
\end{aligned}$$

This forces

(4.32)

$$\begin{aligned}
& 0.25\gamma_0(1-\varsigma) \limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{n=0}^{k-1} \mathbb{P}\{Z_{z_0, i_0}^{\varepsilon, \delta}(nT) \in D_2\} \\
& \leq K_4(2H+1)\sqrt{\varsigma} + K_3 \limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{n=1}^k \mathbb{P}\{Z_{z_0, i_0}^{\varepsilon, \delta}(nT) \notin D\} \mathbb{E}(1 + |Z_{z_0, i_0}^{\varepsilon, \delta}(nT)|).
\end{aligned}$$

In view of Lemma 4.1, we can choose $H = H(\Delta)$ independent of (z_0, i_0) such that

$$(4.33) \quad \limsup_{t \rightarrow \infty} \mathbb{P}\{Z_{z_0, i_0}^{\varepsilon, \delta}(t) \notin D\} \leq \limsup_{t \rightarrow \infty} \frac{\mathbb{E}(|Z_{z_0, i_0}^{\varepsilon, \delta}(t)|)}{H} \leq \frac{\Delta}{6},$$

and

$$\begin{aligned}
& K_4 \limsup_{t \rightarrow \infty} \mathbb{P}\{Z_{z_0, i_0}^{\varepsilon, \delta}(t) \notin D\} \mathbb{E}(1 + |Z_{z_0, i_0}^{\varepsilon, \delta}(t)|) \\
& \leq K_4 \limsup_{t \rightarrow \infty} \frac{\left[\mathbb{E}(1 + |Z_{z_0, i_0}^{\varepsilon, \delta}(t)|) \right]^2}{H} \leq \frac{0.1\gamma_0}{6} \Delta.
\end{aligned}$$

Hence, we have

$$(4.34) \quad K_4 \limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{n=1}^k \mathbb{P}\{Z_{z_0, i_0}^{\varepsilon, \delta}(nT) \notin D\} \mathbb{E}(1 + |Z_{z_0, i_0}^{\varepsilon, \delta}(nT)|) \leq \frac{0.1\gamma_0}{6} \Delta.$$

Choose $\varsigma = \varsigma(H) > 0$ such that $0.25\gamma_0(1-\varsigma) \geq 0.2\gamma_0$ and $K_4(2H+1)\sqrt{\varsigma} \leq \frac{0.1\gamma_0}{6} \Delta$ and let $\varepsilon_0 = \varepsilon_0(\varsigma, H)$, $\delta_0(\varsigma, H)$ such that (4.18), (4.19), and (4.20) hold. As a result, we get from

(4.32) and (4.34) that

$$(4.35) \quad \limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{n=1}^k \mathbb{P}\{Z_{z_0, i_0}^{\varepsilon, \delta}(nT) \in D_2\} \leq \frac{\Delta}{6}.$$

This together with (4.33) and (4.35) shows that for any $\varepsilon < \varepsilon_0, \delta < \delta_0$, we have

$$(4.36) \quad \liminf_{k \rightarrow \infty} \frac{1}{k} \sum_{n=1}^k \mathbb{P}\{Z_{z_0, i_0}^{\varepsilon, \delta}(nT) \in D \setminus D_2\} \geq 1 - \frac{\Delta}{3}.$$

One can conclude the proof by noting that the set $D \setminus D_2$ is a compact subset of $\mathbb{R}_+^{2, \circ}$. \square

Lemma 4.5. *There are $L > 1$, $\varepsilon_1 = \varepsilon(\Delta) > 0$, and $\delta_1 = \delta_1(\Delta) > 0$ such that as long as $0 < \varepsilon < \varepsilon_1, 0 < \delta < \delta_1$, we have*

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \int_0^T \mathbb{P}\{Z_{z_0, i_0}^{\varepsilon, \delta}(t) \in [L^{-1}, L]^2\} \geq 1 - \Delta, (z_0, i_0) \in \mathbb{R}_+^{2, \circ} \times \mathcal{M}.$$

Proof. Let D and T as in Lemma 4.4. Since $D \setminus D_2$ is a compact set in $\mathbb{R}_+^{2, \circ}$, by a modification of the proof of [JJ11, Theorem 2.1], we can show that there is a positive constant $L > 1$ such that $\mathbb{P}\{Z_{z, i}(t) \in [L^{-1}, L]^2\} > 1 - \frac{\Delta}{3}, z \in D \setminus D_2, i \in M, 0 \leq t \leq T$. Hence, it follows from the Markov property of the solution that

$$\begin{aligned} & \mathbb{P}\{Z_{z_0, i_0}^{\varepsilon, \delta}(t) \in [L^{-1}, L]^2\} \\ & \geq \left(1 - \frac{\Delta}{3}\right) \mathbb{P}\{Z_{z_0, i_0}^{\varepsilon, \delta}(jT) \in D \setminus D_2\}, t \in [jT, jT + T]. \end{aligned}$$

Consequently,

$$\begin{aligned} & \liminf_{k \rightarrow \infty} \frac{1}{kT} \int_0^{kT} \mathbb{P}\{Z_{z_0, i_0}^{\varepsilon, \delta}(t) \in [L^{-1}, L]^2\} dt \\ & \geq \left(1 - \frac{\Delta}{3}\right) \liminf_{k \rightarrow \infty} \frac{1}{k} \sum_{j=0}^{k-1} \mathbb{P}\{Z_{z_0, i_0}^{\varepsilon, \delta}(jT) \in D \setminus D_2\} \geq 1 - \Delta. \end{aligned}$$

It is readily seen from this estimate that

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \int_0^T \mathbb{P}\{Z_{z_0, i_0}^{\varepsilon, \delta}(t) \in [L^{-1}, L]^2\} dt \geq 1 - \Delta.$$

\square

Theorem 1.2. *Suppose Assumption 1.3 holds. For sufficiently small δ and ε , the process given by (1.9) has a unique invariant probability measure $\mu^{\varepsilon, \delta}$ with support in $\mathbb{R}_+^{2, \circ}$ (where $\mathbb{R}_+^{2, \circ}$ denotes the interior of \mathbb{R}_+^2). In addition:*

- a) *If $\lim_{\varepsilon \rightarrow 0} \frac{\delta}{\varepsilon} = l \in (0, \infty]$, the family of invariant measures $(\mu^{\varepsilon, \delta})_{\varepsilon > 0}$ converges weakly to μ^0 , the occupation measure of the limit cycle of (1.10), as $\varepsilon \rightarrow 0$ (in the sense of Theorem 1.1).*

b) If $\lim_{\varepsilon \rightarrow 0} \frac{\delta}{\varepsilon} = 0$ and at each critical point (x^*, y^*) of $(\bar{\varphi}(x, y), \bar{\psi}(x, y))$, there is $i^* \in \mathcal{M}$ such that either $\varphi(x^*, y^*, i^*) \neq 0$ or $\psi(x^*, y^*, i^*) \neq 0$, then the family of invariant measures $(\mu^{\varepsilon, \delta})_{\varepsilon > 0}$ converges weakly to μ^0 , the occupation measure of the limit cycle of (1.10), as $\varepsilon \rightarrow 0$.

Proof. The conclusion of Lemma 4.5 is sufficient for the existence of a unique invariant probability measure $\mu^{\varepsilon, \delta}$ in $\mathbb{R}_+^{2,0} \times \mathcal{M}$ of $(Z^{\varepsilon, \delta}(t), \alpha^\varepsilon(t))$ (see [Bel06] or [MT93]). Moreover, the empirical measures

$$\frac{1}{t} \int_0^t \mathbb{P}\{Z_{z_0, i_0}^{\varepsilon, \delta}(s) \in \cdot\} ds, t > 0$$

converge weakly to the invariant measure $\mu^{\varepsilon, \delta}$ as $t \rightarrow \infty$. Applying Fatou's lemma to the above estimate yields

$$\mu^{\varepsilon, \delta}([L^{-1}, L]^2) \geq \Delta, \quad \forall \varepsilon < \varepsilon_0, \delta < \delta_0.$$

This tightness implies Theorem 1.2. □

4.1. An Example. In this section we provide a specific example under the setting of Section 4. We consider the following stochastic predator-prey model with Holling functional response in a switching regime

$$(4.37) \quad \begin{cases} dx^{\varepsilon, \delta}(t) = \left[r(\alpha^\varepsilon(t))x^{\varepsilon, \delta}(t) \left(1 - \frac{x^{\varepsilon, \delta}(t)}{K(\alpha^\varepsilon(t))} \right) - \frac{m(\alpha^\varepsilon(t))x^{\varepsilon, \delta}(t)y^{\varepsilon, \delta}(t)}{a(\alpha^\varepsilon(t)) + b(\alpha^\varepsilon(t))x^{\varepsilon, \delta}(t)} \right] dt \\ \quad + \sqrt{\delta} \lambda(\alpha^\varepsilon(t))x^{\varepsilon, \delta}(t) dW_1(t) \\ dy^{\varepsilon, \delta}(t) = y^{\varepsilon, \delta}(t) \left[-d(\alpha^\varepsilon(t)) + \frac{e(\alpha^\varepsilon(t))m(\alpha^\varepsilon(t))x^{\varepsilon, \delta}(t)}{a(\alpha^\varepsilon(t)) + b(\alpha^\varepsilon(t))x^{\varepsilon, \delta}(t)} - f(\alpha^\varepsilon(t))y^{\varepsilon, \delta}(t) \right] dt \\ \quad + \sqrt{\delta} \rho(\alpha^\varepsilon(t))x^{\varepsilon, \delta}(t) dW_2(t), \end{cases}$$

where W_1 and W_2 are two independent Brownian motions, $\alpha^\varepsilon(t)$ is a Markov chain, that is independent of the Brownian motions, with state space $\mathcal{M} = \{1, 2\}$ and generator Q/ε where

$$Q = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},$$

and $r(1) = 0.9, r(2) = 1.1, K(1) = 4.737, K(2) = 5.238, m(1) = 1.2, m(2) = 0.8, a(1) = a(2) = 1, b(1) = b(2) = 1, d(1) = 0.85, d(2) = 1.15, e(1) = 1, e(2) = 2.5, f(1) = 0.03, f(2) = 0.01, \lambda(1) = 1, \lambda(2) = 2, \rho(1) = 3, \rho(2) = 1$. As ε and δ tend to 0, solutions of equation (4.37) converge to the corresponding solutions of

$$(4.38) \quad \begin{cases} \frac{d}{dt}x(t) = x(t) \left(1 - \frac{x(t)}{5} \right) - \frac{x(t)y(t)}{1+x(t)}, \\ \frac{d}{dt}y(t) = y(t) \left(-1 + \frac{1.6x(t)}{1+x(t)} - 0.02y(t) \right) \end{cases}$$

on any finite time interval $[0, T]$. The system (4.38) has the unique equilibrium $(x^*, y^*) = (1.836, 1.795)$. Modifying [SR93, Theorem 2.6] it can be seen that the solution of equation

(4.38) has a unique limit cycle Γ that attracts all positive solutions except for (x^*, y^*) . Moreover, it is easy to check that the drift

$$(4.39) \quad \begin{pmatrix} r(i)x(t) \left(1 - \frac{x(t)}{K(i)}\right) - \frac{m(i)x(t)y(t)}{a(i) + b(i)x(t)} \\ y(t) \left(-d(i) - \frac{e(i)m(i)x(t)}{a(i) + b(i)x(t)}\right) - f(i)y(t) \end{pmatrix}$$

does not vanish at $(1.836, 1.795)$. The assumptions of Theorem 1.2 hold in this example. As a result, the family $(\mu^{\varepsilon, \delta})_{\varepsilon > 0}$ converges weakly as $\varepsilon \downarrow 0$ to the stationary distribution of (4.38) that is concentrated on the limit cycle Γ . We illustrate this convergence in Figures 1, 2 and 3 below by graphing sample paths of (4.37) for different values of (ε, δ) .

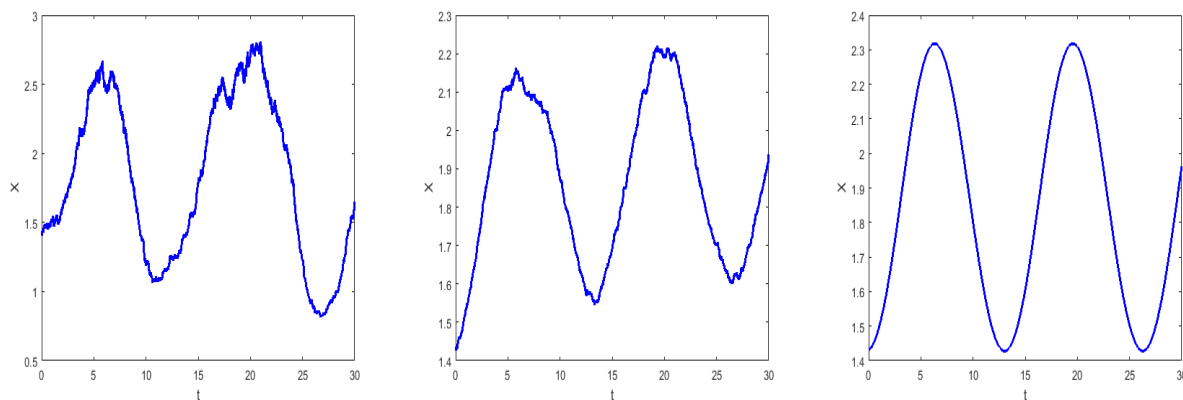


FIGURE 1. From left to right: Graphs of the $x^{\varepsilon, \delta}(t)$ component of (4.37) with $(\varepsilon, \delta) = (0.001, 0.001)$, $(\varepsilon, \delta) = (0.00005, 0.00005)$ and $x(t)$ of the averaged system (4.38) respectively.

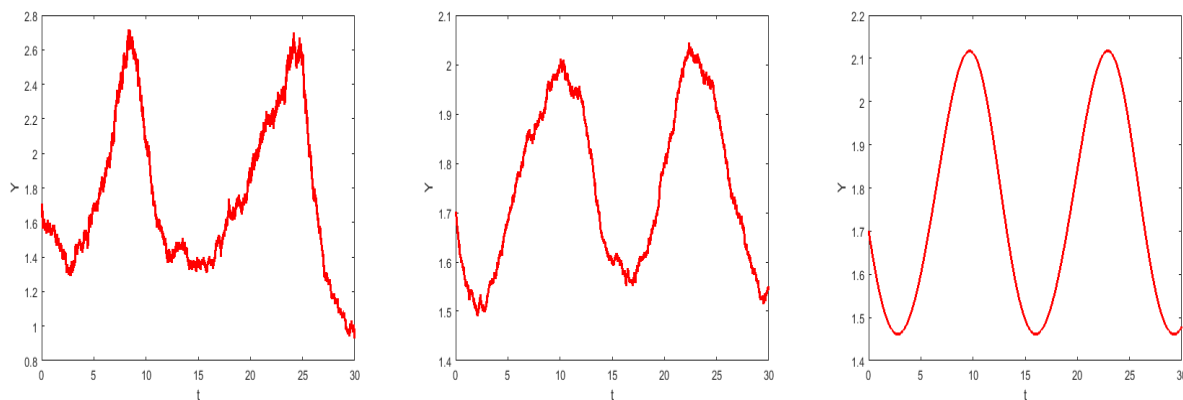


FIGURE 2. From left to right: Graphs of the $y^{\varepsilon, \delta}(t)$ component of (4.37) with $(\varepsilon, \delta) = (0.001, 0.001)$, $(\varepsilon, \delta) = (0.00005, 0.00005)$ and $y(t)$ of the averaged system (4.38) respectively.

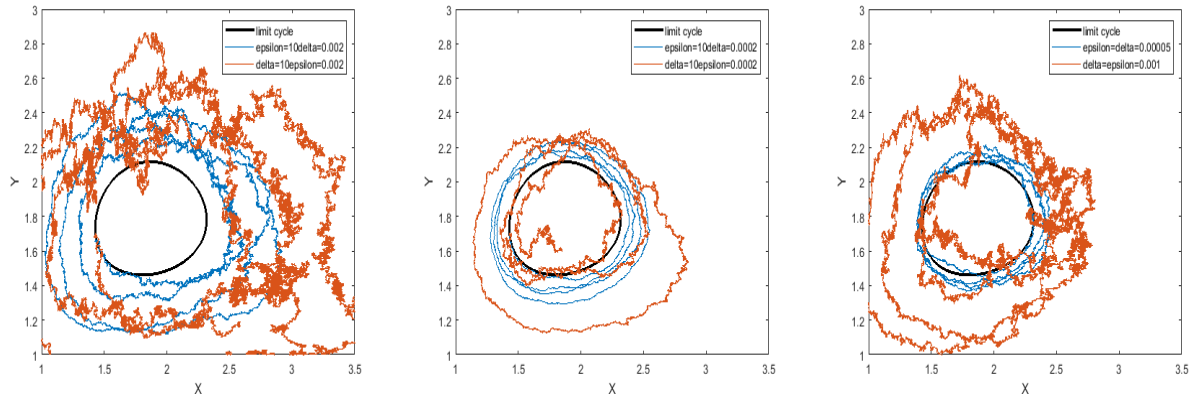


FIGURE 3. Phase portraits of (4.37) for different values of ϵ and δ .

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APPENDIX A. PROOFS OF LEMMAS FROM SECTION 2

Lemma A.1. *For any $R, T, \gamma > 0$, there exists a number $k_1 = k_1(R, T, \gamma) > 0$ such that for all sufficiently small δ ,*

$$\mathbb{P}\{|X_{x,i}^{\varepsilon,\delta}(t) - \xi_{x,i}^\varepsilon(t)| \geq \gamma, \text{ for some } t \in [0, T]\} < \exp\left(-\frac{k_1}{\delta}\right), x \in B_R,$$

where $X_{x,i}^{\varepsilon,\delta}(t)$ and $\xi_{x,i}^\varepsilon(t)$ are the solutions to the systems (1.3) and (1.7) that have initial value (x, i) .

Proof. By (i) and (ii) of Assumption 1.1, we can deduce the existence and boundedness of a unique solution to equation (1.7) using the Lyapunov function method. Moreover, we can find $R_T > R > 0$ such that almost surely

$$(A.1) \quad |\xi_{x,i}^\varepsilon(t)| < R_T - \gamma, \text{ for all } t \in [0, T], x \in B_R.$$

Let $h(\cdot)$ be a twice differentiable function with compact support such that $h(x) = 1$ if $|x| \leq R_T$ and $h(x) = 0$ if $|x| \geq R_T + 1$. Put $f_h(x, i) = h(x)f(x, i)$, $\sigma_h(x, i) = h(x)\sigma(x, i)$ and let $Y_{x,i}^{\varepsilon,\delta}(t)$ be the solution starting at (x, i) of

$$(A.2) \quad dY(t) = f_h(Y(t), \alpha^\varepsilon(t))dt + \sqrt{\delta}\sigma_h(Y(t), \alpha^\varepsilon(t))dW(t)$$

Note that $Y_{x,i}^{\varepsilon,\delta}(t) = X_{x,i}^{\varepsilon,\delta}(t)$ up to the time $\zeta = \inf\{t > 0 : |X_{x,i}^{\varepsilon,\delta}(t)| > R_T\}$. Because of (A.1), the solution $\xi_{x,i}^\varepsilon(t)$ to (1.7) coincides with the solution to

$$dZ(t) = f_h(Z(t), \alpha^\varepsilon(t))dt$$

with starting point $x \in B_R$ and $t \in [0, T]$. We have from the generalized Itô's formula that for all $x \in B_R$ and $t \in [0, T]$,

$$\begin{aligned}
& |Y_{x,i}^{\varepsilon,\delta}(t) - \xi_{x,i}^\varepsilon(t)|^2 \\
& \leq 2 \int_0^t |Y_{x,i}^{\varepsilon,\delta}(s) - \xi_{x,i}^\varepsilon(s)| |f_h(Y_{x,i}^{\varepsilon,\delta}(s), \alpha^\varepsilon(s)) - f_h(\xi_{x,i}^\varepsilon(s), \alpha^\varepsilon(s))| ds \\
(A.3) \quad & + \int_0^t \delta \text{trace}((\sigma_h \sigma_h')(Y_{x,i}^{\varepsilon,\delta}(s), \alpha^\varepsilon(s))) ds \\
& + 2\sqrt{\delta} \left| \int_0^t (Y_{x,i}^{\varepsilon,\delta}(s) - \xi_{x,i}^\varepsilon(s))' \sigma_h(Y_{x,i}^{\varepsilon,\delta}(s), \alpha^\varepsilon(s)) dW(s) \right|.
\end{aligned}$$

Define

$$\begin{aligned}
A = \left\{ \omega \in \Omega : \left| \int_0^t \sqrt{\delta} (Y_{x,i}^{\varepsilon,\delta}(s) - \xi_{x,i}^\varepsilon(s))' \sigma_h(Y_{x,i}^{\varepsilon,\delta}(s), \alpha^\varepsilon(s)) dW(s) \right| \right. \\
\left. - \frac{1}{\delta} \int_0^t \delta |Y_{x,i}^{\varepsilon,\delta}(s) - \xi_{x,i}^\varepsilon(s)|^2 \|\sigma_h \sigma_h'(Y_{x,i}^{\varepsilon,\delta}(s), \alpha^\varepsilon(s))\| ds \leq k_1 \text{ for all } t \in [0, T] \right\}.
\end{aligned}$$

By the exponential martingale inequality, we get that for any $\delta < k_1$

$$\mathbb{P}(A) \geq 1 - 2 \exp\left(-\frac{2k_1}{\delta}\right) \geq 1 - \exp\left(-\frac{k_1}{\delta}\right).$$

Since f_h is Lipschitz and σ_h is bounded, there is an $M_1 > 0$ such that for all $\omega \in A$,

$$\begin{aligned}
& |Y_{x,i}^{\varepsilon,\delta}(t) - \xi_{x,i}^\varepsilon(t)|^2 \\
& \leq 2 \int_0^t |Y_{x,i}^{\varepsilon,\delta}(s) - \xi_{x,i}^\varepsilon(s)| |f_h(Y_{x,i}^{\varepsilon,\delta}(s), \alpha^\varepsilon(s)) - f_h(\xi_{x,i}^\varepsilon(s), \alpha^\varepsilon(s))| ds \\
(A.4) \quad & + 2 \int_0^t |Y_{x,i}^{\varepsilon,\delta}(s) - \xi_{x,i}^\varepsilon(s)|^2 \|\sigma_h \sigma_h'(Y_{x,i}^{\varepsilon,\delta}(s), \alpha^\varepsilon(s))\| ds \\
& + \int_0^t \delta \text{trace}((\sigma_h \sigma_h')(Y_{x,i}^{\varepsilon,\delta}(s), \alpha^\varepsilon(s))) ds + 2 \int_0^t k_1 ds \\
& \leq M_1 \int_0^t |Y_{x,i}^{\varepsilon,\delta}(s) - \xi_{x,i}^\varepsilon(s)|^2 ds + (2k_1 + M_1 \delta)t.
\end{aligned}$$

For each $t \in [0, T]$, an application of Gronwall's inequality implies that on the set A ,

$$|Y_{x,i}^{\varepsilon,\delta}(t) - \xi_{x,i}^\varepsilon(t)|^2 \leq (2k_1 + M_1 \delta)T \exp(M_1 T) < \gamma^2$$

for $0 < \delta < k_1$ sufficiently small. It also follows from this inequality that for $\omega \in A$ and $0 < \delta < k_1$ sufficient small, we have $\zeta > T$, which implies

$$|X_{x,i}^{\varepsilon,\delta}(t) - \xi_{x,i}^\varepsilon(t)|^2 = |Y_{x,i}^{\varepsilon,\delta}(t) - \xi_{x,i}^\varepsilon(t)|^2 < \gamma^2,$$

for all $t \in [0, T]$. □

Lemma A.2. *For each x and γ , we can find $k_{\gamma,x} = k_{\gamma,x}(T) > 0$ such that*

$$\mathbb{P} \left\{ |\xi_{x,i}^\varepsilon(t) - \bar{X}_x(t)| \geq \gamma \text{ for some } t \in [0, T] \right\} \leq \exp\left(-\frac{k_{\gamma,x}}{\varepsilon}\right),$$

where $\bar{X}_x(t)$ is the solution to equation (1.4) with the initial value x .

Proof. This follows from the large deviation principle shown in [HYZ11]. We note that the existence and boundedness of a unique solution to equation (1.7) follows from parts (i) and (ii) of Assumption 1.1. \square

By combining the results of Lemmas A.1 and A.2 we can prove Lemma 2.2.

Proof of Lemma 2.2. By virtue of Lemma A.2, for each x and γ , we have

$$\mathbb{P} \left\{ |\xi_{x,i}^\varepsilon(t) - \bar{X}_x(t)| \geq \frac{\gamma}{6} \text{ for some } t \in [0, T] \right\} \leq \exp \left(-\frac{k_{\gamma/6,x}}{\varepsilon} \right).$$

By part (ii) of Assumption 1.1 together with the Lyapunov method for (1.7), we can find $H_{R,T} > 0$ such that $|\xi_{x,i}^\varepsilon(t)| \leq H_{R,T}$ and $|\bar{X}_x(t)| \leq H_{R,T}$ for all $|x| \leq R$ and $0 \leq t \leq T$. Since $f(\cdot, i)$ is locally Lipschitz for all $i \in \mathcal{M}$, there is a constant $M_2 > 0$ such that $|f(u, i) - f(v, i)| \leq M_2|u - v|$ for all $|u| \vee |v| \leq H_{R,T}$ and $i \in \mathcal{M}$. Using the Gronwall inequality, we have for $|x| \vee |y| \leq R$, $i \in \mathcal{M}$ and any $t \in [0, T]$

$$\begin{aligned} |\xi_{x,i}^\varepsilon(t) - \xi_{y,i}^\varepsilon(t)| &\leq |x - y| \exp(M_2 T), \\ |\bar{X}_x(t) - \bar{X}_y(t)| &\leq |x - y| \exp(M_2 T). \end{aligned}$$

Let $\lambda = \frac{\gamma}{6} \exp(-M_2 T)$. It is easy to see that for $|x - y| < \lambda$,

$$\mathbb{P} \left\{ |\xi_{y,i}^\varepsilon(t) - \bar{X}_y(t)| \geq \frac{\gamma}{2} \text{ for some } t \in [0, T] \right\} \leq \exp \left(-\frac{k_{\gamma/6,x}}{\varepsilon} \right).$$

By the compactness of B_R , for $\gamma > 0$, there is $k_2 = k_2(R, T, \gamma) > 0$ such that for all $x \in B_R$,

$$\mathbb{P} \left\{ |\xi_{x,i}^\varepsilon(t) - \bar{X}_x(t)| \geq \frac{\gamma}{2} \text{ for some } t \in [0, T] \right\} \leq \exp \left(-\frac{k_2}{\varepsilon} \right).$$

Combining this with Lemma A.1, we have

$$\begin{aligned} \mathbb{P} \left\{ |X_{x,i}^{\varepsilon,\delta}(t) - \bar{X}_x(t)| \geq \gamma \text{ for some } t \in [0, T] \right\} &< \exp \left(-\frac{k_1(R, T, \gamma/2)}{\delta} \right) + \exp \left(-\frac{k_2}{\varepsilon} \right) \\ &< \exp \left(-\frac{\kappa}{\varepsilon + \delta} \right) \end{aligned}$$

for a suitable $\kappa = \kappa(R, T, \gamma)$ and for all sufficiently small ε and δ . \square

Proof of Lemma 2.3. Let $n^{\varepsilon,\delta} \in \mathbb{N}$ such that $n^{\varepsilon,\delta} - 1 < \frac{1}{a^{\varepsilon,\delta}} \leq n^{\varepsilon,\delta}$. We consider events $A_k = \{X_{x,i}^{\varepsilon,\delta}(t) \in N, \forall (k-1)\ell < t \leq k\ell\}$. We have $\mathbb{P}(A_1) \leq 1 - a^{\varepsilon,\delta}$. By the Markov property,

$$\begin{aligned} \mathbb{P}(A_k | A_1, \dots, A_{k-1}) &= \int_N \mathbb{P} \left\{ \tilde{\tau}_y^{\varepsilon,\delta} \leq \ell \right\} \mathbb{P} \left\{ X_{x,i}^{\varepsilon,\delta}((k-1)\ell) \in dy \mid A_1, \dots, A_{k-1} \right\} \\ &\leq 1 - a^{\varepsilon,\delta}. \end{aligned}$$

As a result,

$$\mathbb{P}(A_1 A_2 \cdots A_n) \leq (1 - a^{\varepsilon,\delta})^{n^{\varepsilon,\delta}}$$

Since $\lim_{\varepsilon \rightarrow 0} a^{\varepsilon, \delta} = 0$, we deduce that $\lim_{\varepsilon \rightarrow 0} (1 - a^{\varepsilon, \delta})^{n^{\varepsilon, \delta}} = e^{-1}$, which means that $(1 - a^{\varepsilon, \delta})^{n^{\varepsilon, \delta}} < 1/2$ for sufficiently small ε . \square

Proof of Lemma 2.4. The proof is omitted because it states some standard properties of dynamical systems. Interested readers can refer to [Per13]. \square

DEPARTMENT OF MATHEMATICS, MECHANICS AND INFORMATICS, HANOI NATIONAL UNIVERSITY, 334 NGUYEN TRAI, THANH XUAN, HANOI, VIETNAM

E-mail address: `dunh@vnu.edu.vn`

DEPARTMENT OF MATHEMATICS, TUFTS UNIVERSITY, BROMFIELD-PEARSON HALL, 503 BOSTON AVENUE, MEDFORD, MA 02155, UNITED STATES

E-mail address: `alexandru.hening@tufts.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA, 345 GORDON PALMER HALL, BOX 870350, TUSCALOOSA, AL 35487-0350, UNITED STATES

E-mail address: `dangnh.maths@gmail.com`

DEPARTMENT OF MATHEMATICS, WAYNE STATE UNIVERSITY, DETROIT, MI 48202, UNITED STATES

E-mail address: `gyin@math.wayne.edu`