QUASI-STATIONARY DISTRIBUTIONS OF MULTI-DIMENSIONAL DIFFUSION PROCESSES

ALEXANDRU HENING, WEIWEI QI, ZHONGWEI SHEN, AND YINGFEI YI

Abstract. The present paper is devoted to the investigation of the long term behavior of a class of singular multi-dimensional diffusion processes that get absorbed in finite time with probability one. Our focus is on the analysis of quasi-stationary distributions (QSDs), which describe the long term behavior of the system conditioned on not being absorbed. Under natural Lyapunov conditions, we construct a QSD and prove the sharp exponential convergence to this QSD for compactly supported initial distributions. Under stronger Lyapunov conditions ensuring that the diffusion process comes down from infinity, we show the uniqueness of a QSD and the exponential convergence to the QSD for all initial distributions. Our results can be seen as the multi-dimensional generalization of Cattiaux et al (Ann. Prob. 2009) as well as the complement to Hening and Nguyen (Ann. Appl. Prob. 2018) which looks at the long term behavior of multi-dimensional diffusions that can only become extinct asymptotically.

The centerpiece of our approach concerns a uniformly elliptic operator that we relate to the generator, or the Fokker-Planck operator, associated to the diffusion process. This operator only has singular coefficients in its zeroth-order terms and can be handled more easily than the generator. For this operator, we establish the discreteness of its spectrum, its principal spectral theory, the stochastic representation of the semigroup generated by it, and the global regularity for the associated parabolic equation. We show how our results can be applied to most ecological models, among which cooperative, competitive, and predator-prey Lotka-Volterra systems.

Contents

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1. Introduction

Absorbed diffusion processes are often used in population biology to model the evolution of interacting species. Although the eventual extinction of all species is inevitable due to finite population effects (finite resources, finite population sizes, mortality, etc.) species can typically persist for a period of time that is long compared to human timescales [7]. It is important to understand the behavior of the ecosystem before the eventual extinction. This motivates the study of the dynamics of multi-dimensional diffusion processes conditioned on not going extinct.

To be more specific, consider the stochastic Lotka–Volterra competition system:

$$\mathrm{d}Z_t^i = Z_t^i \left(r_i - \sum_{j=1}^{d} c_{ij} Z_t^j \right) \mathrm{d}t + \sqrt{\gamma_i Z_t^i} \mathrm{d}W_t^i, \quad i \in \{1, \dots, d\}, \tag{1.1}$$

where $Z_t = (Z_t^i) \in \overline{\mathcal{U}} := [0,\infty)^d$ are the abundances of the species at time t, $\{r_i\}_i$ are per-capita growth rates, $\{c_{ii}\}_i$ are the intra-specific competition rates, $\{c_{ij}\}_{i\neq j}$ are inter-specific competition rates, $\{\gamma_i\}_i$ are demographic parameters describing ecological timescales (see e.g. [6, 7]), and $\{W^i\}_i$ are independent standard one-dimensional Wiener processes on some probability space. It is well-known (see e.g. [7, 12]) that Z_t reaches the boundary, also called the extinction set, $\Gamma := \left\{z = (z_i) \in \overline{\mathcal{U}} : z_i = 0 \text{ for some } i \in \{1,\dots,d\}\right\}$, of $\overline{\mathcal{U}}$ in finite time almost surely. This corresponds to the extinction of at least one species of the considered community. Nonetheless, typical trajectories or sample paths of Z_t will stay in $\mathcal{U} := (0,\infty)^d$ for a long period before hitting Γ. This can be interpreted as the temporary coexistence of species, before their ultimate extinction. To understand this type of behavior, notions such as quasi-steady states and metastable states have been put forward. These concepts are often formalized in terms of the *quasi-stationary distributions* (QSDs), which are stationary distributions of Z_t conditioned on no species going extinct. In this context, it is of fundamental mathematical importance to analyze the existence, uniqueness, and domains of (exponential) attraction of QSDs.

The purpose of the present paper is to investigate the existence and uniqueness of QSDs and the exponential convergence to QSDs for a class of irreversible diffusion processes given by models of the form

$$\mathrm{d}Z_t^i = b_i(Z_t)\mathrm{d}t + \sqrt{a_i(Z_t^i)}\mathrm{d}W_t^i, \quad i \in \{1,\dots,d\}, \tag{1.2}$$

where $Z_t := (Z_t^i) \in \overline{\mathcal{U}}$, $b_i : \overline{\mathcal{U}} \to \mathbb{R}$ and $a_i : [0,\infty) \to [0,\infty)$. We make the following assumptions.

- **(H1)** $a_i \in C^2([0,\infty))$, $a_i(0) = 0$, $a_i'(0) > 0$, $a_i > 0$ on $(0,\infty)$, $\limsup_{s\to\infty}\left[\frac{|a_i'(s)|^2}{a_i(s)} + a_i''(s)\right] < \infty$ and $\int_1^\infty \frac{\mathrm{d}s}{\sqrt{a_i(s)}} = \infty$ for all $i \in \{1,\dots,d\}$.
- **(H2)** $b_i \in C^1(\overline{\mathcal{U}})$ and $b_i|_{z_i=0} = 0$ for all $i \in \{1,\dots,d\}$, where $z_i = 0$ means the set $\left\{z = (z_i) \in \overline{\mathcal{U}} : z_i = 0\right\}$.
- **(H3)** There exists a positive function $V \in C^2(\overline{\mathcal{U}})$ satisfying the following conditions.
  1. $\lim_{|z|\to\infty} V(z) = \infty$ and $\lim_{|z|\to\infty}(b\cdot\nabla_z V)(z) = -\infty$.
  2. There exists a non-negative and continuous function $\tilde{V} : [0,\infty) \to [0,\infty)$ satisfying
  $$\int_1^\infty \frac{e^{-\beta\tilde{V}}}{a_i}\mathrm{d}s < \infty, \quad \forall \beta > 0 \text{ and } i \in \{1,\dots,d\}$$
  such that $V(z) \geq \sum_{i=1}^d \tilde{V}(z_i)$ for all $z = (z_i) \in \overline{\mathcal{U}}$.

(3) The following limit holds

$$\lim_{|z|\to\infty} \frac{1}{b\cdot\nabla_z V}\sum_{i=1}^{d}\left(|\partial_{z_i} b_i| + \frac{|a_i' b_i|}{a_i} + |a_i'\partial_{z_i} V| + |a_i \partial^2_{z_i z_i} V|\right) = 0.$$

(4) There exist constants $C>0$ and $R>0$ such that

$$\sum_{i=1}^{d}\left(a_i|\partial_{z_i} V|^2 + \frac{b_i^2}{a_i}\right) \le -Cb\cdot\nabla_z V \quad \text{in} \quad \mathcal{U}\setminus B_R^+,$$

where $B_R^+ := \{z=(z_i)\in\mathcal{U} : z_i\in(0,R),\ \forall i\in\{1,\dots,d\}\}$ for $R>0$.

Assumption **(H1)** says that each $a_i(s)$ behaves like $a_i'(0)s$ near $s\approx 0$, and allows each $a_i(s)$ to behave like s^γ for some $\gamma\in(-\infty,2]$ near $s\approx\infty$. Assumption **(H2)** is satisfied if $b_i(z)=z_i f_i(z)$ for $f_i\in C^1(\overline{\mathcal{U}})$. **(H1)** and **(H2)** ensure that (1.2) generates a diffusion process Z_t on $\overline{\mathcal{U}}$ having Γ as an absorbing set. **(H3)**(1) and the condition $\lim_{|z|\to\infty}\frac{\sum_{i=1}^d |a_i\partial^2_{z_i z_i} V|}{b\cdot\nabla_z V}=0$ contained in **(H3)**(3) imply the dissipativity of Z_t, and hence, that it does not explode in finite time almost surely. Other assumptions in **(H3)** are technical ones, but they are made according to examples discussed in Section 6. We note that for a reversible system, the potential function is a natural choice for V. For irreversible systems, polynomials are usually good choices for V, especially when the coefficients are polynomials or rational functions – this is often the case in applications.

We show in Proposition 2.1 that Z_t reaches Γ in finite time almost surely under **(H1)**-**(H3)**, and hence, that Z_t does not admit a stationary distribution that has positive concentration in $\mathcal{U}$. It is then natural to look at Z_t before reaching Γ in order to understand the dynamics of Z_t. This drives us to examine quasi-stationary distributions of Z_t or (1.2) conditioned on coexistence, i.e., $[t<T_\Gamma]$, where $T_\Gamma := \inf\{t>0 : Z_t\in\Gamma\}$ is the first time when Z_t hits Γ. Denote by $\mathbb{P}^\mu$ the law of Z_t with initial distribution μ, and by $\mathbb{E}^\mu$ the expectation with respect to $\mathbb{P}^\mu$.

Definition 1.1 (Quasi-stationary distribution)**.** *A Borel probability measure μ on $\mathcal{U}$ is called a* quasi-stationary distribution *(QSD) of Z_t or* (1.2) *if for each $f\in C_b(\mathcal{U})$, one has*

$$\mathbb{E}^\mu\left[f(Z_t)\middle| t<T_\Gamma\right] = \int_{\mathcal{U}} f\,\mathrm{d}\mu, \quad \forall t\ge 0.$$

The QSDs of Z_t are simply stationary distributions of Z_t conditioned on $[t<T_\Gamma]$. This is why QSDs can be seen as governing the dynamics of Z_t before extinction. It is known from the general theory of QSDs (see e.g. [40, 15]) that if μ is a QSD of Z_t, then there exists a unique $\lambda>0$ such that if $Z_0\sim\mu$ the time T_Γ is exponentially distributed with rate λ, i.e., $\mathbb{P}^\mu[T_\Gamma>t]=e^{-\lambda t}$ for all $t\ge 0$. The number λ is often called the *extinction rate* associated to μ.

Our first result concerning the existence of QSDs and the conditioned dynamics of Z_t is stated in the following theorem. Denote by $\mathcal{P}(\mathcal{U})$ the set of Borel probability measures on $\mathcal{U}$.

Theorem A. *Assume* **(H1)**-**(H3)***. Then, Z_t admits a QSD μ_1, and there exists $r_1 > 0$ such that the following hold.*

- *For any $0 < \epsilon \ll 1$ and $\mu \in \mathcal{P}(\mathcal{U})$ with compact support in $\mathcal{U}$ we have*

$$\lim_{t\to\infty} e^{(r_1-\epsilon)t} \left| \mathbb{E}^{\mu}\left[f(Z_t)\middle| t < T_\Gamma\right] - \int_{\mathcal{U}} f \mathrm{d}\mu_1 \right| = 0, \quad \forall f \in C_b(\mathcal{U}).$$

- *There exists $f \in C_b(\mathcal{U})$ such that for a.e. $x \in \mathcal{U}$, there is a discrete set $\mathcal{I}_x \subset (0,\infty)$ with distances between adjacent points admitting an x-independent positive lower bound, such that for each $0 < \delta \ll 1$ we have*

$$\lim_{\substack{t\to\infty \\ t\in(0,\infty)\setminus\mathcal{I}_{x,\delta}}} e^{(r_1+\epsilon)t} \left| \mathbb{E}^{x}[f(X_t)\middle| t < T_\Gamma] - \int_{\mathcal{U}} f \mathrm{d}\mu_1 \right| = \infty, \quad \forall 0 < \epsilon \ll 1,$$

  where $\mathcal{I}_{x,\delta}$ is the δ-neighbourhood of $\mathcal{I}_x$ in $(0,\infty)$.

Remark 1.1. *The first conclusion in Theorem A actually holds for a much larger class of initial distributions (see Remark 5.2 for more details).*

We point out that the sharp exponential convergence rate r_1 is given by the spectral gap, between the principal eigenvalue and the rest of the spectrum, of the Fokker-Planck operator associated to Z_t in an appropriate weighted function space. The QSD is essentially given by the positive eigenfunction associated to the principal eigenvalue, and the associated extinction rate is just the absolute value of the principal eigenvalue. Such characterizations of the QSD and the exponential convergence rate have been obtained in [6, 7] in the reversible case. Our result is the first of this type for the general setting when Z_t is irreversible. Theorem A applies to a large class of population models including stochastic Lotka-Volterra models, models with Holling type functional responses, and Beddington-DeAngelis models. We refer the reader to Section 6 for more details.

The set $\mathcal{I}_x$ in the second conclusion more or less corresponds to the zeros of the function $t \mapsto \mathbb{E}^x[f(X_t)\big| t < T_\Gamma] - \int_{\mathcal{U}} f \mathrm{d}\nu_1$. For irreversible systems one generally has complex eigenvalues, which give rise to oscillations. As a result, the zeros of the above function exist and form a discrete set as described in the statement of Theorem A.

Although the QSD μ_1 obtained in Theorem A attracts all compactly supported initial distributions, there is no assertion that it is the unique QSD of the process Z_t. To study the uniqueness, we make the following additional assumption.

(H4) There exist positive constants C, γ and R_* such that

$$\lim_{|z|\to\infty} V^{-\gamma-2}\sum_{i=1}^d a_i|\partial_{z_i}V|^2=0 \quad \text{and} \quad \frac{1}{2}\sum_{i=1}^d a_i\partial^2_{z_iz_i}V+b\cdot\nabla_z V\leq -CV^{\gamma+1} \quad \text{in} \quad \mathcal{U}\setminus B^+_{R_*}.$$

Theorem B. *Assume* **(H1)**-**(H4)**. *Let μ_1 and r_1 be as in Theorem A. Then, μ_1 is the unique QSD of Z_t, and for any $0<\epsilon\ll 1$ and $\mu\in\mathcal{P}(\mathcal{U})$, there holds*

$$\lim_{t\to\infty} e^{(r_1-\epsilon)t}\left|\mathbb{E}^\mu\left[f(Z_t)\middle|t<T_\Gamma\right]-\int_{\mathcal{U}} f\mathrm{d}\mu_1\right|=0,\quad \forall f\in C_b(\mathcal{U}).$$

Assumption **(H4)** concerns the strong dissipativity of Z_t near infinity, and implies in particular that Z_t *comes down from infinity* (see Remark 5.3), that is, for each $\lambda>0$, there exists $R=R(\lambda)>0$ such that $\sup_{z\in\mathcal{U}\setminus B^+_R}\mathbb{E}^z\left[e^{\lambda T_R}\right]<\infty$, where $T_R:=\inf\left\{t\geq 0: Z_t\notin\mathcal{U}\setminus B^+_R\right\}$. This property plays a crucial role in the proof of Theorem B. It says that with high probability the process Z_t quickly enters a bounded region. This happens even if the initial distribution of Z_t has a heavy tail near ∞. As a result, it makes no difference to the QSD μ_1 whether the initial distribution of Z_t is compactly supported or not. Theorem B applies to a large class of biological models including in particular the stochastic competition system (1.1) and the stochastic weak cooperation system (i.e., the system (1.1) with $\{-c_{ij}\}_{i\neq j}$ being positive and small in comparison to $\{c_{ii}\}_i$). See Section 6 for more details.

Comparison to existing literature. Due to their popularity in describing non-stationary states that are often observed in applications, QSDs have been attracting significant attention. We refer the reader to [44, 40, 15] and references therein for an overview of the theory, developments and applications of QSD. We next present the current state of the art for diffusion processes. The investigation of QSDs for one-dimensional diffusion processes has attracted a lot of attention. We refer the reader to [37, 16, 39, 48, 30, 51, 8, 9, 10] and references therein for the analysis of the regular case.

For singular diffusion processes including in particular (1.1) and (1.2) in the one-dimensional setting, the work [6] lays the foundation and is generalized in [35, 41, 10, 26]. In contrast, there have not been many studies of QSDs for multi-dimensional diffusion processes. Regular diffusion processes restricted to a bounded domain and killed on the boundary have been studied in [43, 24, 12]. The stochastic competition system (1.1) has been studied in [7] in the reversible case, and in [11] in the irreversible case. In both of the above papers, the exponential convergence to the unique QSD is established. In [7], the authors also deal with the model in the weak cooperation case. The model treated in [11] has a more general deterministic vector field. In [12], the authors study general multi-dimensional diffusion processes, establish the existence and convergence to the QSD. However, they do not look at the uniqueness problem.

The approaches used in [7], [11] and [12] for treating singular diffusion processes in higher dimensions are quite different. The work done in [7] relies on the spectral theory of the generator in the weighted space $L^2(\mathcal{U}, d\mu)$ with μ being the infinite Gibbs measure, and the density of the Markov semigroup with respect to μ. These tools are developed earlier in [6] for one-dimensional singular diffusion processes. We note that the assumptions from [7] make the diffusion process reversible, and much easier to analyze. However, most multi-dimensional diffusion processes would be irreversible. In [11], the authors study general Markov processes and apply their abstract results in particular to (1.1). The main purpose of [11] is to find practical sufficient conditions in order to use the necessary and sufficient condition of Doeblin-type established in their earlier work [8] for the exponential convergence to the unique QSD. Their sufficient conditions are composed of a Lyapunov condition involving a pair of functions, a local Doeblin-type condition and a "coming down from infinity" condition. In [12], the authors almost only impose a commonly used dissipative Lyapunov condition except the Lyapunov constant is assumed to be greater than some exponential rate related to the first exit time.

In comparison to [7, 11, 12], the main novelty of the present paper lies in the approach that allows us to treat Z_t or (1.2) under elementary Lyapunov conditions. The results we obtain are as strong as those established in [6, 7] for reversible diffusion processes, even though we work in the much more general setting of irreversible processes. The centerpiece of our approach is the spectral theory of an elliptic operator derived from the Fokker-Planck operator associated to Z_t through a two-stage change of variables. We establish the discreteness of the spectrum of this operator as well as its principal spectral theory. These results allow us to use the semigroup generated by this operator in order to establish its stochastic representation and to study the fine dynamical properties of Z_t. A direct consequence of our approach is the characterization of the sharp exponential convergence rate in Theorem A – this was previously unknown for singular diffusion processes in higher dimensions. Moreover, since spectral theory and the stochastic representation of semigroups are important tools, our results go beyond the study of QSD and are of independent interest.

Demographic and environmental stochasticity. Consider an isolated ecosystem of interacting species. Due to finite population effects and demographic stochasticity, extinction of all species is certain to occur in finite time for all populations. However, the time to extinction can be large and the species densities can fluctuate before extinction occurs.

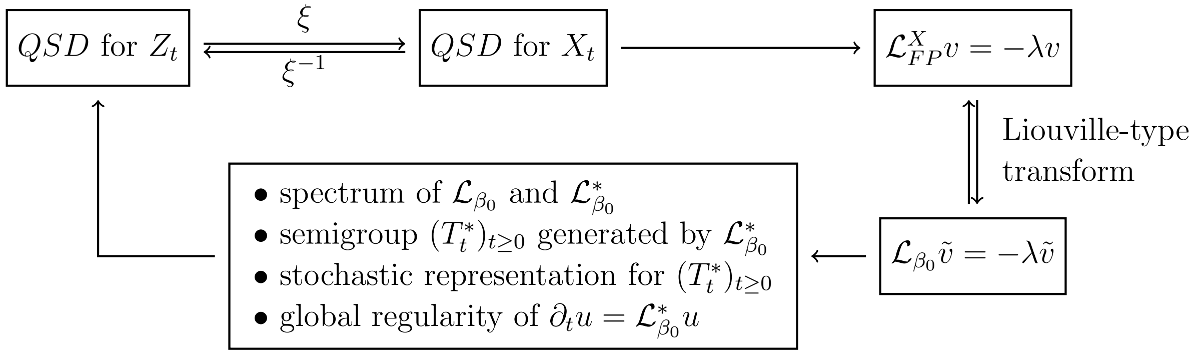

FIGURE 1. Overview of proofs.

One way of capturing this behaviour is ignoring the effects of demographic stochasticity (i.e. finite population effects) and focusing on models with environmental stochasticity where extinction can only be asymptotic as $t \to \infty$. This approach led to the development of the field of modern coexistence theory (MCT), started by Lotka [36] and Volterra [50], and later developed by Chesson [13, 14] and other authors [49, 22, 47, 4]. Recently, there have been powerful results that have led to a general theory of coexistence and extinction [27, 3, 28].

A second way of analyzing the long term dynamics of the species is by including demographic stochasticity and studying the QSDs of the system - this is the approach we took in this paper. Our work can be seen as complementary to the work done for systems with environmental stochasticity.

Overview of proofs. The proofs of Theorem A and Theorem B use techniques from PDE, spectral theory, semigroup theory and probability theory, and are rather involved. For the reader's convenience, we outline the strategy of the proofs with the help of Figure 1.

- (Equivalent formalism) Theoretically, the study of QSDs of Z_t can be accomplished by investigating the (principal) spectral theory of $\mathcal{L}^Z_{\mathbf{FP}}$, the Fokker-Planck operator associated to Z_t. However, the degeneracy of $\mathcal{L}^Z_{\mathbf{FP}}$ on Γ would cause significant drawbacks. To circumvent this, we introduce a homeomorphism $\xi : \overline{\mathcal{U}} \to \overline{\mathcal{U}}$ and define a new process $X_t = \xi(Z_t)$ whose Fokker-Planck operator $\mathcal{L}^X_{\mathbf{FP}}$ has $\frac{1}{2}\Delta$ as its second-order term.

  Although $\mathcal{L}^X_{\mathbf{FP}}$ has the best possible second-order term, the coefficients of its first-order terms unfortunately have blow-up singularities on Γ. Introducing a Liouville-type transform, we convert $\mathcal{L}^X_{\mathbf{FP}}$ into a uniformly elliptic operator

$$\mathcal{L}_{\beta_0} := e^{\frac{Q}{2}+\beta_0 U} \mathcal{L}^X_{\mathbf{FP}} e^{-\frac{Q}{2}-\beta_0 U}$$

whose blow-up singularities on Γ only appear in the coefficients of the zeroth-order terms. Here $U = V \circ \xi^{-1}$ and Q, given in (2.5), has singularities near Γ (see Remark A.1).

The number β_0 is chosen so that $\mathcal{L}_{\beta_0}$ satisfies certain a priori estimates. The details are presented in Subsection 2.2. The number β_0 is fixed in Lemma 3.2 (3).

- (Spectral analysis) Our spectral analysis focuses on the operator $\mathcal{L}_{\beta_0}$ in $L^2(\mathcal{U};\mathbb{C})$ as well as its adjoint $\mathcal{L}^*_{\beta_0}$. According to the behavior of the coefficients of $\mathcal{L}_{\beta_0}$ near Γ and infinity, we design a weight function and define a weighted first-order Sobolev space $\mathcal{H}^1(\mathcal{U};\mathbb{C})$ that is compactly embedded into $L^2(\mathcal{U};\mathbb{C})$. Applying the a priori estimates of $\mathcal{L}_{\beta_0}$, we are able to solve the elliptic problem for $\mathcal{L}_{\beta_0}-M$ for some $M \gg 1$ in $\mathcal{H}^1(\mathcal{U};\mathbb{C})$. The discreteness of the spectrum and principal spectral theory of $\mathcal{L}_{\beta_0}$ and $\mathcal{L}^*_{\beta_0}$ then follow.

  The details are given in Subsection 3.3 and Subsection 3.4.
- (Semigroup and stochastic representation) The operator $\mathcal{L}^*_{\beta_0}$ generates an analytic and eventually compact semigroup $(T^*_t)_{t\geq 0}$ on $L^2(\mathcal{U};\mathbb{C})$ that can be "block-diagonalized" according to spectral projections. We establish the representation of $(T^*_t)_{t\geq 0}$ in terms of X_t before reaching Γ, and therefore, connect the dynamics of $(T^*_t)_{t\geq 0}$ with that of X_t conditioned on $[t < S_\Gamma]$, where S_Γ is the first time that X_t hits Γ. More precisely, we show that for each $f \in C_b(\mathcal{U};\mathbb{C})$ satisfying $\tilde{f} := f e^{-\frac{Q}{2}-\beta_0 U} \in L^2(\mathcal{U};\mathbb{C})$, there holds

$$T^*_t \tilde{f} = e^{-\frac{Q}{2}-\beta_0 U} \mathbb{E}^\bullet \left[f(X_t) \mathbb{1}_{\{t<S_\Gamma\}} \right], \quad \forall t \geq 0.$$

  The semigroups are given in Subsection 3.3 and Subsection 3.4. The stochastic representation of $(T^*_t)_{t\geq 0}$ is established in Subsection 4.3.
- (Global regularity and conclusions) The spectral theory and stochastic representation allow us to prove the results as in Theorem A and Theorem B for the process X_t. While proving the existence of QSDs is pretty straightforward, we run into significant technical difficulties trying to establish the convergence even for compactly supported initial distributions. This is due to: (i) the limitations of the stochastic representation because of the unboundedness of the Liouville-type transform and its inverse (i.e., $e^{\frac{Q(x)}{2}+\beta_0 U(x)}$ grows to ∞ as $|x| \to \infty$ and $e^{-\frac{Q}{2}-\beta_0 U}$ blows up at Γ); (ii) the requirement of L^∞ properties of $(T^*_t)_{t\geq 0}$. These issues are overcome by establishing the global regularity of solutions of $\partial_t u = \mathcal{L}^*_{\beta_0} u$ leading in particular to the global regularity of $(T^*_t)_{t\geq 0}$.

  The details are given in Section 5.

The rest of the paper is organized as follows. In Section 2, we provide some preliminaries including the proof of Z_t being absorbed by Γ in finite time almost surely, the derivation of the operator $\mathcal{L}_{\beta_0}$, and results related to the approximation of S_Γ. In Section 3, we study the spectral theory of $\mathcal{L}_{\beta_0}$ and its adjoint operator $\mathcal{L}^*_{\beta_0}$, and establish the associated semigroups $(T_t)_{t\geq 0}$ and $(T^*_t)_{t\geq 0}$. Section 4 is devoted to the stochastic representation of $(T^*_t)_{t\geq 0}$. In Section 5, we investigate the existence and uniqueness of QSDs and the exponential convergence to QSDs of X_t conditioned on the coexistence. Theorem A and Theorem B are proven in this section. In the last section, Section 6, we discuss applications of Theorem A and Theorem B to a wider variety of ecological models including stochastic Lotka-Volterra systems, and models with Holling type or Beddington-DeAngelis functional responses. Appendix A is included to provide the proof of some technical lemmas.

2. Preliminaries

In Subsection 2.1, we show that Z_t hits Γ in finite time almost surely. In Subsection 2.2, we present equivalent formulations for studying the existence of QSDs, and derive the operator we shall focus on in later sections. In Subsection 2.3, we fix a family of first exit times and present an approximation result.

2.1. Hitting the absorbing boundary.

We prove that Z_t reaches Γ in finite time almost surely. Denote by $\mathcal{L}^Z$ the diffusion operator associated to Z_t, namely,

$$\mathcal{L}^Z = \frac{1}{2}\sum_{i=1}^d a_i \partial^2_{z_i z_i} + b\cdot\nabla_z.$$

Proposition 2.1. *Assume* **(H1)**-**(H3)**. *Then,* $\mathbb{P}^z[T_\Gamma < \infty] = 1$ *for each* $z \in \mathcal{U}$.

Proof. Note that **(H1)-(H2)** ensure the pathwise uniqueness as well as the strong Markov property of solutions of (1.2) until the explosion time. Recall that for $R > 0$,

$$B^+_R = \{z = (z_i) \in \mathcal{U} : z_i \in (0, R),\ \forall i \in \{1, \dots, d\}\}.$$

The result is proven in four steps.

Step 1. We claim the existence of $R > 0$ such that $\mathbb{P}^z[T_R < \infty] = 1$ for each $z \in \mathcal{U}$, where $T_R := \inf\left\{t \in [0, T_\Gamma] : Z_t \in B^+_R\right\}$.

By the assumptions **(H3)**(1)(3), there is $R > 0$ such that $\mathcal{L}^Z V \leq -1$ in $\mathcal{U}\setminus B^+_R$. For each $z \in \mathcal{U}$, Itô's formula gives

$$\mathbb{E}^z\left[V(Z_{t\wedge T_R})\right] = V(z) + \mathbb{E}^z\left[\int_0^{t\wedge T_R} \mathcal{L}^Z V(Z_s)\mathrm{d}s\right] \leq V(z) - \mathbb{E}^z\left[t\wedge T_R\right], \quad \forall t \geq 0.$$

Passing to the limit $t \to \infty$ yields $\mathbb{E}^z\left[T_R\right] \leq V(z) < \infty$. The claim follows.

Step 2. We prove $\mathbb{P}^z[\tau_{2R} < \infty] = 1$ for each $z \in B^+_{2R}$, where $\tau_{2R} := \inf\left\{t \geq 0 : Z_t \notin B^+_{2R}\right\}$.

For each $i \in \{1, \ldots, d\}$, we set $\overline{b}_i := \sup_{B^+_{2R}} b_i$, denote by Y^{i,y_i}_t the solution of the following one-dimensional SDE

$$\mathrm{d}Y^i_t = \overline{b}_i \mathrm{d}t + \sqrt{a_i(Y_i)}\mathrm{d}W^i_t$$

with initial condition $Y^{i,y_i}_0 = y_i \in [0, \infty)$, and let $\tau^{y_i}_i$ be the first time that Y^{i,y_i}_t hits 0, namely, $\tau^{y_i}_i = \inf\left\{t \geq 0 : Y^{i,y_i}_t = 0\right\}$. The assumptions on a_i and [29, Theorem VI-3.2] guarantee that $\mathbb{P}\left[\tau^{y_i}_i < \infty\right] = 1$ for all $y_i \in [0, \infty)$ and $i \in \{1, \ldots, d\}$.

Let $z = (z_i) \in B^+_{2R}$. By the comparison theorem for one-dimensional SDEs (see e.g. [29, Theorem VI-1.1]) and the fact that $\mathbb{P}\left[\tau^{z_i}_i < \infty\right] = 1$ for each $i \in \{1, \ldots, d\}$, we find up to a set of probability zero,

$$[\tau_{2R} = \infty] \subset \left[Z^i_t \leq Y^{i,z_i}_t,\ \forall t \in [0, \tau^{z_i}_i],\ i \in \{1, \ldots, d\}\right] \subset [\tau_{2R} < \infty].$$

From this we conclude that $\mathbb{P}^z[\tau_{2R} = \infty] = 0$.

Step 3. We show that $\inf_{z \in \overline{B^+_R}} \mathbb{P}^z[Z_{\tau_{2R}} \in \Gamma] > 0$.

Fix $i \in \{1, \ldots, d\}$. Calculating the probability that the process $Y^{i,R}_t$ first exits the interval $(0, \frac{3R}{2})$ through $\frac{3R}{2}$ (see [29, Theorem VI-3.1]), we find $\mathbb{P}\left[Y^{i,R}_t \in [0, 3R/2)\,,\ \forall t \in [0, \tau^R_i]\right] > 0$. Since

$$\mathbb{P}\left[Y^{i,y_i}_t \leq Y^{i,R}_t,\ \forall t \in [0, \tau^{y_i}_i]\right] = 1, \quad \forall y_i \in [0, R]$$

due to the comparison theorem (see e.g. [29, Theorem VI-1.1]), we deduce

$$\inf_{y_i \in [0,R]} \mathbb{P}\left[Y^{i,y_i}_t \in [0, 3R/2)\,,\ \forall t \in [0, \tau^{y_i}_i]\right] \geq \mathbb{P}\left[Y^{i,R}_t \in [0, 3R/2)\,,\ \forall t \in [0, \tau^R_i]\right] > 0.$$

This together with the comparison theorem yields for each $z = (z_i) \in \overline{B^+_R}$,

$$\begin{aligned}
\mathbb{P}^z[Z_{\tau_{2R}} \in \Gamma] &\geq \mathbb{P}\left[Y^{i,z_i}_t \in [0, 3R/2)\,,\ \forall t \in [0, \tau^{z_i}_i],\ i \in \{1, \ldots, d\}\right] \\
&= \prod_{i=1}^d \mathbb{P}\left[Y^{i,z_i}_t \in [0, 3R/2)\,,\ \forall t \in [0, \tau^{z_i}_i]\right] \\
&\geq \prod_{i=1}^d \inf_{y_i \in [0,R]} \mathbb{P}\left[Y^{i,y_i}_t \in [0, 3R/2)\,,\ \forall t \in [0, \tau^{y_i}_i]\right] > 0,
\end{aligned}$$

where we used the independence of Y^{i,z_i}_t, $i \in \{1, \ldots, d\}$ in the equality. The claim follows.

Step 4. We finish the proof of the proposition. By **Step 3**, $p := \inf_{z\in\partial B_R^+\setminus\Gamma}\mathbb{P}^z[Z_{\tau_{2R}}\in\Gamma]>0$. Set

$$T_R^{(1)} := \inf\left\{t\in[0,T_\Gamma] : Z_t\in B_R^+\right\} \quad \text{and} \quad S_{2R}^{(1)} := \inf\left\{t\geq T_R^{(1)} : Z_t\notin B_{2R}^+\right\},$$

and recursively define for each $n\geq 1$,

$$T_R^{(n+1)} := \inf\left\{t\in[S_{2R}^{(n)},T_\Gamma] : Z_t\in B_R^+\right\} \quad \text{and} \quad S_{2R}^{(n+1)} := \inf\left\{t\geq T_R^{(n+1)} : Z_t\notin B_{2R}^+\right\}.$$

Fix $z\in\mathcal{U}$. Since **Step 1**, **Step 2** and the strong Markov property ensure $\mathbb{P}^z\left[T_R^{(n)}<\infty\right]=1$ and $\mathbb{P}^z\left[S_{2R}^{(n)}<\infty\right]=1$ for all $n\in\mathbb{N}$, we find $\mathbb{P}^z\left[Z_{S_{2R}^{(n)}}\in\partial B_{2R}^+\setminus\Gamma\right]\leq(1-p)^n$ for all $n\in\mathbb{N}$. As a result

$$\mathbb{P}^z\left[T_\Gamma=\infty\right]=\mathbb{P}^z\left[S_{2R}^{(n)}<\infty,\forall n\in\mathbb{N}\right]\leq\lim_{n\to\infty}(1-p)^n=0.$$

This completes the proof. □

Remark 2.1. *The assumptions* **(H3)***(2)(4) are not needed in the proof of Proposition 2.1.*

2.2. **Equivalent formulation.** Denote by $\mathcal{L}^Z_{\mathbf{FP}}$ the Fokker-Planck operator associated to Z_t or (1.2), namely,

$$\mathcal{L}^Z_{\mathbf{FP}}u := \frac{1}{2}\sum_{i=1}^d \partial^2_{z_iz_i}(a_iu)-\nabla_z\cdot(bu) \quad \text{in} \quad \mathcal{U}. \tag{2.1}$$

Proposition 2.2. *Assume* **(H1)**-**(H2)**. *Let μ be a QSD of Z_t. Then, μ admits a positive density $u\in C^2(\mathcal{U})$ that satisfies $-\mathcal{L}^Z_{\mathbf{FP}}u=\lambda_1u$ in $\mathcal{U}$, where λ_1 is the extinction rate associated to μ.*

Proof. It follows from [40, Proposition 4] and the elliptic regularity theory (see e.g. [5]). □

Proposition 2.2 suggests studying the principal spectral theory of the operator $-\mathcal{L}^Z$ in order to find a QSD for Z_t. Direct analysis of the operator $-\mathcal{L}^Z$ is however difficult due to the degeneracy of the diffusion matrix $\operatorname{diag}\{a_1,\dots,a_d\}$ on the boundary Γ of $\mathcal{U}$. To resolve this issue, we define a new process that is equivalent to Z_t and whose Fokker-Planck operator or diffusion operator is uniformly non-degenerate in $\mathcal{U}$. We proceed as follow.

For each $i\in\{1,\dots,d\}$, we define $\xi_i:[0,\infty)\to[0,\infty)$ by setting

$$\xi_i(z_i) := \int_0^{z_i}\frac{1}{\sqrt{a_i(s)}}\mathrm{d}s, \quad z_i\in[0,\infty).$$

By **(H1)**, each ξ_i is increasing and onto, and thus, ξ_i^{-1} is well-defined. Set

$$\xi := (\xi_i) : \overline{\mathcal{U}} \to \overline{\mathcal{U}} \quad \text{and} \quad \xi^{-1} := (\xi_i^{-1}) : \overline{\mathcal{U}} \to \overline{\mathcal{U}}.$$

Clearly, $\xi : \overline{\mathcal{U}} \to \overline{\mathcal{U}}$ is a homeomorphism with inverse ξ^{-1}, and satisfies $\xi(\Gamma) = \Gamma$ and $\xi(\mathcal{U}) = \mathcal{U}$.

Define a new process $X_t = (X_t^i)$ by setting

$$X_t^i := \xi_i(Z_t^i), \quad i \in \{1, \dots, d\}, \quad \text{or simply,} \quad X_t = \xi(Z_t), \quad t \geq 0.$$

It is clear that Γ is also an absorbing set for the process X_t, and X_t reaches Γ in finite time almost surely. Moreover, QSDs of Z_t and X_t are in an one-to-one correspondence as shown in the next result whose proof is straightforward.

Proposition 2.3. *Let μ be a Borel probability measure on $\mathcal{U}$. Then, μ is a QSD of Z_t if and only if $\xi_*\mu$ is a QSD of X_t, where ξ_* is the pushforward operator induced by ξ. Moreover, μ and $\xi_*\mu$ have the same extinction rates.*

Itô's formula gives

$$\mathrm{d}X_t^i = \big[p_i(X_t) - q_i(X_t^i)\big]\,\mathrm{d}t + \mathrm{d}W_t^i, \quad i \in \{1, \dots, d\} \quad \text{in} \quad \mathcal{U}, \tag{2.2}$$

where $p_i : \mathcal{U} \to \mathbb{R}$ and $q_i : (0, \infty) \to \mathbb{R}$ are given by

$$p_i(x) := \frac{b_i(\xi^{-1}(x))}{\sqrt{a_i(\xi_i^{-1}(x_i))}} \quad \text{and} \quad q_i(x_i) := \frac{a_i'(\xi_i^{-1}(x_i))}{4\sqrt{a_i(\xi_i^{-1}(x_i))}}, \quad x = (x_i) \in \mathcal{U}.$$

Denote by $\mathcal{L}_{\mathbf{FP}}^X$ the Fokker-Planck operator associated to (2.2), namely,

$$\mathcal{L}_{\mathbf{FP}}^X v = \frac{1}{2}\Delta v - \nabla \cdot ((p - q)v) \quad \text{in} \quad \mathcal{U},$$

where $p = (p_i)$ and $q = (q_i)$. Then, Proposition 2.2 has a counterpart for QSDs of X_t.

Proposition 2.4. *Assume* **(H1)**-**(H2)***. Let ν be a QSD of X_t with extinction rate λ_1. Then, ν admits a positive density $v \in C^2(\mathcal{U})$ that satisfies $-\mathcal{L}_{\mathbf{FP}}^X v = \lambda_1 v$ in $\mathcal{U}$.*

Remark 2.2. *Note that the process X_t and the process generated by solutions of (2.2) are not really the same, as (2.2) is only defined in $\mathcal{U}$. However, the two processes agree as long as X_t stays in $\mathcal{U}$. More precisely, if we denote by S_Γ the first time that X_t reaches Γ, that is, $S_\Gamma = \inf\{t \geq 0 : X_t \in \Gamma\}$, then X_t satisfies (2.2) on the event $[t < S_\Gamma]$.*

As indicated by Proposition 2.4, QSDs of X_t are closely related to positive eigenfunctions of $-\mathcal{L}_{\mathbf{FP}}^X$, and therefore, it is natural to investigate the associated eigenvalue problem, namely,

$$-\mathcal{L}_{\mathbf{FP}}^X v = \lambda v \quad \text{in} \quad \mathcal{U}. \tag{2.3}$$

Note that the operator $\mathcal{L}_{\mathbf{FP}}^X$ is uniformly elliptic in $\mathcal{U}$, but the functions q_i, $i \in \{1, \dots, d\}$ appearing in its first-order terms satisfy $q_i(x_i) \to \infty$ as $x_i \to 0^+$ for each $i \in \{1, \dots, d\}$. Such blow-up singularities make the investigation of the above eigenvalue problem very hard. In the following, we transform (2.3) into the eigenvalue problem of another elliptic operator that has blow-up singularities only in the zeroth-order term and thus is easier to deal with.

Set

$$U := V \circ \xi^{-1} \quad \text{in} \quad \mathcal{U}, \tag{2.4}$$

where V is given in **(H3)**, and

$$Q(x) := \sum_{i=1}^{d} \int_1^{x_i} 2q_i(s)ds = \frac{1}{2}\sum_{i=1}^{d}\left[\ln a_i(\xi_i^{-1}(x_i)) - \ln a_i(\xi_i^{-1}(1))\right], \quad x \in \mathcal{U}. \tag{2.5}$$

For each $\beta > 0$, we use the Liouville-type transform to define the differential operator

$$\mathcal{L}_\beta := e^{\frac{Q}{2}+\beta U}\mathcal{L}_{\mathbf{FP}}^X e^{-\frac{Q}{2}-\beta U}.$$

It is straightforward to check that

$$\mathcal{L}_\beta = \frac{1}{2}\Delta - (p + \beta\nabla U)\cdot\nabla - e_\beta \quad \text{in} \quad \mathcal{U}, \tag{2.6}$$

where

$$e_\beta = \frac{1}{2}\left(\beta\Delta U - \beta^2|\nabla U|^2\right) - \beta p\cdot\nabla U + \frac{1}{2}\sum_{i=1}^{d}(q_i^2 - q_i') - p\cdot q + \nabla\cdot p. \tag{2.7}$$

Note that the coefficient of the first-order term $-(p + \beta\nabla U)$ behaves nicely near Γ, and the term $\frac{1}{2}\sum_{i=1}^{d}(q_i^2 - q_i') - p\cdot q$ blows up at the boundary Γ, but it appears in the zeroth-order term of $\mathcal{L}_\beta$.

The following proposition establishes the "equivalence" between the eigenvalue problem (2.3) and the eigenvalue problem associated to the operator $\mathcal{L}_\beta$.

Proposition 2.5. *Suppose $v \in W_{loc}^{2,1}(\mathcal{U})$ and $\lambda \in \mathbb{R}$. Set $\tilde{v} := ve^{\frac{Q}{2}+\beta U}$. Then, (v, λ) solves (2.3) if and only if $-\mathcal{L}_\beta\tilde{v} = \lambda\tilde{v}$ in $\mathcal{U}$.*

According to Proposition 2.5, the investigation of QSDs of X_t is reduced to the exploration of the principal spectral theory of $-\mathcal{L}_\beta$ (with a fixed β), something which we will do by choosing an appropriate function space.

2.3. **Approximation by first exit times.** Let $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ be a sequence of arbitrarily fixed bounded, connected and open sets in $\mathcal{U}$ with C^2 boundaries that satisfy

$$\mathcal{U}_n \subset\subset \mathcal{U}_{n+1} \subset\subset \mathcal{U}, \quad \forall n \in \mathbb{N} \quad \text{and} \quad \mathcal{U} = \bigcup_{n\in\mathbb{N}} \mathcal{U}_n.$$

For each $n \in \mathbb{N}$, denote by τ_n the first time that X_t exits $\mathcal{U}_n$, namely,

$$\tau_n = \inf\{t \geq 0 : X_t \notin \mathcal{U}_n\}.$$

Recall that S_Γ is the first time that X_t hits Γ. The following result turns out to be useful.

Lemma 2.1. *Assume* **(H1)**-**(H3)***. For each* $x \in \mathcal{U}$*, one has* $\mathbb{P}^x[\lim_{n\to\infty} \tau_n = S_\Gamma] = 1$ *and*

$$\lim_{n\to\infty} \mathbb{E}^x\left[f(X_t)\mathbb{1}_{\{t<\tau_n\}}\right] = \mathbb{E}^x\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right], \quad \forall f \in C_b(\mathcal{U}).$$

Proof. Fix $x \in \mathcal{U}$. Obviously, $\tau_n < \tau_{n+1}$ for each $n \in \mathbb{N}$. Set $\tau := \lim_{n\to\infty} \tau_n$. The first conclusion follows if we show $\mathbb{P}^x[\tau = S_\Gamma] = 1$.

Clearly, $\tau_n < S_\Gamma$ for each $n \in \mathbb{N}$, leading to $\tau \leq S_\Gamma$. Since $X_t = \xi(Z_t)$ for $t \geq 0$, we find from Proposition 2.1 that $\mathbb{P}^x[S_\Gamma < \infty] = \mathbb{P}^{\xi^{-1}(x)}[T_\Gamma < \infty] = 1$. Therefore, $\mathbb{P}^x[\tau < \infty] = 1$.

Noting that arguments in the proof of Proposition 2.1 ensure that Z_t and X_t do not explode in finite time, we derive $|X_\tau| = \lim_{n\to\infty} |X_{\tau_n}| < \infty$. Moreover, since $X_{\tau_n} \in \partial\mathcal{U}_n$ and $\mathcal{U} = \cup_{n\in\mathbb{N}}\mathcal{U}_n$, it follows that $X_\tau \in \Gamma$. As S_Γ is the first hitting time of the boundary Γ and $\tau \leq S_\Gamma$, one has $\tau = S_\Gamma$.

Since τ_n increases to S_Γ $\mathbb{P}$-a.s., we find $\lim_{n\to\infty} \mathbb{1}_{\{t<\tau_n\}} = \mathbb{1}_{\{t<S_\Gamma\}}$ for each $t \geq 0$. The second conclusion then follows from the dominated convergence theorem. This completes the proof. □

3. Spectral theory and semigroup

In this section we study for some appropriately fixed β the spectral theory of $-\mathcal{L}_\beta$ in an appropriate function space. We also analyze the semigroup generated by $\mathcal{L}_\beta$. In Subsection 3.1 we define a weighted Hilbert space. In Subsection 3.2 we derive some important estimates and meanwhile fix a special β, denoted by β_0. In Subsection 3.3 we study the (principal) spectral theory of $-\mathcal{L}_{\beta_0}$ and the semigroup generated by $\mathcal{L}_{\beta_0}$. In Subsection 3.4 the spectral theory of $-\mathcal{L}^*_{\beta_0}$, where $\mathcal{L}^*_{\beta_0}$ is the adjoint operator of $\mathcal{L}_{\beta_0}$, and the semigroup generated by $\mathcal{L}^*_{\beta_0}$ are investigated.

3.1. **A weighted Hilbert space.** For $\delta \in (0,1)$, let

$$\Gamma_\delta := \{x=(x_i)\in \mathcal{U} : x_i \le \delta \text{ for some } i\in\{1,\dots,d\}\}.$$

It is easy to see from **(H3)**(1) that there exists $R_0 \gg 1$ such that $-(b\cdot\nabla_z V)\circ\xi^{-1} > 0$ in $\mathcal{U}\setminus B^+_{R_0}$, where we recall $B^+_R = \{x=(x_i)\in\mathcal{U} : x_i\in(0,R),\ \forall i\in\{1,\dots,d\}\}$ for $R>0$. Fix some $\delta_0\in(0,1)$. Let $\alpha:\mathcal{U}\to\mathbb{R}$ be defined by

$$\alpha(x) := \begin{cases} \sum_{i=1}^d \max\left\{\frac{1}{x_i^2},1\right\}, & x\in\Gamma_{\delta_0}\bigcap B^+_{R_0},\\ \sum_{i=1}^d \max\left\{\frac{1}{x_i^2},1\right\} - (b\cdot\nabla_z V)(\xi^{-1}(x)), & x\in\Gamma_{\delta_0}\bigcap(\mathcal{U}\setminus B^+_{R_0}),\\ -(b\cdot\nabla_z V)(\xi^{-1}(x)), & x\in(\mathcal{U}\setminus\Gamma_{\delta_0})\bigcap(\mathcal{U}\setminus B^+_{R_0}),\\ 1, & \text{otherwise}. \end{cases} \tag{3.1}$$

Obviously, $\inf_{\mathcal{U}}\alpha>0$. This α is defined according to the behavior of the coefficients of $-\mathcal{L}_\beta$ near Γ and ∞. Its significance is partially reflected in Lemma 3.2 below. See Remark 3.1 after Lemma 3.2 for more comments.

Denote by $\mathcal{H}^1(\mathcal{U};\mathbb{C})$ the space of all weakly differentiable complex-valued functions $\phi:\mathcal{U}\to\mathbb{C}$ satisfying

$$\|\phi\|_{\mathcal{H}^1} := \left(\int_{\mathcal{U}}\alpha|\phi|^2\mathrm{d}x\right)^{\frac12} + \left(\int_{\mathcal{U}}|\nabla\phi|^2\mathrm{d}x\right)^{\frac12} < \infty.$$

It is not hard to verify that $\mathcal{H}^1(\mathcal{U};\mathbb{C})$ is a Hilbert space with the inner product:

$$\langle\phi,\psi\rangle_{\mathcal{H}^1} := \int_{\mathcal{U}}\alpha\phi\overline{\psi}\mathrm{d}x + \int_{\mathcal{U}}\nabla\phi\cdot\nabla\overline{\psi}\mathrm{d}x, \quad \forall\phi \text{ and } \psi\in\mathcal{H}^1(\mathcal{U};\mathbb{C}),$$

where $\overline{\psi}$ denotes the complex conjugate of ψ.

Lemma 3.1. *Assume* **(H3)**. *Then, $\mathcal{H}^1(\mathcal{U};\mathbb{C})$ is compactly embedded into $L^2(\mathcal{U};\mathbb{C})$.*

Proof. Let $\{\phi_n\}_{n\in\mathbb{N}}\subset\mathcal{H}^1(\mathcal{U};\mathbb{C})$ satisfy $\sup_{n\in\mathbb{N}}\|\phi_n\|_{\mathcal{H}^1}\le 1$. Fix $R>0$. Since $H^1(B^+_R;\mathbb{C})$ is compactly embedded into $L^2(B^+_R;\mathbb{C})$, there is a subsequence, still denoted by $\{\phi_n\}_{n\in\mathbb{N}}$, and a measurable function $\phi_R\in L^2(B^+_R;\mathbb{C})$, such that $\phi_n(x)\to\phi_R(x)$ for a.e. $x\in B^+_R$ and $\lim_{n\to\infty}\int_{B^+_R}|\phi_n-\phi_R|^2\mathrm{d}x=0$.

Let $\{R_m\}_m\subset(0,\infty)$ satisfy $R_m\to\infty$ as $m\to\infty$. Then, the above results hold for each R_m in place of R. We apply the standard diagonal argument to find a subsequence, still denoted by $\{\phi_n\}_{n\in\mathbb{N}}$, and a measurable function $\phi:\mathcal{U}\to\mathbb{C}$ such that $\phi_n\to\phi$ a.e. in $\mathcal{U}$ as $n\to\infty$, and

$$\lim_{n\to\infty}\int_{B^+_R}|\phi_n-\phi|^2\mathrm{d}x=0,\quad\forall R>0. \tag{3.2}$$

Applying Fatou's lemma, we find $\int_{\mathcal{U}} \alpha\phi^2 \mathrm{d}x \le \liminf_{n\to\infty} \int_{\mathcal{U}} \alpha\phi_n^2 \mathrm{d}x \le 1$. It follows from (3.2) that

$$\limsup_{n\to\infty} \int_{\mathcal{U}} |\phi_n - \phi|^2 \mathrm{d}x \le \limsup_{n\to\infty} \int_{\mathcal{U}\setminus B_R^+} |\phi_n - \phi|^2 \mathrm{d}x, \quad \forall R > 0.$$

Note that

$$\int_{\mathcal{U}\setminus B_R^+} |\phi_n - \phi|^2 \mathrm{d}x \le \frac{2}{\inf_{\mathcal{U}\setminus B_R^+} \alpha} \int_{\mathcal{U}\setminus B_R^+} \alpha \left(\phi_n^2 + \phi^2\right) \mathrm{d}x \le \frac{2}{\inf_{\mathcal{U}\setminus B_R^+} \alpha},$$

which together with the fact $\alpha(x) \to \infty$ as $|x| \to \infty$ yields $\limsup_{n\to\infty} \int_{\mathcal{U}\setminus B_R^+} |\phi_n - \phi|^2 \mathrm{d}x = 0$, and hence, $\lim_{n\to\infty} \int_{\mathcal{U}} |\phi_n - \phi|^2 \mathrm{d}x = 0$. This completes the proof. □

3.2. **Some estimates.** We recall from (2.7) the definition of e_β and define for $N \ge 1$,

$$\begin{aligned} e_{\beta,N} :&= e_\beta - \frac{N-1}{N}(\nabla\cdot p + \beta\Delta U) \\ &= \left(\frac{1}{N} - \frac{1}{2}\right)\beta\Delta U - \frac{\beta^2}{2}|\nabla U|^2 - \beta p\cdot\nabla U + \frac{1}{2}\sum_{i=1}^d (q_i^2 - q_i') - p\cdot q + \frac{\nabla\cdot p}{N}. \end{aligned} \tag{3.3}$$

Obviously, $e_{\beta,1} = e_\beta$ for all $\beta > 0$. The main reason for introducing $e_{\beta,N}$ is that they arise naturally in deriving a priori estimates for both sesquilinear forms and partial differential equations related to $\mathcal{L}_\beta$ or its adjoint (see Lemma 3.3 and Lemma 4.1).

Lemma 3.2. *Assume* **(H1)-(H3)**. *Then, the following hold.*

(1) *There exists $C > 0$ such that*

$$|\nabla U|^2 + |p|^2 \le C\alpha \quad \textit{in} \quad \mathcal{U},$$

where α is defined in (3.1).

(2) *For each $\beta > 0$, there exists $C(\beta) > 0$ such that*

$$|e_{\beta,N}| \le C(\beta)\alpha \quad \textit{in} \quad \mathcal{U}, \quad \forall N \ge 1.$$

(3) *There exist positive constants β_0, M and C_* such that,*

$$e_{\beta_0,N} + M \ge C_*\alpha \quad \textit{in} \quad \mathcal{U}, \quad \forall N \ge 1.$$

Since the proof of this lemma is long and relatively independent, we postpone it to Appendix A.1 for the sake of readability.

Remark 3.1. *Note that $e_{\beta,1} = e_\beta$ is the zeroth-order term of the operator $\mathcal{L}_\beta$ (see* (2.6)*) that has blow-up singularities at Γ as mentioned earlier. Lemma 3.2 says in particular that e_β is well-controlled by the weight function α, laying the foundation for our analysis.*

In what follows, the positive constants β_0, M and C_* are fixed such that the conclusion in Lemma 3.2 (3) holds.

3.3. **Spectrum and semigroup.** We investigate the spectral theory of $-\mathcal{L}_{\beta_0}$ and the semigroup generated by $\mathcal{L}_{\beta_0}$. Corresponding results are stated in Theorem 3.1 and Theorem 3.2.

Denote by $\mathcal{E}_{\beta_0} : \mathcal{H}^1(\mathcal{U};\mathbb{C}) \times \mathcal{H}^1(\mathcal{U};\mathbb{C}) \to \mathbb{C}$ the sesquilinear form associated to $-\mathcal{L}_{\beta_0}$, namely,

$$\mathcal{E}_{\beta_0}(\phi,\psi) = \frac{1}{2}\int_{\mathcal{U}} \nabla\phi\cdot\nabla\overline{\psi}\mathrm{d}x + \int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot\nabla\phi\overline{\psi}\mathrm{d}x + \int_{\mathcal{U}} e_{\beta_0}\phi\overline{\psi}\mathrm{d}x, \quad \forall\phi,\psi\in\mathcal{H}^1(\mathcal{U};\mathbb{C}).$$

The following lemma addresses the boundedness and "coercivity" of $\mathcal{E}_{\beta_0}$.

Lemma 3.3. *Assume* **(H1)**-**(H3)**.

1. *There exists $C>0$ such that $|\mathcal{E}_{\beta_0}(\phi,\psi)| \le C\|\phi\|_{\mathcal{H}^1}\|\psi\|_{\mathcal{H}^1}$ for all $\phi,\psi\in\mathcal{H}^1(\mathcal{U};\mathbb{C})$.*
2. *For each $\phi=\phi_1+i\phi_2\in\mathcal{H}^1(\mathcal{U};\mathbb{C})$, we have*

$$\mathcal{E}_{\beta_0}(\phi,\phi) = \frac{1}{2}\int_{\mathcal{U}}|\nabla\phi|^2\mathrm{d}x + \int_{\mathcal{U}} e_{\beta_0,2}|\phi|^2\mathrm{d}x + i\int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot(\phi_1\nabla\phi_2-\phi_2\nabla\phi_1)\mathrm{d}x,$$

where $e_{\beta_0,2}$ is defined in (3.3). In particular,

$$\Re\mathcal{E}_{\beta_0}(\phi,\phi) + M\|\phi\|_{L^2}^2 \ge \min\left\{\frac{1}{2}, C_*\right\}\|\phi\|_{\mathcal{H}^1}^2, \quad \forall\phi\in\mathcal{H}^1(\mathcal{U};\mathbb{C}).$$

Proof. (1) Let $\phi,\psi\in\mathcal{H}^1(\mathcal{U};\mathbb{C})$. Applying Hölder's inequality, we derive

$$\begin{aligned}|\mathcal{E}_{\beta_0}(\phi,\psi)| \le{}& \frac{1}{2}\left(\int_{\mathcal{U}}|\nabla\phi|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathcal{U}}|\nabla\psi|^2\mathrm{d}x\right)^{\frac{1}{2}}\\ &+\left(\int_{\mathcal{U}}|\nabla\phi|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathcal{U}}|p+\beta_0\nabla U|^2|\psi|^2\mathrm{d}x\right)^{\frac{1}{2}}\\ &+\left(\int_{\mathcal{U}}|e_{\beta_0}||\phi|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathcal{U}}|e_{\beta_0}||\psi|^2\mathrm{d}x\right)^{\frac{1}{2}}.\end{aligned}$$

Since Lemma 3.2 (1) ensures the existence of $C>0$ such that

$$\int_{\mathcal{U}}|p+\beta_0\nabla U|^2|\psi|^2\mathrm{d}x \le C(1+\beta_0^2)\int_{\mathcal{U}}\alpha|\psi|^2\mathrm{d}x,$$

the conclusion follows readily from Lemma 3.2 (2) and the definition of the norm $\|\cdot\|_{\mathcal{H}^1}$.

(2) Let $\{\eta_n\}_{n\in\mathbb{N}}$ be a sequence of smooth functions on $\mathcal{U}$ taking values in $[0,1]$ and satisfying

$$\eta_n(x) = \begin{cases}1, & x\in\left(\mathcal{U}\setminus\Gamma_{\frac{2}{n}}\right)\bigcap B^+_{\frac{n}{2}},\\ 0, & x\in\Gamma_{\frac{1}{n}}\bigcup(\mathcal{U}\setminus B^+_n),\end{cases} \quad \text{and} \quad |\nabla\eta_n(x)| \le \begin{cases}2n, & x\in\Gamma_{\frac{2}{n}}\setminus\Gamma_{\frac{1}{n}},\\ 4, & x\in\left(\mathcal{U}\setminus\Gamma_{\frac{2}{n}}\right)\bigcap\left(B^+_n\setminus B^+_{\frac{n}{2}}\right).\end{cases}$$

Fix $\phi \in \mathcal{H}^1(\mathcal{U};\mathbb{R})$. We calculate

$$\begin{aligned}\mathcal{E}_{\beta_0}(\phi,\eta_n^2\phi) &= \frac{1}{2}\int_{\mathcal{U}}\nabla\phi\cdot\nabla(\eta_n^2\overline{\phi})\mathrm{d}x + \int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot\nabla\phi(\eta_n^2\overline{\phi})\mathrm{d}x + \int_{\mathcal{U}} e_{\beta_0}\eta_n^2|\phi|^2\mathrm{d}x\\ &= \frac{1}{2}\int_{\mathcal{U}}\eta_n^2|\nabla\phi|^2\mathrm{d}x + \int_{\mathcal{U}}\eta_n\overline{\phi}\nabla\phi\cdot\nabla\eta_n\mathrm{d}x\\ &\quad + \int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot\nabla\phi(\eta_n^2\overline{\phi})\mathrm{d}x + \int_{\mathcal{U}} e_{\beta_0}\eta_n^2|\phi|^2\mathrm{d}x\\ &=: I_1(n)+I_2(n)+I_3(n)+I_4(n).\end{aligned} \tag{3.4}$$

We find $\lim_{n\to\infty} I_1(n) = \frac{1}{2}\int_{\mathcal{U}}|\nabla\phi|^2\mathrm{d}x$ from $\int_{\mathcal{U}}|\nabla\phi|^2\mathrm{d}x < \infty$ and the dominated convergence theorem.

Applying Hölder's inequality, we find $|I_2(n)| \le \left(\int_{\mathcal{U}}\eta_n^2|\nabla\phi|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathcal{U}}|\nabla\eta_n|^2|\phi|^2\mathrm{d}x\right)^{\frac{1}{2}}$. From the construction of η_n, we see that

$$|\nabla\eta_n|^2|\phi|^2 \le \begin{cases} 4n^2|\phi|^2 & \text{in } \Gamma_{\frac{2}{n}}\setminus\Gamma_{\frac{1}{n}},\\ 16|\phi|^2 & \text{in } \left(\mathcal{U}\setminus\Gamma_{\frac{2}{n}}\right)\bigcap\left(B_n^+\setminus B_{\frac{n}{2}}^+\right),\\ 0 & \text{otherwise.}\end{cases}$$

Since $n^2 \le \sum_{i=1}^d \max\left\{\frac{1}{x_i^2},1\right\}$ in $\Gamma_{\frac{2}{n}}\setminus\Gamma_{\frac{1}{n}}$ for $n\ge 1$, the definition of α yields the existence of $C_1>0$ such that $|\nabla\eta_n|^2|\phi|^2 \le C_1\alpha|\phi|^2$ in $\mathcal{U}$ for all $n\gg 1$. Since $\lim_{n\to\infty}|\nabla\eta_n| = 0$, we apply the dominated convergence theorem to conclude

$$\lim_{n\to\infty}\int_{\mathcal{U}}|\nabla\eta_n|^2|\phi|^2\mathrm{d}x = 0, \tag{3.5}$$

which leads to $\lim_{n\to\infty} I_2(n) = 0$.

Denote $\phi = \phi_1 + i\phi_2$. We calculate for each $j\in\{1,\dots,d\}$,

$$(\partial_j\phi)\overline{\phi} = (\partial_j\phi_1 + i\partial_j\phi_2)(\phi_1 - i\phi_2) = \frac{1}{2}\partial_j|\phi|^2 + i(\phi_1\partial_j\phi_2 - \phi_2\partial_j\phi_1).$$

Integration by parts yields

$$\begin{aligned}I_3(n) &= \frac{1}{2}\int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot\eta_n^2\nabla|\phi|^2\mathrm{d}x + i\int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot\eta_n^2(\phi_1\nabla\phi_2-\phi_2\nabla\phi_1)\mathrm{d}x\\ &= \frac{1}{2}\int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot\nabla(\eta_n^2|\phi|^2)\mathrm{d}x - \int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot\nabla\eta_n(\eta_n|\phi|^2)\mathrm{d}x\\ &\quad + i\int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot\eta_n^2(\phi_1\nabla\phi_2-\phi_2\nabla\phi_1)\mathrm{d}x\\ &= -\frac{1}{2}\int_{\mathcal{U}}(\nabla\cdot p+\beta_0\Delta U)\eta_n^2|\phi|^2\mathrm{d}x - \int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot\nabla\eta_n(\eta_n|\phi|^2)\mathrm{d}x\\ &\quad + i\int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot\eta_n^2(\phi_1\nabla\phi_2-\phi_2\nabla\phi_1)\mathrm{d}x.\end{aligned}$$

It follows that

$$\begin{aligned}I_3(n)+I_4(n) &= -\frac{1}{2}\int_{\mathcal{U}}(\nabla\cdot p+\beta_0\Delta U)\eta_n^2|\phi|^2\mathrm{d}x - \int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot\nabla\eta_n(\eta_n|\phi|^2)\mathrm{d}x\\ &\quad + \int_{\mathcal{U}}e_{\beta_0}\eta_n^2|\phi|^2\mathrm{d}x + i\int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot\eta_n^2(\phi_1\nabla\phi_2-\phi_2\nabla\phi_1)\mathrm{d}x\\ &= -\int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot\nabla\eta_n(\eta_n|\phi|^2)\mathrm{d}x + \int_{\mathcal{U}}e_{\beta_0,2}\eta_n^2|\phi|^2\mathrm{d}x\\ &\quad + i\int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot\eta_n^2(\phi_1\nabla\phi_2-\phi_2\nabla\phi_1)\mathrm{d}x.\end{aligned} \tag{3.6}$$

We apply Hölder's inequality to find

$$\begin{aligned}\left|\int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot\nabla\eta_n(\eta_n|\phi|^2)\mathrm{d}x\right| &\leq \left(\int_{\mathcal{U}}|p+\beta_0\nabla U|^2\eta_n^2|\phi|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathcal{U}}|\nabla\eta_n|^2|\phi|^2\mathrm{d}x\right)^{\frac{1}{2}}\\ &\leq \left(\int_{\mathcal{U}}|p+\beta_0\nabla U|^2|\phi|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathcal{U}}|\nabla\eta_n|^2|\phi|^2\mathrm{d}x\right)^{\frac{1}{2}}.\end{aligned}$$

Note that Lemma 3.2 (1) gives $\int_{\mathcal{U}}|p+\beta_0\nabla U|^2|\phi|^2\mathrm{d}x \leq C_2\int_{\mathcal{U}}\alpha|\phi|^2\mathrm{d}x$ for some $C_2>0$, which together with (3.5), yields

$$\lim_{n\to\infty}\int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot\nabla\eta_n(\eta_n|\phi|^2)\mathrm{d}x = 0. \tag{3.7}$$

It follows from Lemma 3.2 (2) that $|e_{\beta_0,2}|\eta_n^2|\phi|^2 \leq C_3\alpha|\phi|^2$ for some $C_3>0$. Together with the fact $\phi\in\mathcal{H}^1(\mathcal{U};\mathbb{R})$ and the dominated convergence theorem this yields

$$\lim_{n\to\infty}\int_{\mathcal{U}}e_{\beta_0,2}\eta_n^2|\phi|^2\mathrm{d}x = \int_{\mathcal{U}}e_{\beta_0,2}|\phi|^2\mathrm{d}x. \tag{3.8}$$

Since Young's inequality gives

$$\begin{aligned}\left|(p+\beta_0\nabla U)\cdot\eta_n^2(\phi_1\nabla\phi_2-\phi_2\nabla\phi_1)\right| &\leq \frac{1}{2}\eta_n^2|\nabla\phi|^2+\frac{1}{2}|p+\beta_0\nabla U|^2\eta_n^2|\phi|^2\\ &\leq \frac{1}{2}|\nabla\phi|^2+\frac{1}{2}|p+\beta_0\nabla U|^2|\phi|^2,\end{aligned}$$

we deduce from the dominated convergence theorem that

$$\lim_{n\to\infty}\int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot\eta_n^2(\phi_1\nabla\phi_2-\phi_2\nabla\phi_1)\mathrm{d}x=\int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot(\phi_1\nabla\phi_2-\phi_2\nabla\phi_1)\mathrm{d}x.$$

Letting $n\to\infty$ in (3.6), we conclude from (3.7), (3.8) and the above equality that

$$\lim_{n\to\infty}[I_3(n)+I_4(n)]=\int_{\mathcal{U}}e_{\beta_0,2}\phi^2\mathrm{d}x+i\int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot(\phi_1\nabla\phi_2-\phi_2\nabla\phi_1)\mathrm{d}x.$$

Passing to the limit $n\to\infty$ in (3.4), we derive the expected identity from the limits of $I_1(n)$, $I_2(n)$, $I_3(n)$ and $I_4(n)$ as $n\to\infty$.

The last part of claim (2) is an immediate consequence of Lemma 3.2 (3). □

For $f\in L^2(\mathcal{U};\mathbb{C})$, we consider the following problem:

$$(-\mathcal{L}_{\beta_0}+M)u=f\quad\text{in}\quad\mathcal{U},\tag{3.9}$$

and look for solutions in $\mathcal{H}^1(\mathcal{U};\mathbb{C})$.

Definition 3.1. A function $u\in\mathcal{H}^1(\mathcal{U};\mathbb{C})$ is called a *weak solution* of (3.9) if

$$\mathcal{E}_{\beta_0}(u,\phi)+M\langle u,\phi\rangle_{L^2}=\langle f,\phi\rangle_{L^2},\quad\forall\phi\in\mathcal{H}^1(\mathcal{U};\mathbb{C}),$$

where $\langle\cdot,\cdot\rangle_{L^2}$ is the usual inner product on $L^2(\mathcal{U};\mathbb{C})$.

Lemma 3.4. *Assume* **(H1)-(H3)**. *Then, for any $f\in L^2(\mathcal{U};\mathbb{C})$, (3.9) admits a unique weak solution u_f in $\mathcal{H}^1(\mathcal{U};\mathbb{C})$. Moreover, the following hold.*

1. *There is a constant $C>0$ such that $\|u_f\|_{\mathcal{H}^1}\leq C\|f\|_{L^2}$ for all $f\in L^2(\mathcal{U};\mathbb{C})$.*
2. *We have $u_f\in H^2_{loc}(\mathcal{U};\mathbb{C})$ and*

$$(-\mathcal{L}_{\beta_0}+M)u_f=f\quad\textit{a.e. in}\quad\mathcal{U},$$

and

$$\mathcal{E}_{\beta_0}(u_f,\phi)=\langle-\mathcal{L}_{\beta_0}u_f,\phi\rangle_{L^2},\quad\forall\phi\in\mathcal{H}^1(\mathcal{U};\mathbb{C}).$$

3. *If $f\in L^2(\mathcal{U};\mathbb{C})$ satisfies $f\geq 0$ a.e. in $\mathcal{U}$, then $u_f\geq 0$ a.e. in $\mathcal{U}$. If in addition $f>0$ on a set of positive Lebesgue measure, then $u_f>0$ in $\mathcal{U}$.*

Proof. Fix $f \in L^2(\mathcal{U};\mathbb{C})$. Hölder's inequality gives

$$|\langle f, \phi\rangle_{L^2}| \leq \left(\int_{\mathcal{U}} \frac{1}{\alpha}|f|^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathcal{U}} \alpha|\phi|^2 \mathrm{d}x\right)^{\frac{1}{2}} \leq \frac{1}{(\inf_{\mathcal{U}} \alpha)^{\frac{1}{2}}} \|f\|_{L^2}\|\phi\|_{\mathcal{H}^1}, \quad \forall \phi \in \mathcal{H}^1(\mathcal{U};\mathbb{C}). \tag{3.10}$$

Hence, $\phi \mapsto \langle f, \phi\rangle_{L^2} : \mathcal{H}^1(\mathcal{U};\mathbb{C}) \to \mathbb{C}$ is a continuous linear functional.

By Lemma 3.3 and the fact $\|\phi\|_{L^2} \leq (\inf_{\mathcal{U}} \alpha)^{-\frac{1}{2}} \|\phi\|_{\mathcal{H}^1}$ for $\phi \in \mathcal{H}^1(\mathcal{U};\mathbb{C})$ one has

$$|\mathcal{E}_{\beta_0}(\phi,\psi)| + M|\langle \phi, \psi\rangle_{L^2}| \leq C_1 \|\phi\|_{\mathcal{H}^1}\|\psi\|_{\mathcal{H}^1}, \quad \forall \phi \text{ and } \psi \in \mathcal{H}^1(\mathcal{U};\mathbb{C})$$

for some $C_1 > 0$, and

$$\Re\mathcal{E}_{\beta_0}(\phi,\phi) + M\|\phi\|^2_{L^2} \geq \min\left\{\frac{1}{2}, C_*\right\}\|\phi\|^2_{\mathcal{H}^1}, \quad \forall \phi \in \mathcal{H}^1(\mathcal{U};\mathbb{C}). \tag{3.11}$$

We apply the Lax-Milgram theorem (see e.g. [23]) to find a unique $u_f \in \mathcal{H}^1(\mathcal{U};\mathbb{C})$ such that

$$\mathcal{E}_{\beta_0}(u_f,\phi) + M\langle u_f, \phi\rangle_{L^2} = \langle f, \phi\rangle_{L^2}, \quad \forall \phi \in \mathcal{H}^1(\mathcal{U};\mathbb{C}). \tag{3.12}$$

This shows that u_f is the unique weak solution of (3.9).

(1) Setting $\phi = u$ in (3.12), we derive from (3.10) and (3.11) that

$$\min\left\{\frac{1}{2}, C_*\right\}\|u_f\|^2_{\mathcal{H}^1} \leq \Re\mathcal{E}_{\beta_0}(u_f,u_f) + M\|u_f\|^2_{L^2} \leq \frac{1}{(\inf_{\mathcal{U}} \alpha)^{\frac{1}{2}}}\|f\|_{L^2}\|u_f\|_{\mathcal{H}^1}.$$

(2) The classical regularity theory of elliptic equations ensures $u_f \in H^2_{loc}(\mathcal{U};\mathbb{C})$. Hence, u_f is a strong solution.

Fix $\phi \in \mathcal{H}^1(\mathcal{U};\mathbb{C})$. Let $\{\eta_n\}_{n\in\mathbb{N}}$ be the sequence of non-negative functions constructed in the proof of Lemma 3.3. Then, $0 \leq \eta_n \leq 1$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} \eta_n = 1$ in $\mathcal{U}$. Integration by parts gives

$$\begin{aligned}\langle -\mathcal{L}_{\beta_0}u_f, \eta_n^2\phi\rangle_{L^2} &= \frac{1}{2}\int_{\mathcal{U}} \nabla u_f \cdot \nabla(\eta_n^2\overline{\phi})\mathrm{d}x + \int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot\nabla u_f(\eta_n^2\overline{\phi})\mathrm{d}x + \int_{\mathcal{U}} e_{\beta_0}\eta_n^2 u_f\overline{\phi}\mathrm{d}x \\ &= \frac{1}{2}\int_{\mathcal{U}} \eta_n^2\nabla u_f\cdot\nabla\overline{\phi}\mathrm{d}x + \int_{\mathcal{U}} \eta_n\overline{\phi}\nabla u_f\cdot\nabla\eta_n\mathrm{d}x \\ &\quad + \int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot\nabla u_f(\eta_n^2\overline{\phi})\mathrm{d}x + \int_{\mathcal{U}} e_{\beta_0}\eta_n^2 u_f\overline{\phi}\mathrm{d}x.\end{aligned} \tag{3.13}$$

As $\mathcal{L}_{\beta_0}u_f \in L^2(\mathcal{U};\mathbb{C})$ and $\phi \in \mathcal{H}^1(\mathcal{U};\mathbb{C}) \subset L^2(\mathcal{U};\mathbb{C})$, we derive from the dominated convergence theorem that $\lim_{n\to\infty}\langle -\mathcal{L}_{\beta_0}u_f, \eta_n^2\phi\rangle_{L^2} = \langle -\mathcal{L}_{\beta_0}u_f, \phi\rangle_{L^2}$. Since $\nabla u_f, \nabla\overline{\phi} \in L^2(\mathcal{U};\mathbb{C})$, the dominated convergence theorem ensures that $\lim_{n\to\infty}\frac{1}{2}\int_{\mathcal{U}}\eta_n^2\nabla u_f\cdot\nabla\overline{\phi}\mathrm{d}x = \frac{1}{2}\int_{\mathcal{U}}\nabla u_f\cdot\nabla\overline{\phi}\mathrm{d}x$. Arguing as in the proof of Lemma 3.3, we find $\lim_{n\to\infty}\int_{\mathcal{U}}\eta_n\overline{\phi}\nabla u_f\cdot$

$\nabla\eta_n \mathrm{d}x = 0$. Thanks to Lemma 3.2, we apply Hölder's inequality to find positive constants C_2 and C_3 such that

$$\left|\int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot\nabla u_f\overline{\phi}\mathrm{d}x\right| \le \left(\int_{\mathcal{U}}|\nabla u_f|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathcal{U}}|p+\beta_0\nabla U|^2|\phi|^2\mathrm{d}x\right)^{\frac{1}{2}}$$
$$\le C_2\left(\int_{\mathcal{U}}\alpha|u_f|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathcal{U}}\alpha|\phi|^2\mathrm{d}x\right)^{\frac{1}{2}},$$

and

$$\left|\int_{\mathcal{U}}e_{\beta_0}u_f\overline{\phi}\mathrm{d}x\right| \le C_3\left(\int_{\mathcal{U}}\alpha|u_f|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathcal{U}}\alpha|\phi|^2\mathrm{d}x\right)^{\frac{1}{2}}.$$

Therefore, the dominated convergence theorem yields

$$\lim_{n\to\infty}\int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot\nabla u_f(\eta_n^2\overline{\phi})\mathrm{d}x+\int_{\mathcal{U}}e_{\beta_0}\eta_n^2u_f\overline{\phi}\mathrm{d}x = \int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot\nabla u_f\overline{\phi}\mathrm{d}x+\int_{\mathcal{U}}e_{\beta_0}u_f\overline{\phi}\mathrm{d}x.$$

Letting $n\to\infty$ in (3.13), the result follows.

(3) Suppose $f\ge 0$ a.e. in $\mathcal{U}$. In this case, u_f must be real-valued. It is easy to verify that $u_f^- \in \mathcal{H}^1(\mathcal{U};\mathbb{C})$. Thanks to (2), we obtain $\mathcal{E}_{\beta_0}(u_f,u_f^-)+M\langle u_f,u_f^-\rangle_{L^2} = \langle f,u_f^-\rangle_{L^2}\ge 0$. It follows from Lemma 3.3 (2) and the fact $\mathcal{E}_{\beta_0}(u_f,u_f^-) = -\mathcal{E}_{\beta_0}(u_f^-,u_f^-)$ and $\langle u_f,u_f^-\rangle_{L^2} = -\langle u_f^-,u_f^-\rangle_{L^2}$ that

$$\min\left\{\frac{1}{2},C_*\right\}\|u_f^-\|_{\mathcal{H}^1}^2 \le \mathcal{E}_{\beta_0}(u_f^-,u_f^-)+M\|u_f^-\|_{L^2}^2\le 0.$$

This implies $u_f^- = 0$, and hence that $u_f\ge 0$. The last part follows from the weak Harnack's inequality of weak solutions of elliptic equations (see e.g. [23, Theorem 8.18]). □

By Lemma 3.4 and Lemma 3.1, the operator

$$(-\mathcal{L}_{\beta_0}+M)^{-1}: L^2(\mathcal{U};\mathbb{C})\to L^2(\mathcal{U};\mathbb{C}),\quad f\mapsto u_f$$

is linear, positive and compact. In light of Lemma 3.4, we define the domain of $\mathcal{L}_{\beta_0}$ as follows:

$$\mathcal{D} := (-\mathcal{L}_{\beta_0}+M)^{-1}L^2(\mathcal{U};\mathbb{C}) = \left\{\phi\in\mathcal{H}^1(\mathcal{U};\mathbb{C}) : \mathcal{L}_{\beta_0}\phi\in L^2(\mathcal{U};\mathbb{C})\right\}.$$

The next result collects basic spectral properties of $-\mathcal{L}_{\beta_0}$.

Theorem 3.1. *Assume* **(H1)**-**(H3)**. *Then, the following hold.*

(1) *The operator $-\mathcal{L}_{\beta_0}$ has a discrete spectrum and is contained in $\{\lambda\in\mathbb{C} : \Re\lambda > -M\}$.*
(2) *The number $\lambda_1 := \inf\{\Re\lambda : \lambda\in\sigma(-\mathcal{L}_{\beta_0})\}$ is a simple eigenvalue of $-\mathcal{L}_{\beta_0}$, and is dominating, in the sense that $\inf\{\Re\lambda : \lambda\in\sigma(-\mathcal{L}_{\beta_0})\setminus\{\lambda_1\}\} > \lambda_1$.*

(3) *The eigenspace of* λ_1 *is spanned over* $\mathbb{C}$ *by* $\tilde{v}_1$ *for some* $\tilde{v}_1 \in \mathcal{D}$ *a.e. positive in* $\mathcal{U}$.

Proof. Due to compactness, $(-\mathcal{L}_{\beta_0}+M)^{-1}$ has a discrete spectrum. Since Lemma 3.4 (3) ensures the a.e. positivity of $(-\mathcal{L}_{\beta_0}+M)^{-1}\phi$ in $\mathcal{U}$ for each $\phi \in L^2_+(\mathcal{U}) \setminus \{0\}$, the operator $(-\mathcal{L}_{\beta_0}+M)^{-1}$ is in particular positive and nonsupporting (in the language of I. Sawashima [45]). As a result, we are able to apply the results in [45] (also see [38, Theeorem 2.3]) to conclude

- the spectral radius r_1 of $(-\mathcal{L}_{\beta_0}+M)^{-1}$ is a positive and simple eigenvalue of $(-\mathcal{L}_{\beta_0}+M)^{-1}$;
- the eigenspace of r_1 is spanned over $\mathbb{C}$ by $\tilde{v}_1$ for some $\tilde{v}_1 > 0$ a.e. in $\mathcal{U}$;
- r_1 is dominating in the sense that $\sup\{|\lambda| : \lambda \in \sigma((-\mathcal{L}_{\beta_0}+M)^{-1}) \setminus \{r_1\}\} < r_1$.

The theorem then follows from the spectral mapping theorem. □

Remark 3.2. *We point out that the positive cone* $L^2_+(\mathcal{U})$ *has empty interior so that the celebrated Kreĭn-Rutman theorem* [31] *for compact and strongly positive operators, often used to treat elliptic operators on bounded domains, does not apply here. Restricting* $-\mathcal{L}_{\beta_0}$ *to a smaller space does not help as* $\mathcal{U}$ *is unbounded.*

The number λ_1 *is often called* the principal eigenvalue *of* $-\mathcal{L}_{\beta_0}$*. So far, it is not clear whether* λ_1 *is positive. The positivity of* λ_1 *is shown later by means of the absorbing properties of the process* X_t.

The following result concerns the semigroup generated by $\mathcal{L}_{\beta_0}$.

Theorem 3.2. *Assume* **(H1)**-**(H3)**. *Then,* $(\mathcal{L}_{\beta_0}, \mathcal{D})$ *generates a* C_0*-semigroup* $(T_t)_{t\geq 0}$ *on* $L^2(\mathcal{U};\mathbb{C})$*. Moreover,* $(T_t)_{t\geq 0}$ *is positive (i.e.,* $T_t L^2_+(\mathcal{U}) \subset L^2_+(\mathcal{U})$ *for all* $t \geq 0$*), extends to an analytic semigroup and is immediately compact.*

Proof. Note that it is equivalent to study the operator $\mathcal{L}_{\beta_0} - M$ with domain $\mathcal{D}$. The proof is broken into two steps.

Step 1. We claim:

(a) $(\mathcal{L}_{\beta_0} - M, \mathcal{D})$ is closed and $\mathcal{D}$ is dense in $L^2(\mathcal{U};\mathbb{C})$;

(b) The resolvent set of $\mathcal{L}_{\beta_0} - M$ contains $(0,\infty)$, and for each $\lambda > 0$ one has

$$\|(\lambda + M - \mathcal{L}_{\beta_0})^{-1}\|_{L^2\to L^2} \leq \frac{1}{\lambda}.$$

Thus, $(\mathcal{L}_{\beta_0} - M, \mathcal{D})$ generates a C_0-semigroup of contractions $\{T_t\}_{t\geq 0}$ in $L^2(\mathcal{U};\mathbb{C})$ by the Hille-Yosida theorem (see e.g. [42, 19]). By Lemma 3.4 (3), this semigroup must be positive.

(a) Obviously, $C_0^\infty(\mathcal{U};\mathbb{C}) \subset \mathcal{D}$. The density of $\mathcal{D}$ in $L^2(\mathcal{U};\mathbb{C})$ follows readily. To see the closedness of $(\mathcal{L}_{\beta_0} - M, \mathcal{D})$, we take sequences of functions $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathcal{D}$ and

$\{f_n\}_{n\in\mathbb{N}} \subset L^2(\mathcal{U};\mathbb{C})$ that converge respectively to ϕ and f in $L^2(\mathcal{U};\mathbb{C})$ and satisfy $(\mathcal{L}_{\beta_0}-M)\phi_n = f_n$ for all $n\in\mathbb{N}$. By Lemma 3.4, we find $\tilde{\phi} := (\mathcal{L}_{\beta_0}-M)^{-1}f \in \mathcal{D}$ and $\|\phi_n - \tilde{\phi}\|_{\mathcal{H}^1} \le C\|f_n - f\|_{L^2}$ for all $n\in\mathbb{N}$, where $C>0$ is independent of n. Passing to the limit as $n\to\infty$, we conclude $\phi = \tilde{\phi}\in\mathcal{D}$, and hence, $(\mathcal{L}_{\beta_0}-M)\phi = f$. This proves (a).

(b) Fix $\lambda > 0$. By Theorem 3.1, λ is in the resolvent set of $\mathcal{L}_{\beta_0}-M$. To establish the upper bound of $\|(\lambda+M-\mathcal{L}_{\beta_0})^{-1}\|_{L^2\to L^2}$, we let $f\in L^2(\mathcal{U};\mathbb{C})$ and $u\in\mathcal{D}$ be such that $(\lambda+M-\mathcal{L}_{\beta_0})u = f$. It follows from Lemma 3.4 (2) that $\mathcal{E}_{\beta_0}(u,\phi)+(\lambda+M)\langle u,\phi\rangle_{L^2} = \langle f,\phi\rangle_{L^2}$ for all $\phi\in\mathcal{H}^1(\mathcal{U};\mathbb{C})$. As a result, Lemma 3.3 (2) ensures

$$\min\left\{\frac{1}{2}, C_*\right\}\|u\|^2_{\mathcal{H}^1} + \lambda\|u\|^2_{L^2} \le \Re\mathcal{E}_{\beta_0}(u,\phi) + \lambda\|u\|^2_{L^2} \le \|f\|_{L^2}\|u\|_{L^2},$$

yielding the expected upper bound.

Step 2. To show that $(T_t)_{t\ge 0}$ extends to an analytic semigroup, we set

$$S := \left\{\langle -(\mathcal{L}_{\beta_0}-M)u, u\rangle_{L^2} : u\in\mathcal{D},\ \|u\|_{L^2} = 1\right\}.$$

We claim there exists $\theta\in(0,\frac{\pi}{2})$ such that $S\subset\{\lambda\in\mathbb{C} : |\arg\lambda|\le\theta\}$. Then, for fixed $\theta_*\in(\theta,\frac{\pi}{2})$, $\Sigma_{\theta_*} := \{\lambda\in\mathbb{C} : |\arg\lambda| > \theta_*\}\subset\mathbb{C}\setminus\overline{S}$ and there is $C_1>0$ such that $d(\lambda,\overline{S})\ge C_1|\lambda|$ for all $\lambda\in\Sigma_{\theta_*}$.

Since $(-\infty,0)$ is contained in the resolvent of $\mathcal{L}_{\beta_0}-M$, an application of [42, Theorem 1.3.9] yields that Σ_{θ_*} is contained in the resolvent of $\mathcal{L}_{\beta_0}-M$ and

$$\|(\lambda+M-\mathcal{L}_{\beta_0})^{-1}\|_{L^2\to L^2} \le \frac{1}{d(\lambda,\overline{S})} \le \frac{1}{C_1|\lambda|},\quad \forall\lambda\in\Sigma_{\theta_*}.$$

As a result, [42, Theorem 2.5.2] enables us to extend $(T_t)_{t\ge0}$ to an analytic semigroup. Moreover, as $(\mathcal{L}_{\beta_0}-M)^{-1}$ is compact by Lemma 3.1, the immediate compactness of $(T_t)_{t\ge0}$ follows from [19, Theorem II.4.29].

It suffices to prove the claim. To do so, we fix $u\in\mathcal{D}$. Note that Lemma 3.4 (2) gives

$$\langle -(\mathcal{L}_{\beta_0}-M)u,u\rangle_{L^2} = \mathcal{E}_{\beta_0}(u,u) + M\|u\|^2_{L^2}.$$

We see from Lemma 3.3 (2) that

$$\Re\langle -(\mathcal{L}_{\beta_0}-M)u,u\rangle_{L^2} = \Re\mathcal{E}_{\beta_0}(u,u) + M\|u\|^2_{L^2} \ge \min\left\{\frac{1}{2}, C_*\right\}\|u\|_{\mathcal{H}^1}.$$

Applying Young's inequality, we derive from Lemma 3.2 that

$$\begin{aligned}|\Im\langle -(\mathcal{L}_{\beta_0}-M)u,u\rangle_{L^2}| &= |\Im\mathcal{E}_{\beta_0}(u,u)|\\ &= \left|\int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot(u_1\nabla u_2-u_2\nabla u_1)\right|\\ &\leq \frac{1}{2}\int_{\mathcal{U}}|\nabla u|^2\mathrm{d}x+\frac{1}{2}\int_{\mathcal{U}}|p+\beta_0\nabla U|^2|u|^2\mathrm{d}x\\ &\leq \frac{1}{2}\int_{\mathcal{U}}|\nabla u|^2\mathrm{d}x+\frac{C_2}{2}\int_{\mathcal{U}}\alpha|u|^2\mathrm{d}x,\end{aligned}$$

where $u:=u_1+iu_2$ and $C_2>0$ is independent of $u\in\mathcal{D}$. Therefore,

$$0\leq\frac{|\Im\langle -(\mathcal{L}_{\beta_0}-M)u,u\rangle_{L^2}|}{\Re\langle -(\mathcal{L}_{\beta_0}-M)u,u\rangle_{L^2}}\leq\frac{\frac{1}{2}+\frac{C_2}{2}}{\min\left\{\frac{1}{2},C_*\right\}}.$$

The claim follows. This completes the proof. □

3.4. **Adjoint operator and semigroup.** Let $(\mathcal{L}^*_{\beta_0},\mathcal{D}^*)$ be the adjoint operator of $(\mathcal{L}_{\beta_0},\mathcal{D})$ in $L^2(\mathcal{U};\mathbb{C})$. Then, $\mathcal{D}^*$ is given by

$$\mathcal{D}^*:=\left\{w\in L^2(\mathcal{U};\mathbb{C}):\exists f\in L^2(\mathcal{U};\mathbb{C})\text{ s.t. }\langle w,\mathcal{L}_{\beta_0}\phi\rangle_{L^2}=\langle f,\phi\rangle_{L^2},\ \forall\phi\in\mathcal{D}\right\}.$$

For each $w\in\mathcal{D}^*$, $\mathcal{L}^*_{\beta_0}w$ is the unique element in $L^2(\mathcal{U};\mathbb{C})$ such that $\langle w,\mathcal{L}_{\beta_0}\phi\rangle_{L^2}=\langle\mathcal{L}^*_{\beta_0}w,\phi\rangle_{L^2}$ for all $\phi\in\mathcal{D}$. Integration by parts yields

$$\mathcal{L}^*_{\beta_0}w=\frac{1}{2}\Delta w+\nabla\cdot((p+\beta_0\nabla U)w)-e_{\beta_0}w,\quad w\in C_0^\infty(\mathcal{U};\mathbb{C}).$$

The following lemma summarizes some properties of the operator $-\mathcal{L}^*_{\beta_0}$.

Lemma 3.5. *Assume* **(H1)-(H3)***. Then, the following hold.*

(1) $\sigma(-\mathcal{L}^*_{\beta_0})=\sigma(-\mathcal{L}_{\beta_0})\subset\{\lambda\in\mathbb{C}:\Re\lambda>-M\}$.
(2) $\mathcal{D}^*=\left\{w\in\mathcal{H}^1(\mathcal{U};\mathbb{C}):\mathcal{L}^*_{\beta_0}w\in L^2(\mathcal{U};\mathbb{C})\right\}$.
(3) *For each* $\phi\in\mathcal{H}^1(\mathcal{U};\mathbb{C})$ *and* $w\in\mathcal{D}^*$ *one has* $\langle\phi,-\mathcal{L}^*_{\beta_0}w\rangle_{L^2}=\mathcal{E}_{\beta_0}(\phi,w)$.
(4) λ_1 *is a simple and dominating eigenvalue of* $-\mathcal{L}^*_{\beta_0}$ *with the associated eigenspace spanned over* $\mathbb{C}$ *by* $\tilde{v}_1^*$ *for some* $\tilde{v}_1^*\in\mathcal{D}^*$ *a.e. positive in* $\mathcal{U}$.

Proof. (1) Note that $\sigma(-\mathcal{L}^*_{\beta_0})=\overline{\sigma(-\mathcal{L}_{\beta_0})}$. Since the spectrum of $-\mathcal{L}_{\beta_0}$ consists of eigenvalues due to Lemma 3.1 (1), and the coefficients of $-\mathcal{L}_{\beta_0}$ are real-valued, we have $\Lambda\in\sigma(-\mathcal{L}_{\beta_0})$ if and only if $\overline{\Lambda}\in\sigma(-\mathcal{L}_{\beta_0})$. Hence, $\overline{\sigma(-\mathcal{L}_{\beta_0})}=\sigma(-\mathcal{L}_{\beta_0})$, which leads to $\sigma(-\mathcal{L}^*_{\beta_0})=\sigma(-\mathcal{L}_{\beta_0})$.

(2) Let $\mathcal{D}^*_1:=\left\{w\in\mathcal{H}^1(\mathcal{U};\mathbb{C}):\mathcal{L}^*_{\beta_0}w\in L^2(\mathcal{U};\mathbb{C})\right\}$. It is easy to verify that $\mathcal{D}^*_1\subset\mathcal{D}^*$. To prove the converse, we take $u^*\in\mathcal{D}^*$. Then, there exists a unique $f\in L^2(\mathcal{U};\mathbb{C})$ such that $-\mathcal{L}^*_{\beta_0}u^*=f$. Fix $\lambda>0$. Setting $f^*:=f+(M+\lambda)u^*\in L^2(\mathcal{U};\mathbb{C})$, we find

$(-\mathcal{L}^*_{\beta_0}+M+\lambda)u^*=f^*$. It follows from (1) that $-\lambda$ belongs to the resolvent set of $-\mathcal{L}^*_{\beta_0}+M$, and therefore, $u^*=(-\mathcal{L}^*_{\beta_0}+M+\lambda)^{-1}f^*$.

Arguing as in the proof of Lemma 3.4, we deduce $(-\mathcal{L}^*_{\beta_0}+M+\lambda)^{-1}L^2(\mathcal{U};\mathbb{C})\subset\mathcal{D}^*_1$, and hence, $u^*\in\mathcal{D}^*_1$. It follows that $\mathcal{D}^*=\mathcal{D}^*_1$.

(3) Note that $\langle\phi,-\mathcal{L}^*_{\beta_0}w\rangle_{L^2}=\langle-\mathcal{L}_{\beta_0}\phi,w\rangle_{L^2}$ for all $\phi\in C_0^\infty(\mathcal{U};\mathbb{C})$ and $w\in\mathcal{D}^*$. It follows from Lemma 3.4 (2) that $\langle\phi,-\mathcal{L}^*_{\beta_0}w\rangle_{L^2}=\mathcal{E}_{\beta_0}(\phi,w)$ for all $\phi\in C_0^\infty(\mathcal{U};\mathbb{C})$ and $w\in\mathcal{D}^*$. Since $C_0^\infty(\mathcal{U};\mathbb{C})$ is dense in $\mathcal{H}^1(\mathcal{U};\mathbb{C})$, the conclusion follows from approximation arguments as in the proof of Lemma 3.4 (2).

(4) This follows from (1) and arguments as in the proof of Theorem 3.1. □

Denote by $(T^*_t)_{t\geq0}$ the dual semigroup of $(T_t)_{t\geq0}$. It is well-known (see e.g. [42, Corollary 1.10.6]) that $(T^*_t)_{t\geq0}$ is a C_0-semigroup with infinitesimal generator $(\mathcal{L}^*_{\beta_0},\mathcal{D}^*)$.

Theorem 3.3. *Assume* **(H1)**-**(H3)**. *Then,* $(T^*_t)_{t\geq0}$ *is an analytic semigroup. Moreover, it is positive, i.e.,* $T^*_tL^2_+(\mathcal{U})\subset L^2_+(\mathcal{U})$ *for all* $t\geq0$*, and immediately compact.*

Proof. Note that the resolvent set of $\mathcal{L}^*_{\beta_0}-M$ coincides with that of $\mathcal{L}_{\beta_0}-M$. Thanks to [42, Theorem 2.5.2], the conclusion is a straightforward consequence of the analyticity of $(T_t)_{t\geq0}$ and the fact $\|(\lambda+M-\mathcal{L}^*_{\beta_0})^{-1}\|_{L^2\to L^2}=\|(\overline{\lambda}+M-\mathcal{L}_{\beta_0})^{-1}\|_{L^2\to L^2}$ for each $\lambda\in\mathbb{C}$ with $\Re\lambda>0$. The positivity and immediate compactness follow from arguments as in the proof of Theorem 3.2. □

4. Stochastic representation of semigroups

In this section, we study the stochastic representation of the semigroup $(T^*_t)_{t\geq0}$. Subsection 4.1 and Subsection 4.2 are respectively devoted to the stochastic representation and estimates of semigroups generated by $\mathcal{L}^*_{\beta_0}$ restricted to bounded domains with zero Dirichlet boundary condition. In Subsection 4.3, we establish the stochastic representation for $(T^*_t)_{t\geq0}$.

4.1. Stochastic representation in bounded domains.

Let $\Omega\subset\subset\mathcal{U}$ be a bounded and connected subdomain with C^2 boundary. Denote by $\mathcal{L}^X$ the diffusion operator associated to X_t or (2.2), namely,

$$\mathcal{L}^X=\frac{1}{2}\Delta+(p-q)\cdot\nabla.$$

For each $N>1$, let $\mathcal{L}^X_N|_\Omega$ be $\mathcal{L}^X$ considered as an operator in $L^N(\Omega;\mathbb{C})$ with domain $W^{2,N}(\Omega;\mathbb{C})\cap W^{1,N}_0(\Omega;\mathbb{C})$. It is well-known (see e.g. [23, 42, 19]) that the spectrum of $-\mathcal{L}^X_N|_\Omega$ is discrete and contained in $\{\lambda\in\mathbb{C}:\Re\lambda>0\}$ and $\mathcal{L}^X_N|_\Omega$ generates an analytic semigroup $(S^{(\Omega,N)}_t)_{t\geq0}$ of contractions on $L^N(\Omega;\mathbb{C})$ that satisfies $S^{(\Omega,N)}_tL^N_+(\Omega)\subset L^N_+(\Omega)$

for all $t \geq 0$. Moreover, the following stochastic representation holds: for each $f \in C(\overline{\Omega}; \mathbb{C})$,

$$S_t^{(\Omega,N)} f(x) = \mathbb{E}^x \left[f(X_t) \mathbb{1}_{\{t<\tau_\Omega\}} \right], \quad \forall (x,t) \in \overline{\Omega} \times [0,\infty), \tag{4.1}$$

where $\tau_\Omega := \inf\{t \geq 0 : X_t \notin \Omega\}$ is the first time that X_t exits Ω.

For $N > 1$, let $\mathcal{L}^{*,N}_{\beta_0}|_\Omega$ be $\mathcal{L}^*_{\beta_0}$ considered as an operator in $L^N(\Omega;\mathbb{C})$ with domain $W^{2,N}(\Omega;\mathbb{C}) \cap W^{1,N}_0(\Omega;\mathbb{C})$.

Proposition 4.1. *The following statements hold.*

(1) *The spectrum of $-\mathcal{L}^{*,N}_{\beta_0}|_\Omega$ is discrete and is contained in $\{\lambda \in \mathbb{C} : \Re\lambda > 0\}$.*

(2) *$\mathcal{L}^{*,N}_{\beta_0}|_\Omega$ generates an analytic semigroup of contractions $(T_t^{(*,\Omega,N)})_{t\geq 0}$ on $L^N(\Omega;\mathbb{C})$ that is positive, namely, $T_t^{(*,\Omega,N)} L^N_+(\Omega) \subset L^N_+(\Omega)$ for all $t \geq 0$.*

(3) *For each $f \in L^N(\Omega;\mathbb{C})$ one has*

$$T_t^{(*,\Omega,N)} \tilde{f} = e^{-\frac{Q}{2}-\beta_0 U} S_t^{(\Omega,N)} f, \quad \forall t \geq 0,$$

where $\tilde{f} := e^{-\frac{Q}{2}-\beta_0 U} f$.

(4) *For each $f \in C(\overline{\Omega};\mathbb{C})$ one has*

$$T_t^{(*,\Omega,N)} \tilde{f}(x) = e^{-\frac{Q(x)}{2}-\beta_0 U(x)} \mathbb{E}^x \left[f(X_t) \mathbb{1}_{\{t<\tau_\Omega\}} \right], \quad \forall (x,t) \in \overline{\Omega} \times [0,\infty),$$

where $\tilde{f} := e^{-\frac{Q}{2}-\beta_0 U} f$.

(5) *For any $N_1, N_2 > 1$, $T_t^{(*,\Omega,N_1)}$ and $T_t^{(*,\Omega,N_2)}$ coincide on $L^{N_1}(\Omega;\mathbb{C}) \cap L^{N_2}(\Omega;\mathbb{C})$ for all $t \geq 0$.*

Proof. Since straightforward calculations give

$$\mathcal{L}^{*,N}_{\beta_0}|_\Omega \tilde{f} = e^{-\frac{Q}{2}-\beta_0 U} \mathcal{L}^X_N|_\Omega f, \quad \forall f \in W^{2,N}(\Omega;\mathbb{C}) \cap W^{1,N}_0(\Omega;\mathbb{C}),$$

where $\tilde{f} := e^{-\frac{Q}{2}-\beta_0 U} f$, the conclusions (1)-(4) follow immediately from the corresponding properties of $\mathcal{L}^X_N|_\Omega$ and $(S_t^{(\Omega,N)})_{t\geq 0}$.

In particular, for any $N_1, N_2 > 1$ we have $T_t^{(*,\Omega,N_1)} \tilde{f} = T_t^{(*,\Omega,N_2)} \tilde{f}$ for all $\tilde{f} \in C(\overline{\Omega};\mathbb{C})$. Statement (5) then follows from the density of $C(\Omega;\mathbb{C})$ in $L^N(\Omega;\mathbb{C})$ for any $N > 1$. □

4.2. **Estimates of semigroups in bounded domains.** We prove two useful lemmas concerning some estimates of the semigroup $(T_t^{(*,\Omega,N)})_{t\geq 0}$.

Lemma 4.1. *Let $N \geq 2$ and $\tilde{f} \in L^N(\Omega)$. Then, $\tilde{w} := T_\bullet^{(*,\Omega,N)} \tilde{f}$ satisfies the following inequalities:*

$$\begin{aligned} &\frac{1}{N} \int_\Omega |\tilde{w}|^N(\cdot,t) \mathrm{d}x + \frac{N-1}{2} \int_{t_1}^t \int_\Omega |\tilde{w}|^{N-2} |\nabla \tilde{w}|^2 \mathrm{d}x\mathrm{d}s + C_* \int_{t_1}^t \int_\Omega \alpha |\tilde{w}|^N \mathrm{d}x\mathrm{d}s \\ &\quad \leq \frac{1+e^{NM(t-t_1)}}{N} \int_\Omega |\tilde{w}(\cdot,t_1)|^N \mathrm{d}x, \quad \forall t > t_1 \geq 0, \end{aligned}$$

and

$$\frac{1}{N}\int_\Omega |\tilde w|^N(\cdot,t)\mathrm{d}x+\frac{N-1}{2}\int_{t_2}^t\int_\Omega |\tilde w|^{N-2}|\nabla\tilde w|^2\mathrm{d}x\mathrm{d}s+C_*\int_{t_2}^t\int_\Omega \alpha|\tilde w|^N\mathrm{d}x\mathrm{d}s$$
$$\leq \frac{2}{N(t_2-t_1)}\int_{t_1}^{t_2}\int_\Omega |\tilde w|^N\mathrm{d}x\mathrm{d}s,\quad \forall t>t_2>t_1\geq 0.$$

Proof. Fix $N\geq 2$ and $\tilde f\in L^N(\Omega)$. Then, $\tilde w:=T_\bullet^{(*,\Omega,N)}\tilde f$ satisfies

$$\partial_t\tilde w=\mathcal{L}_{\beta_0}^{*,N}|_\Omega\tilde w\quad\text{in}\quad \Omega\times(0,\infty).$$

Multiplying the above equation by $|\tilde w|^{N-2}\tilde w$ and integrating by parts, we find for $t>0$

$$\begin{aligned}
&\int_\Omega |\tilde w|^{N-2}\tilde w\partial_t\tilde w\mathrm{d}x\\
&=-\frac{N-1}{2}\int_\Omega |\tilde w|^{N-2}|\nabla\tilde w|^2\mathrm{d}x-(N-1)\int_\Omega (p+\beta_0\nabla U)\cdot|\tilde w|^{N-2}\tilde w\nabla\tilde w\mathrm{d}x-\int_\Omega e_{\beta_0}|\tilde w|^N\mathrm{d}x\\
&=-\frac{N-1}{2}\int_\Omega |\tilde w|^{N-2}|\nabla\tilde w|^2\mathrm{d}x-\frac{N-1}{N}\int_\Omega (p+\beta_0\nabla U)\cdot\nabla|\tilde w|^N\mathrm{d}x-\int_\Omega e_{\beta_0}|\tilde w|^N\mathrm{d}x\\
&=-\frac{N-1}{2}\int_\Omega |\tilde w|^{N-2}|\nabla\tilde w|^2\mathrm{d}x+\frac{N-1}{N}\int_\Omega (\nabla\cdot p+\beta_0\Delta U)|\tilde w|^N\mathrm{d}x-\int_\Omega e_{\beta_0}|\tilde w|^N\mathrm{d}x\\
&=-\frac{N-1}{2}\int_\Omega |\tilde w|^{N-2}|\nabla\tilde w|^2\mathrm{d}x-\int_\Omega e_{\beta_0,N}|\tilde w|^N\mathrm{d}x.
\end{aligned}\tag{4.2}$$

Since $|\tilde w|^{N-2}\tilde w\partial_t\tilde w=\frac{1}{N}\partial_t|\tilde w|^N$, we integrate the above equality on $[t_1,t]\subset[0,\infty)$ to derive

$$\frac{1}{N}\int_\Omega |\tilde w|^N(\cdot,t)\mathrm{d}x+\frac{N-1}{2}\int_{t_1}^t\int_\Omega |\tilde w|^{N-2}|\nabla\tilde w|^2\mathrm{d}x\mathrm{d}s$$
$$+\int_{t_1}^t\int_\Omega e_{\beta_0,N}|\tilde w|^N\mathrm{d}x\mathrm{d}s=\frac{1}{N}\int_\Omega |\tilde w|^N(\cdot,t_1)\mathrm{d}x.$$

As Lemma 3.2 (3) gives $e_{\beta_0,N}+M\geq C_*\alpha$ for all $N\geq 2$, we find

$$\begin{aligned}
&\frac{1}{N}\int_\Omega |\tilde w|^N(\cdot,t)\mathrm{d}x+\frac{N-1}{2}\int_{t_1}^t\int_\Omega |\tilde w|^{N-2}|\nabla\tilde w|^2\mathrm{d}x\mathrm{d}s+C_*\int_{t_1}^t\int_\Omega \alpha|\tilde w|^N\mathrm{d}x\mathrm{d}s\\
&\quad\leq M\int_{t_1}^t\int_\Omega |\tilde w|^N\mathrm{d}x\mathrm{d}s+\frac{1}{N}\int_\Omega |\tilde w|^N(\cdot,t_1)\mathrm{d}x,\quad \forall t>t_1\geq 0.
\end{aligned}\tag{4.3}$$

Setting $g(t):=\int_{t_1}^t\int_\Omega |\tilde w|^N\mathrm{d}x\mathrm{d}s$ for $t\geq t_1$, we arrive at $\frac{1}{N}g'\leq Mg+\frac{1}{N}\|\tilde w(\cdot,t_1)\|_{L^N}^N$ for all $t>t_1$. Gronwall's inequality gives $g(t)\leq\frac{e^{NM(t-t_1)}}{NM}\|\tilde w(\cdot,t_1)\|_{L^N}^N$ for all $t>t_1$. Inserting this into (4.3) yields the first inequality.

Now, we prove the second inequality. Fix $t_1, t_2 \in [0,\infty)$ with $t_1 < t_2$. Let $\eta \in C^\infty((0,\infty))$ be non-negative and non-decreasing such that

$$\eta(t) = \begin{cases} 0, & t \in [0, t_1], \\ 1, & t \in [t_2, \infty), \end{cases} \qquad \text{and} \qquad \max_{t\in[t_1,t_2]} \eta'(t) \le \frac{2}{t_2 - t_1}.$$

Multiplying (4.2) by η and integrating by parts, we find for $t > t_2$,

$$\begin{aligned} &\frac{1}{N}\int_\Omega \eta(t)|\tilde{w}|^N(\cdot,t)\mathrm{d}x - \frac{1}{N}\int_0^t\int_\Omega \eta'|\tilde{w}|^N \mathrm{d}x\mathrm{d}s \\ &\quad = -\frac{N-1}{2}\int_0^t\int_\Omega \eta|\tilde{w}|^{N-2}|\nabla\tilde{w}|^2\mathrm{d}x\mathrm{d}s - \int_0^t\int_\Omega \eta e_{\beta_0,N}|\tilde{w}|^N\mathrm{d}x\mathrm{d}s. \end{aligned}$$

The definition of η then gives

$$\begin{aligned} &\frac{1}{N}\int_\Omega |\tilde{w}|^N(\cdot,t)\mathrm{d}x + \frac{N-1}{2}\int_{t_2}^t\int_\Omega |\tilde{w}|^{N-2}|\nabla\tilde{w}|^2\mathrm{d}x\mathrm{d}s + \int_{t_2}^t\int_\Omega e_{\beta_0,N}|\tilde{w}|^N\mathrm{d}x\mathrm{d}s \\ &\quad \le \frac{1}{N}\int_{t_1}^{t_2}\int_\Omega \eta'|\tilde{w}|^N\mathrm{d}x\mathrm{d}s \le \frac{2}{N(t_2-t_1)}\int_{t_1}^{t_2}\int_\Omega |\tilde{w}|^N\mathrm{d}x\mathrm{d}s, \quad \forall t > t_2. \end{aligned}$$

This completes the proof. □

Lemma 4.2. *For each $t > 0$, there exists $C = C(t)$, independent of the domain Ω, such that*

$$\|T_t^{(*,\Omega,2_*)}\tilde{f}\|_{L^2(\Omega)} \le C\|\tilde{f}\|_{L^{2_*}(\Omega)}, \quad \forall \tilde{f} \in L^{2_*}(\Omega),$$

where $2_* := \frac{2(d+2)}{d+4} \in (1,2)$ is the dual exponent of $2 + \frac{4}{d}$.

Proof. Take $N \in (1,2]$. Then, $N' := \frac{N}{N-1} \ge 2$. Denote by $(T_t^{(\Omega,N')})_{t\ge 0}$ the semigroup on $L^{N'}(\Omega)$ that is dual to $(T_t^{(*,\Omega,N)})_{t\ge0}$. Let $\mathcal{L}_{\beta_0}^{N'}|_\Omega$ be $\mathcal{L}_{\beta_0}$ considered as an operator in $L^{N'}(\Omega)$ with domain $W^{2,N'}(\Omega)\cap W_0^{1,N'}(\Omega)$. It is not hard to check that $\mathcal{L}_{\beta_0}^{N'}|_\Omega$, being $\mathcal{L}_{\beta_0}$ considered as an operator in $L^{N'}(\Omega)$ with domain $W^{2,N'}(\Omega;\mathbb{C})\cap W_0^{1,N'}(\Omega;\mathbb{C})$, is the generator of $(T_t^{(\Omega,N')})_{t\ge0}$.

Take $\tilde{g} \in L^{N'}(\Omega)$ and denote $\tilde{v} := T_\bullet^{(\Omega,N')}\tilde{g}$. Then, $\tilde{v}$ is the solution of

$$\partial_t\tilde{v} = \mathcal{L}_{\beta_0}^{N'}|_\Omega\tilde{v} \quad \text{in} \quad \Omega\times(0,\infty).$$

Multiplying the above equation by $|\tilde{v}|^{N'-2}\tilde{v}$ and integrating by parts, we find for $t>0$,

$$\begin{aligned}&\int_\Omega |\tilde{v}|^{N'-2}\tilde{v}\partial_t\tilde{v}\mathrm{d}x\\ &\quad= -\frac{N'-1}{2}\int_\Omega |\tilde{v}|^{N'-2}|\nabla\tilde{v}|^2\mathrm{d}x - \int_\Omega (p+\beta_0\nabla U)\cdot\nabla\tilde{v}|\tilde{v}|^{N'-2}\tilde{v}\mathrm{d}x - \int_\Omega e_{\beta_0}|\tilde{v}|^{N'}\mathrm{d}x\\ &\quad= -\frac{N'-1}{2}\int_\Omega |\tilde{v}|^{N'-2}|\nabla\tilde{v}|^2\mathrm{d}x + \frac{1}{N'}\int_\Omega (\nabla\cdot p+\beta_0\Delta U)|\tilde{v}|^{N'}\mathrm{d}x - \int_\Omega e_{\beta_0}|\tilde{v}|^{N'}\mathrm{d}x\\ &\quad= -\frac{N'-1}{2}\int_\Omega |\tilde{v}|^{N'-2}|\nabla\tilde{v}|^2\mathrm{d}x - \int_\Omega e^*_{\beta_0,N'}|\tilde{v}|^{N'}\mathrm{d}x,\end{aligned}$$

where $e^*_{\beta_0,N'} := e_{\beta_0} - \frac{1}{N'}(\nabla\cdot p+\beta_0\Delta U)$. We can follow the proof of Lemma 3.2 (3) to show $e^*_{\beta_0,N} + M \geq C_*\alpha$ in $\mathcal{U}$ for all $N\geq 1$. Then, arguing as in the proof of Lemma 4.1 yields

$$\begin{aligned}&\frac{1}{N'}\int_\Omega |\tilde{v}|^{N'}(\cdot,t)\mathrm{d}x + \frac{N'-1}{2}\int_0^t\int_\Omega |\tilde{v}|^{N'-2}|\nabla\tilde{v}|^2\mathrm{d}x\mathrm{d}s + C_*\int_0^t\int_\Omega \alpha|\tilde{v}|^{N'}\mathrm{d}x\mathrm{d}s\\ &\quad\leq \frac{1+e^{N'Mt}}{N'}\int_\Omega |\tilde{g}|^{N'}\mathrm{d}x, \quad \forall t>0,\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}&\frac{1}{N'}\int_\Omega |\tilde{v}|^{N'}(\cdot,t)\mathrm{d}x + \frac{N'-1}{2}\int_{t_2}^t\int_\Omega |\tilde{v}|^{N'-2}|\nabla\tilde{v}|^2\mathrm{d}x\mathrm{d}s + C_*\int_{t_2}^t\int_\Omega \alpha|\tilde{v}|^{N'}\mathrm{d}x\mathrm{d}s\\ &\quad\leq \frac{2}{N'(t_2-t_1)}\int_{t_1}^{t_2}\int_\Omega |\tilde{v}|^{N'}\mathrm{d}x\mathrm{d}s, \quad \forall t>t_2>t_1\geq 0.\end{aligned} \tag{4.5}$$

The Sobolev embedding theorem gives

$$\|\tilde{v}^{\frac{N'}{2}}\|_{L^{2\kappa}(\Omega\times[0,t])} \leq C_1\left(\sup_{s\in[0,t]}\|\tilde{v}^{\frac{N'}{2}}(\cdot,s)\|_{L^2(\Omega)} + \|\nabla\tilde{v}^{\frac{N'}{2}}\|_{L^2(\Omega\times[0,t])}\right),$$

where $\kappa = \frac{d+2}{d}$ and $C_1>0$ depends only on d. This together with (4.4) gives rise to

$$\begin{aligned}\left(\int_0^t\int_\Omega |\tilde{v}|^{\kappa N'}\mathrm{d}x\mathrm{d}s\right)^{\frac{1}{\kappa}} &= \|\tilde{v}^{\frac{N'}{2}}\|^2_{L^{2\kappa}(\Omega\times[0,t])}\\ &\leq 2C_1^2\left(\sup_{s\in[0,t]}\int_\Omega |\tilde{v}(x,s)|^{N'}\mathrm{d}x + \frac{|N'|^2}{4}\int_0^t\int_\Omega |\tilde{v}|^{N'-2}|\nabla\tilde{v}|^2\mathrm{d}x\mathrm{d}s\right)\\ &\leq 2C_1^2\left(1+\frac{N'}{2(N'-1)}\right)(1+e^{N'Mt})\int_\Omega |\tilde{g}|^{N'}\mathrm{d}x\\ &= C_2(1+e^{N'Mt})\int_\Omega |\tilde{g}|^{N'}\mathrm{d}x, \quad \forall t>0,\end{aligned}$$

where $C_2 := 2C_1^2\left(1+\frac{N'}{2(N'-1)}\right)$. We then deduce from (4.5) (with $\kappa N'$ instead of N') that

$$\frac{1}{\kappa N'}\int_\Omega |\tilde{v}|^{\kappa N'}(\cdot,t)\mathrm{d}x + \frac{\kappa N'-1}{2}\int_{t_2}^t\int_\Omega |\tilde{v}|^{\kappa N'-2}|\nabla\tilde{v}|^2\mathrm{d}x\mathrm{d}s + C_*\int_{t_2}^t\int_\Omega \alpha|\tilde{v}|^{\kappa N'}\mathrm{d}x\mathrm{d}s$$
$$\le \frac{2}{\kappa N'(t_2-t_1)}\int_{t_1}^{t_2}\int_\Omega |\tilde{v}|^{\kappa N'}\mathrm{d}x\mathrm{d}s \le \frac{2}{\kappa N'(t_2-t_1)}C_2^\kappa(1+e^{N'Mt_2})^\kappa\|\tilde{g}\|_{L^{N'}(\Omega)}^{\kappa N'}$$

for all $t>t_2>t_1\ge 0$, where we used (4.4) in the second inequality.

As a consequence, for each $t>0$, there exists $C_3 = C_3(d,N',t)>0$ such that

$$\|T_t^{(\Omega,N')}\tilde{g}\|_{L^{\kappa N'}(\Omega)} = \|\tilde{v}(\cdot,t)\|_{L^{\kappa N'}(\Omega)} \le C_3\|\tilde{g}\|_{L^{N'}(\Omega)}.$$

Since $T^{(\Omega,N')}$ and $T_t^{(*,\Omega,N)}$ are adjoint to each other, it follows that

$$\|T_t^{(*,\Omega,N)}\tilde{f}\|_{L^N(\Omega)} \le C_3\|\tilde{f}\|_{L^{N_*}(\Omega)}, \quad \forall \tilde{f}\in L^{N_*}(\Omega)\cap L^N(\Omega).$$

where $N_* := \frac{\kappa N'}{\kappa N'-1}$. Thanks to Proposition 4.1 (5), we deduce from standard approximations that

$$\|T_t^{(*,\Omega,N_*)}\tilde{f}\|_{L^N(\Omega)} \le C_3\|\tilde{f}\|_{L^{N_*}(\Omega)}, \quad \forall \tilde{f}\in L^{N_*}(\Omega).$$

Setting $N=2$ yields $2_* = \frac{2(d+2)}{d+4}\in(1,2)$. This completes the proof. □

4.3. **Stochastic representation.** We prove the following theorem concerning the stochastic representation of $(T_t^*)_{t\ge 0}$.

Theorem 4.1. *Assume* **(H1)**-**(H3)**. *For each $f\in C_b(\mathcal{U};\mathbb{C})$ satisfying $\tilde{f} := fe^{-\frac{Q}{2}-\beta_0 U}\in L^2(\mathcal{U};\mathbb{C})$ we have*

$$T_t^*\tilde{f}(x) = e^{-\frac{Q(x)}{2}-\beta_0 U(x)}\mathbb{E}^x\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right], \quad \forall (x,t)\in\mathcal{U}\times[0,\infty).$$

Consider the following initial value problem associated to the operator $\mathcal{L}_{\beta_0}^*$:

$$\begin{cases}\partial_t\tilde{w} = \frac{1}{2}\Delta\tilde{w} + \nabla\cdot((p+\beta_0\nabla U)\tilde{w}) - e_{\beta_0}\tilde{w} & \text{in} \quad \mathcal{U}\times[0,\infty),\\ \tilde{w}(\cdot,0)=\tilde{f} & \text{in} \quad \mathcal{U}.\end{cases} \tag{4.6}$$

Definition 4.1. *A function $\tilde{w}\in C(\mathcal{U}\times[0,\infty))\bigcap L^2_{loc}([0,\infty),\mathcal{H}^1(\mathcal{U}))$ is called a* weak solution *of (4.6) if for each $\phi\in C_0^{1,1}(\mathcal{U}\times[0,\infty))$ one has*

$$\int_\mathcal{U}\tilde{w}(\cdot,t)\phi(\cdot,t)\mathrm{d}x - \int_\mathcal{U}\tilde{f}\phi(\cdot,0)\mathrm{d}x - \int_0^t\int_\mathcal{U}\tilde{w}\partial_t\phi\mathrm{d}x\mathrm{d}s$$
$$= -\frac{1}{2}\int_0^t\int_\mathcal{U}\nabla\tilde{w}\cdot\nabla\phi\mathrm{d}x\mathrm{d}s - \int_0^t\int_\mathcal{U}(p+\beta_0\nabla U)\cdot\tilde{w}\nabla\phi\mathrm{d}x\mathrm{d}s - \int_0^t\int_\mathcal{U}e_{\beta_0}\tilde{w}\phi\mathrm{d}x\mathrm{d}s$$

for all $t\in[0,\infty)$.

Lemma 4.3. *Assume* **(H1)**-**(H3)**. *For each* $\tilde{f} \in C(\mathcal{U}) \cap L^2(\mathcal{U})$, (4.6) *admits at most one weak solution.*

The proof of the above lemma follows from energy methods and approximation arguments. Since it is somewhat standard we present its proof in Appendix A.2.

Now, we prove Theorem 4.1.

Proof of Theorem 4.1. Treating the real and imaginary parts separately, we only need to prove the theorem for $f \in C_b(\mathcal{U})$ such that $\tilde{f} := fe^{-\frac{Q}{2}-\beta_0 U} \in L^2(\mathcal{U})$. Fix such an f.

We show $T^*_\bullet \tilde{f}$ is a weak solution of (4.6). Due to the analyticity of $(T^*_t)_{t\geq 0}$ (see Theorem 3.3) and Lemma 3.5, we find

(1) $T^*_\bullet \tilde{f} \in C([0,\infty), L^2(\mathcal{U})) \bigcap C^1((0,\infty), L^2(\mathcal{U}))$;
(2) $T^*_t \tilde{f} \in \mathcal{D}^* \subset \mathcal{H}^1(\mathcal{U}) \bigcap H^2_{loc}(\mathcal{U})$ for all $t > 0$;
(3) $\frac{\mathrm{d}}{\mathrm{d}t} T^*_t \tilde{f} = \mathcal{L}^*_{\beta_0} T^*_t \tilde{f}$ for all $t > 0$.

Since $\tilde{f} \in C(\mathcal{U})$, the classical regularity theory of parabolic equations yields that $T^*_\bullet \tilde{f} \in C(\mathcal{U} \times [0,\infty))$. Applying Lemma 3.3 (2) and Lemma 3.5, we find for each $t > 0$,

$$\begin{aligned}
\min\left\{\frac{1}{2}, C_*\right\} \|T^*_t \tilde{f}\|^2_{\mathcal{H}^1} &\leq \mathcal{E}_{\beta_0}(T^*_t \tilde{f}, T^*_t \tilde{f}) + M\|T^*_t \tilde{f}\|^2_{L^2} \\
&= -\langle T^*_t \tilde{f}, \mathcal{L}^*_{\beta_0} T^*_t \tilde{f}\rangle_{L^2} + M\|T^*_t \tilde{f}\|^2_{L^2} \\
&= -\langle T^*_t \tilde{f}, \frac{\mathrm{d}}{\mathrm{d}t} T^*_t \tilde{f}\rangle_{L^2} + M\|T^*_t \tilde{f}\|^2_{L^2} \\
&= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|T^*_t \tilde{f}\|^2_{L^2} + M\|T^*_t \tilde{f}\|^2_{L^2}.
\end{aligned}$$

It follows that

$$\min\left\{\frac{1}{2}, C_*\right\} \int_0^t \|T^*_s \tilde{f}\|^2_{\mathcal{H}^1} \mathrm{d}s \leq \frac{1}{2}\|\tilde{f}\|^2_{L^2} + M\int_0^t \|T^*_s \tilde{f}\|^2_{L^2} \mathrm{d}s, \quad \forall t > 0.$$

This yields $T^*_\bullet \tilde{f} \in L^2_{loc}([0,\infty), \mathcal{H}^1(\mathcal{U}))$. By (3), it is easy to check that the integral identity in Definition 4.1 holds with $\tilde{w}$ replaced by $T^*_\bullet \tilde{f}$. As a consequence, $T^*_\bullet \tilde{f}$ is a weak solution of (4.6).

Define

$$\tilde{w}(x,t) := e^{-\frac{Q(x)}{2}-\beta_0 U(x)} \mathbb{E}^x \left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right], \quad (x,t) \in \mathcal{U} \times [0,\infty).$$

We claim that $\tilde{w}$ is also a weak solution of (4.6). Then, Lemma 4.3 yields $T^*_\bullet \tilde{f} = \tilde{w}$, leading to the conclusion of the theorem.

The continuity of $\tilde{w}$ in $\mathcal{U} \times [0,\infty)$ follows from the definition and continuity properties of X_t. We show $\tilde{w} \in L^2_{loc}([0,\infty), \mathcal{H}^1(\mathcal{U}))$. Let $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ be as in Subsection 2.3. It follows from Lemma 2.1 and Proposition 4.1 (4) that $\tilde{w} = \lim_{n\to\infty} T^{(*,\mathcal{U}_n,2)}_\bullet \tilde{f}|_{\mathcal{U}_n}$ in

$\mathcal{U}\times[0,\infty)$, where we recall from Subsection 4.1 that $(T_t^{(*,\mathcal{U}_n,2)})_{t\geq 0}$ is the positive analytic semigroup of contractions on $L^2(\mathcal{U}_n;\mathbb{C})$ generated by $\mathcal{L}^{*,2}_{\beta_0}|_{\mathcal{U}_n}$ with domain $W^{2,2}(\mathcal{U}_n;\mathbb{C})\cap W^{1,2}_0(\mathcal{U}_n;\mathbb{C})$.

For convenience, we define $\tilde{w}_n := T_\bullet^{(*,\mathcal{U}_n,2)}\tilde{f}|_{\mathcal{U}_n}$ for $n\in\mathbb{N}$. Then, $\lim_{n\to\infty}\tilde{w}_n=\tilde{w}$. Lemma 4.1 (with $t_1=0$) gives for each $t\in[0,\infty)$ and $n\in\mathbb{N}$,

$$\frac{1}{2}\int_{\mathcal{U}_n}\tilde{w}_n^2(\cdot,t)\mathrm{d}x+\frac{1}{2}\int_0^t\int_{\mathcal{U}_n}|\nabla\tilde{w}_n|^2\mathrm{d}x\mathrm{d}s+C_*\int_0^t\int_{\mathcal{U}_n}\alpha\tilde{w}_n^2\mathrm{d}x\mathrm{d}s\leq\frac{1+e^{2Mt}}{2}\int_{\mathcal{U}_n}\tilde{f}^2\mathrm{d}x.$$

Letting $n\to\infty$ yields $\tilde{w}\in L^2_{loc}([0,\infty),\mathcal{H}^1(\mathcal{U}))$. Since $\partial_t\tilde{w}_n=\mathcal{L}^{*,2}_{\beta_0}|_{\mathcal{U}_n}\tilde{w}_n$ for all $t>0$ and $n\in\mathbb{N}$, standard approximation arguments ensure that $\tilde{w}$ is a weak solution of (4.6). This finishes the proof. □

5. QSD: existence, uniqueness and convergence

In this section, we study the existence and uniqueness of QSDs of X_t, as well as the exponential convergence of the process X_t conditioned on the event $[t<S_\Gamma]$ to QSDs. In Subsection 5.1, we show the existence of QSDs of X_t. In Subsection 5.2, we study the sharp exponential convergence of X_t with compactly supported initial distributions. In Subsection 5.3, we investigate the uniqueness of QSDs of X_t and the exponential convergence of X_t with arbitrary initial distribution. The proofs of Theorems A and B are outlined in Subsection 5.4.

5.1. Existence.

We construct QSDs for X_t. Recall that λ_1 and $\tilde{v}_1$ are given in Theorem 3.1.

Theorem 5.1. *Assume* **(H1)-(H3)**. *Then, the following statements hold.*

(1) $\lambda_1>0$ *and* $\int_{\mathcal{U}}\tilde{v}_1 e^{-\frac{Q}{2}-\beta_0 U}\mathrm{d}x<\infty$.
(2) *For each* $f\in C_b(\mathcal{U})$,

$$\mathbb{E}^{\nu_1}\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right]=e^{-\lambda_1 t}\int_{\mathcal{U}}f\mathrm{d}\nu_1,\quad\forall t\geq 0,$$

where $\mathrm{d}\nu_1:=\frac{\tilde{v}_1 e^{-\frac{Q}{2}-\beta_0 U}}{\int_{\mathcal{U}}\tilde{v}_1 e^{-\frac{Q}{2}-\beta_0 U}\mathrm{d}x}\mathrm{d}x$.
(3) ν_1 *is a QSD of* X_t *with extinction rate* λ_1.

We need the following lemma. Recall that the weight function α is defined in (3.1).

Lemma 5.1. *Assume* **(H1)-(H3)**. *For each* $\tilde{v}\in L^2(\mathcal{U},\alpha\mathrm{d}x;\mathbb{C})$ *one has* $\int_{\mathcal{U}}|\tilde{v}|e^{-\frac{Q}{2}-\beta_0 U}\mathrm{d}x<\infty$.

Proof. Fix $\tilde{v} \in L^2(\mathcal{U}, \alpha \mathrm{d}x; \mathbb{C})$. By Hölder's inequality,

$$\int_{\mathcal{U}} |\tilde{v}| e^{-\frac{Q}{2} - \beta_0 U} \mathrm{d}x \leq \left(\int_{\mathcal{U}} \alpha |\tilde{v}|^2 \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{\mathcal{U}} \frac{1}{\alpha} e^{-Q - 2\beta_0 U} \mathrm{d}x \right)^{\frac{1}{2}}.$$

It suffices to verify

$$\int_{\mathcal{U}} \frac{1}{\alpha} e^{-Q - 2\beta_0 U} \mathrm{d}x < \infty. \tag{5.1}$$

Let $\tilde{\alpha}(t) := \max\{\frac{1}{t^2}, 1\}$ for $t > 0$. According to the definition of α given in (3.1) and the fact that $\inf_{\mathcal{U}} \alpha > 0$, there exists $C_1 > 0$ such that $\alpha(x) \geq C_1 \sum_{i=1}^d \tilde{\alpha}(x_i)$ for $x \in \mathcal{U}$. Since $U(x) = V(\xi^{-1}(x)) \geq \sum_{i=1}^d \tilde{V}(\xi_i^{-1}(x_i))$ for $x \in \mathcal{U}$ due to **(H3)**(2) and $e^{-Q} = \frac{\left[\prod_{i=1}^d a_i(\xi_i^{-1}(1))\right]^{\frac{1}{2}}}{\left[\prod_{i=1}^d a_i(\xi_i^{-1}(x_i))\right]^{\frac{1}{2}}}$, we derive

$$\int_{\mathcal{U}} \frac{1}{\alpha} e^{-Q - 2\beta_0 U} \mathrm{d}x \leq \frac{\left[\prod_{i=1}^d a_i(\xi_i^{-1}(1))\right]^{\frac{1}{2}}}{C_1} \int_{\mathcal{U}} \frac{\prod_{i=1}^d \exp\left\{-2\beta_0 \tilde{V}(\xi_i^{-1}(x_i))\right\}}{\left(\sum_{i=1}^d \tilde{\alpha}(x_i)\right) \times \left[\prod_{i=1}^d a_i(\xi_i^{-1}(x_i))\right]^{\frac{1}{2}}} \mathrm{d}x.$$

For each $k \in \{1, \dots, d\}$, we denote by Σ_k the collection of all subsets of $\{1, \dots, d\}$ with k elements, and set

$$A_k := \sup_{\sigma \in \Sigma_k} \int_{\{x_\sigma = (x_i)_{i \in \sigma}: x_i > 0,\ \forall i \in \sigma\}} \frac{\prod_{i \in \sigma} \exp\left\{-2\beta_0 \tilde{V}(\xi_i^{-1}(x_i))\right\}}{\left(\sum_{i \in \sigma} \tilde{\alpha}(x_i)\right) \times \left[\prod_{i \in \sigma} a_i(\xi_i^{-1}(x_i))\right]^{\frac{1}{2}}} \mathrm{d}x_\sigma.$$

Clearly, (5.1) holds if $A_d < \infty$. We show this by induction.

First, we show $A_1 < \infty$. Following the arguments leading to (A.2), we can find $C_2 > 0$ such that $a_i(\xi_i^{-1}(x_i)) \geq C_2^2 x_i^2$ for $x_i \in [0, 1]$ and $i \in \{1, \dots, d\}$. It follows that for each $i \in \{1, \dots, d\}$,

$$\begin{aligned}
\int_0^\infty \frac{e^{-2\beta_0 \tilde{V}(\xi_i^{-1}(x_i))}}{\tilde{\alpha}(x_i) \left[a_i(\xi_i^{-1}(x_i))\right]^{\frac{1}{2}}} \mathrm{d}x_i &\leq \frac{1}{C_2} \int_0^1 \frac{e^{-2\beta_0 \tilde{V}(\xi_i^{-1}(x_i))}}{\tilde{\alpha}(x_i) x_i} \mathrm{d}x_i + \int_1^\infty \frac{e^{-2\beta_0 \tilde{V}(\xi_i^{-1}(x_i))}}{\tilde{\alpha}(x_i) \left[a_i(\xi_i^{-1}(x_i))\right]^{\frac{1}{2}}} \mathrm{d}x_i \\
&\leq \frac{1}{C_2} \int_0^1 x_i e^{-2\beta_0 \tilde{V}(\xi_i^{-1}(x_i))} \mathrm{d}x_i + \int_1^\infty \frac{e^{-2\beta_0 \tilde{V}(\xi_i^{-1}(x_i))}}{\left[a_i(\xi_i^{-1}(x_i))\right]^{\frac{1}{2}}} \mathrm{d}x_i \\
&\leq \frac{1}{2C_2} + \int_1^\infty \frac{e^{-2\beta_0 \tilde{V}(\xi_i^{-1}(x_i))}}{\left[a_i(\xi_i^{-1}(x_i))\right]^{\frac{1}{2}}} \mathrm{d}x_i,
\end{aligned}$$

where we used the definition of $\tilde{\alpha}$ in the second inequality and the non-negativity of $\tilde{V}$ in the last inequality. Changing variable, we find from **(H3)**(2) that

$$\int_1^\infty \frac{e^{-2\beta_0 \tilde{V}(\xi_i^{-1}(x_i))}}{\left[a_i(\xi_i^{-1}(x_i))\right]^{\frac{1}{2}}}\mathrm{d}x_i = \int_{\xi_i^{-1}(1)}^\infty \frac{e^{-2\beta_0\tilde{V}(z_i)}}{a_i(z_i)}\mathrm{d}z_i < \infty,$$

leading to $A_1 < \infty$.

Suppose $A_k < \infty$ for some $k \in \{1, \dots, d-1\}$, we show $A_{k+1} < \infty$. Without loss of generality, we only prove

$$A_{k+1}^1 := \int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^{k+1} \exp\left\{-2\beta_0\tilde{V}(\xi_i^{-1}(x_i))\right\}}{\left(\sum_{i=1}^{k+1}\tilde{\alpha}(x_i)\right) \times \left[\prod_{i=1}^{k+1} a_i(\xi_i^{-1}(x_i))\right]^{\frac{1}{2}}}\mathrm{d}x_1\cdots\mathrm{d}x_{k+1} < \infty;$$

integrals corresponding to other $\sigma \in \Sigma_{k+1}$ can be treated in exactly the same way. Note that

$$\begin{aligned}
&\int_0^1 \cdots \int_0^1 \frac{\prod_{i=1}^{k+1} \exp\left\{-2\beta_0\tilde{V}(\xi_i^{-1}(x_i))\right\}}{\left(\sum_{i=1}^{k+1}\tilde{\alpha}(x_i)\right) \times \prod_{i=1}^{k+1} \left[a_i(\xi_i^{-1}(x_i))\right]^{\frac{1}{2}}}\mathrm{d}x_1\cdots\mathrm{d}x_{k+1}\\
&\quad\le \frac{1}{C_2^{k+1}}\int_0^1\cdots\int_0^1 \frac{1}{\left(\sum_{i=1}^{k+1}\frac{1}{x_i^2}\right)\times\prod_{i=1}^{k+1} x_i}\mathrm{d}x_1\cdots\mathrm{d}x_{k+1}\\
&\quad\le \frac{1}{(k+1)C_2^{k+1}}\int_0^1\cdots\int_0^1 \frac{1}{\left(\prod_{i=1}^{k+1} x_i\right)^{1-\frac{2}{k+1}}}\mathrm{d}x_1\cdots\mathrm{d}x_{k+1} < \infty.
\end{aligned}$$

Straightforward calculations give

$$\begin{aligned}
&\int_0^\infty\cdots\int_1^\infty\cdots\int_0^\infty \frac{\prod_{i=1}^{k+1} \exp\left\{-2\beta_0\tilde{V}(\xi_i^{-1}(x_i))\right\}}{\left(\sum_{i=1}^{k+1}\tilde{\alpha}(x_i)\right) \times \prod_{i=1}^{k+1} \left[a_i(\xi_i^{-1}(x_i))\right]^{\frac{1}{2}}}\mathrm{d}x_1\cdots\mathrm{d}x_j\cdots\mathrm{d}x_{k+1}\\
&\quad\le \int_1^\infty \frac{e^{-2\beta_0\tilde{V}(\xi_j^{-1}(x_j))}}{\left[a_j(\xi_j^{-1}(x_j))\right]^{\frac{1}{2}}}\mathrm{d}x_j\\
&\qquad\times\int_0^\infty\cdots\int_0^\infty \frac{\prod_{\substack{i=1\\ i\neq j}}^{k+1} \exp\left\{-2\beta_0\tilde{V}(\xi_i^{-1}(x_i))\right\}}{\left(\sum_{\substack{i=1\\ i\neq j}}^{k+1}\tilde{\alpha}(x_i)\right) \times \prod_{\substack{i=1\\ i\neq j}}^{k+1} \left[a_i(\xi_i^{-1}(x_i))\right]^{\frac{1}{2}}}\mathrm{d}x_1\cdots\widehat{\mathrm{d}x_j}\cdots\mathrm{d}x_{k+1}\\
&\quad\le A_k\int_1^\infty \frac{e^{-2\beta_0\tilde{V}(\xi_j^{-1}(x_j))}}{\left[a_j(\xi_j^{-1}(x_j))\right]^{\frac{1}{2}}}\mathrm{d}x_j = A_k\int_{\xi_j^{-1}(1)}^\infty \frac{e^{-2\beta_0\tilde{V}(z_j)}}{a_j(z_j)}\mathrm{d}z_j < \infty,
\end{aligned}$$

where we used **(H3)**(2) in the last inequality. It follows that

$$\begin{aligned} A^1_{k+1} \le & \int_0^1 \cdots \int_0^1 \frac{\prod_{i=1}^{k+1} \exp\left\{-2\beta_0 \tilde{V}(\xi_i^{-1}(x_i))\right\}}{\left(\sum_{i=1}^{k=1} \tilde{\alpha}(x_i)\right) \times \prod_{i=1}^{k+1} \left[a_i(\xi_i^{-1}(x_i))\right]^{\frac{1}{2}}} \mathrm{d}x_1 \cdots \mathrm{d}x_{k+1} \\ & + \sum_{j=1}^{k+1} \int_0^\infty \cdots \int_1^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^{k+1} \exp\left\{-2\beta_0 \tilde{V}(\xi_i^{-1}(x_i))\right\}}{\left(\sum_{i=1}^{d} \tilde{\alpha}(x_i)\right) \times \prod_{i=1}^{d} \left[a_i(\xi_i^{-1}(x_i))\right]^{\frac{1}{2}}} \mathrm{d}x_1 \cdots \mathrm{d}x_j \cdots \mathrm{d}x_{k+1} \\ & < \infty. \end{aligned}$$

This completes the proof. □

Proof of Theorem 5.1. (1) Since $\tilde{v}_1 \in L^2(\mathcal{U}, \alpha \mathrm{d}x)$, Lemma 5.1 yields $\int_{\mathcal{U}} \tilde{v}_1 e^{-\frac{Q}{2} - \beta_0 U} \mathrm{d}x < \infty$.

To see $\lambda_1 > 0$, we fix $f \in C_0^\infty(\mathcal{U})$ and set $\tilde{f} := f e^{-\frac{Q}{2} - \beta_0 U}$. Obviously, $\tilde{f} \in L^2(\mathcal{U})$. Theorem 4.1 gives

$$e^{\frac{-Q(x)}{2} - \beta_0 U(x)} \mathbb{E}^x \left[f(X_t) \mathbb{1}_{\{t<S_\Gamma\}} \right] = T_t^* \tilde{f}(x), \quad \forall (x,t) \in \mathcal{U} \times [0,\infty).$$

Setting $v_1 := C \tilde{v}_1 e^{-\frac{Q}{2} - \beta_0 U}$, where $C := \left(\int_{\mathcal{U}} \tilde{v}_1 e^{-\frac{Q}{2} - \beta_0 U} \mathrm{d}x \right)^{-1}$, we deduce

$$\int_{\mathcal{U}} v_1 \mathbb{E}^\bullet \left[f(X_t) \mathbb{1}_{\{t<S_\Gamma\}} \right] \mathrm{d}x = C \langle \tilde{v}_1, T_t^* \tilde{f} \rangle_{L^2} = C \langle T_t \tilde{v}_1, \tilde{f} \rangle_{L^2}, \quad \forall t \ge 0,$$

which together with $T_t \tilde{v}_1 = e^{-\lambda_1 t} \tilde{v}_1$ yields

$$\int_{\mathcal{U}} v_1 \mathbb{E}^\bullet \left[f(X_t) \mathbb{1}_{\{t<S_\Gamma\}} \right] \mathrm{d}x = C e^{-\lambda_1 t} \int_{\mathcal{U}} \tilde{v}_1 \tilde{f} \mathrm{d}x = e^{-\lambda_1 t} \int_{\mathcal{U}} v_1 f \mathrm{d}x, \quad \forall t \ge 0. \tag{5.2}$$

For each $x \in \mathcal{U}$, the fact $\mathbb{P}^x [S_\Gamma < \infty] = 1$ implies $\lim_{t\to\infty} \mathbb{E}^x \left[f(X_t) \mathbb{1}_{\{t<S_\Gamma\}} \right] = 0$. This together with the fact $\sup_{x\in\mathcal{U}} \left| \mathbb{E}^x \left[f(X_t) \mathbb{1}_{\{t<S_\Gamma\}} \right] \right| \le \|f\|_\infty$ for all $t \ge 0$ and the dominated convergence theorem implies $\lim_{t\to\infty} \int_{\mathcal{U}} v_1 \mathbb{E}^\bullet \left[f(X_t) \mathbb{1}_{\{t<S_\Gamma\}} \right] \mathrm{d}x = 0$. From which, we conclude $\lambda_1 > 0$, otherwise a contradiction can be easily derived from (5.2).

(2) Fix $f \in C_b(\mathcal{U})$ and take a sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subset C_0^\infty(\mathcal{U})$ that locally uniformly converges to f as $n \to \infty$ and satisfies $\|f_n\|_\infty \le \|f\|_\infty$ for all $n \in \mathbb{N}$. It follows from (5.2) that for each $t \ge 0$,

$$\int_{\mathcal{U}} \mathbb{E}^\bullet \left[f_n(X_t) \mathbb{1}_{\{t<S_\Gamma\}} \right] \mathrm{d}\nu_1 = e^{-\lambda_1 t} \int_{\mathcal{U}} f_n \mathrm{d}\nu_1, \quad \forall n \in \mathbb{N},$$

where $\mathrm{d}\nu_1 := v_1 \mathrm{d}x$. Letting $n \to \infty$, we conclude the result from the dominated convergence theorem.

(3) Applying (2) with $f = \mathbb{1}_{\mathcal{U}}$, we find $\mathbb{P}^{\nu_1}[t < S_\Gamma] = \mathbb{E}^{\nu_1}\left[\mathbb{1}_{\{t<S_\Gamma\}}\right] = e^{-\lambda_1 t}$ for all $t \geq 0$. Applying (2) again, we conclude

$$\frac{\mathbb{E}^{\nu_1}\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right]}{\mathbb{P}^{\nu_1}[t < S_\Gamma]} = \int_{\mathcal{U}} f \mathrm{d}\nu_1, \quad \forall f \in C_b(\mathcal{U}).$$

That is, ν_1 is a QSD of X_t and λ_1 is the associated extinction rate. □

5.2. **Sharp exponential convergence.** We study the long-time dynamics of X_t before reaching the boundary Γ. Ahead of stating the result, we recall and introduce some notation.

Recall that the spectra of $-\mathcal{L}^*_{\beta_0}$ and $-\mathcal{L}_{\beta_0}$ coincide, are discrete and contained in $\{\lambda \in \mathbb{C} : \Re\lambda > 0 \text{ and } \arg\lambda \leq \theta\}$ for some $\theta \in (0, \frac{\pi}{2})$. The number λ_1 is the principal eigenvalue of both $-\mathcal{L}^*_{\beta_0}$ and $-\mathcal{L}_{\beta_0}$. Let $\tilde{v}^*_1$ be as in Lemma 3.5 (4) and suppose it satisfies the normalization

$$\langle \tilde{v}_1, \tilde{v}^*_1 \rangle_{L^2} = \int_{\mathcal{U}} \tilde{v}_1 e^{-\frac{Q}{2} - \beta_0 U} \mathrm{d}x, \tag{5.3}$$

where $\tilde{v}_1$ is given in Theorem 3.1. The last integral converges thanks to Lemma 5.1. Note that $\mathrm{ran}(\mathcal{P}_1|_{L^2(\mathcal{U})})$ and $\mathrm{ran}(\mathcal{P}^*_1|_{L^2(\mathcal{U})})$ are respectively spanned over $\mathbb{R}$ by $\tilde{v}_1$ and $\tilde{v}^*_1$.

Set $\lambda_2 := \min\left\{\Re\lambda : \lambda \in \sigma(-\mathcal{L}^*_{\beta_0}) \text{ and } \Re\lambda > \lambda_1\right\}$. Then, $\lambda_2 > \lambda_1$ and $\{\lambda \in \sigma(-\mathcal{L}^*_{\beta_0}) : \Re\lambda = \lambda_2\}$ consists of finitely many elements. For $k = 1, 2$, let $\mathcal{P}^*_k$ and $\mathcal{P}_k$ be respectively the spectral projections of $-\mathcal{L}^*_{\beta_0}$ and $-\mathcal{L}_{\beta_0}$ corresponding to $\{\lambda \in \sigma(-\mathcal{L}^*_{\beta_0}) : \Re\lambda = \lambda_k\}$. Clearly, $\mathcal{P}^*_1$ and $\mathcal{P}_1$ are adjoint to each other. Since the coefficients of $-\mathcal{L}^*_{\beta_0}$ and $-\mathcal{L}_{\beta_0}$ are real-valued resulting in the symmetry of the set $\{\lambda \in \sigma(-\mathcal{L}^*_{\beta_0}) : \Re\lambda = \lambda_2\}$ with respect to the real axis, $\mathcal{P}^*_2$ and $\mathcal{P}_2$ are also adjoint to each other.

Suppose the set $\{\lambda \in \sigma(-\mathcal{L}^*_{\beta_0}) : \Re\lambda = \lambda_2\}$ consists of N_* elements and is enumerated as

$$\lambda_{2,i}, \quad i \in \{0, \dots, N_* - 1\}.$$

Denote by $\mathcal{P}^*_{2,i}$ and $\mathcal{P}_{2,i}$ the spectral projections of $-\mathcal{L}^*_{\beta_0}$ and $-\mathcal{L}_{\beta_0}$ corresponding to $\lambda_{2,i}$ and $\overline{\lambda_{2,i}}$, respectively. Note that $\mathcal{P}^*_{2,i}$ and $\mathcal{P}_{2,i}$ are adjoint to each other. Obviously, $\mathcal{P}^*_2 = \sum_{i=0}^{N_*-1} \mathcal{P}^*_{2,i}$ and $\mathcal{P}_2 = \sum_{i=0}^{N_*-1} \mathcal{P}_{2,i}$.

For $i \in \{0, \dots, N_* - 1\}$, we let

- N_i be the order of the pole $\lambda_{2,i}$ of the resolvent of $-\mathcal{L}^*_{\beta_0}$,
- $d_i = \dim(\mathrm{ran}\mathcal{P}^*_{2,i})$,
- $\{\tilde{v}^{(*,2)}_{i,j} : j \in \{1, \dots, d_i\}\}$ and $\{\tilde{v}^{(2)}_{i,j} : j \in \{1, \dots, d_i\}\}$ be generalized eigenfunctions of $-\mathcal{L}^*_{\beta_0}$ and $-\mathcal{L}_{\beta_0}$ that form bases of $\mathrm{ran}\mathcal{P}^*_{2,i}$ and $\mathrm{ran}\mathcal{P}_{2,i}$, respectively, and satisfy the normalization

$$\langle \tilde{v}^{(2)}_{i,j}, \tilde{v}^{(*,2)}_{i,k} \rangle_{L^2} = \delta_{jk}, \quad \forall j, k \in \{1, \dots, d_i\}. \tag{5.4}$$

Recall that ν_1 is the QSD of X_t obtained in Theorem 5.1, and $\{T_t\}_{t\geq 0}$ and $\{T_t^*\}_{t\geq 0}$ are positive and analytic semigroups of contractions on $L^2(\mathcal{U};\mathbb{C})$ generated by $\mathcal{L}_{\beta_0}$ and $\mathcal{L}_{\beta_0}^*$, respectively.

The main result in this subsection is stated in the next theorem.

Theorem 5.2. *Assume* **(H1)**-**(H3)**. *For each $\nu\in\mathcal{P}(\mathcal{U})$ with compact support in $\mathcal{U}$, there holds for each $f\in C_b(\mathcal{U})$,*

$$
\begin{aligned}
&\mathbb{E}^{\nu}[f(X_t)\big|t<S_\Gamma]\\
&\quad=\int_{\mathcal{U}} f\mathrm{d}\nu_1+\frac{e^{\lambda_1 t}}{\int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}\tilde{v}_1^*\mathrm{d}\nu}\int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}T_t^*\mathcal{P}_2^*\left(\tilde{f}-\tilde{\mathbb{1}}_{\mathcal{U}}\int_{\mathcal{U}} f\mathrm{d}\nu_1\right)\mathrm{d}\nu+o(e^{-(\lambda_2-\lambda_1)t})\\
&\quad=\int_{\mathcal{U}} f\mathrm{d}\nu_1+\frac{e^{-(\lambda_2-\lambda_1)t}}{\int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}\tilde{v}_1^*\mathrm{d}\nu}\\
&\qquad\qquad\times\int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}\sum_{j=0}^{N_*-1}e^{-i\Im\lambda_{2,j}t}\sum_{k=0}^{N_j-1}\frac{t^k}{k!}(\mathcal{L}_{\beta_0}^*+\lambda_{2,j})^k\mathcal{P}_{2,j}^*\left(\tilde{f}-\tilde{\mathbb{1}}_{\mathcal{U}}\int_{\mathcal{U}} f\mathrm{d}\nu_1\right)\mathrm{d}\nu\\
&\qquad+o(e^{-(\lambda_2-\lambda_1)t})\quad as\quad t\to\infty,
\end{aligned}
$$

where $\tilde{f}:=e^{-\frac{Q}{2}-\beta_0 U}f$ and $\tilde{\mathbb{1}}_{\mathcal{U}}:=e^{-\frac{Q}{2}-\beta_0 U}\mathbb{1}_{\mathcal{U}}$.

In particular, the following hold:

- *For each $0<\epsilon\ll 1$,*
$$\lim_{t\to\infty} e^{(\lambda_2-\lambda_1-\epsilon)t}\left|\mathbb{E}^x[f(X_t)\big|t<S_\Gamma]-\int_{\mathcal{U}} f\mathrm{d}\nu_1\right|=0,\quad \forall f\in C_b(\mathcal{U})\ and\ x\in\mathcal{U}.$$
- *If $f\in C_b(\mathcal{U})$ is such that $\mathcal{P}_2^*\left(\tilde{f}-\tilde{\mathbb{1}}_{\mathcal{U}}\int_{\mathcal{U}} f\mathrm{d}\nu_1\right)\neq 0$, then for a.e. $x\in\mathcal{U}$, there is a discrete set $\mathcal{I}_x\subset(0,\infty)$ with distances between adjacent points admitting an x-independent positive lower bound, such that for each $0<\delta\ll 1$ there holds*
$$\lim_{\substack{t\to\infty\\ t\in(0,\infty)\setminus\mathcal{I}_{x,\delta}}} e^{(\lambda_2-\lambda_1+\epsilon)t}\left|\mathbb{E}^x[f(X_t)\big|t<S_\Gamma]-\int_{\mathcal{U}} f\mathrm{d}\nu_1\right|=\infty,\quad \forall 0<\epsilon\ll 1,$$
where $\mathcal{I}_{x,\delta}$ is the δ-neighbourhood of $\mathcal{I}_x$ in $(0,\infty)$.

Remark 5.1. *We make some remarks about Theorem 5.2.*

(1) *Theorem 5.2 appears to be a direct consequence of the decomposition of $(T_t^*)_{t\geq 0}$ according to spectral projections ensured by Theorem 3.3 and the stochastic representation given in Theorem 4.1. This is however deceptive due to the following two reasons: (i) the stochastic representation given in Theorem 4.1 is only true for $f\in C_b(\mathcal{U})$ such that $e^{-\frac{Q}{2}-\beta_0 U}f\in L^2(\mathcal{U})$; this is indeed a restriction as*

$e^{-\frac{Q}{2}-\beta_0 U}$ and $e^{\frac{Q}{2}+\beta_0 U}$ are respectively unbounded near Γ and ∞; (ii) the semigroup $(T_t^*)_{t\geq 0}$ is naturally defined on $L^2(\mathcal{U})$, but we need its L^∞ properties.

(2) *For $f \in C_b(\mathcal{U})$, the function $\tilde{f} := e^{-\frac{Q}{2}-\beta_0 U} f$ does not necessarily belong to $L^2(\mathcal{U})$. Neither does $\tilde{f} - \tilde{\mathbb{1}}_{\mathcal{U}} \int_{\mathcal{U}} f \mathrm{d}\nu_1$. Its projections under $\mathcal{P}_2^*$ and $\mathcal{P}_{2,j}^*$ are justified in Lemma 5.3 (2).*

(3) *Theorem 5.2 actually holds for all initial distributions $\nu \in \mathcal{P}(\mathcal{U})$ satisfying the condition $\int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U} \mathrm{d}\nu < \infty$. See Remark 5.2 for more details.*

We prove several lemmas before proving Theorem 5.2.

Lemma 5.2. *Assume* **(H1)**-**(H3)**.

(1) *One has*

$$T_t^*\mathcal{P}_2^* = \sum_{j=0}^{N_*-1} T_t^*\mathcal{P}_{2,j}^* = e^{-\lambda_2 t} \sum_{j=0}^{N_*-1} e^{-i\Im\lambda_{2,j} t} \sum_{k=0}^{N_j-1} \frac{t^k}{k!} (\mathcal{L}_{\beta_0}^* + \lambda_{2,j})^k \mathcal{P}_{2,j}^*, \quad \forall t \geq 0.$$

(2) *For each $0 < \epsilon \ll 1$, there exists $C = C(\epsilon) > 0$ such that*

$$\left\| T_t^* - e^{-\lambda_1 t}\mathcal{P}_1^* - T_t^*\mathcal{P}_2^* \right\|_{L^2 \to L^2} \leq C e^{-(\lambda_2+\epsilon)t}, \quad \forall t \geq 0.$$

(3) *For each $0 < \epsilon \ll 1$ and $f \in \operatorname{ran}\mathcal{P}_2^* \setminus \{0\}$ we have*

$$\lim_{t\to\infty} e^{(\lambda_2-\epsilon)t} |T_t^* f| = 0 \quad in \quad \mathcal{U}.$$

(4) *Let $f \in \operatorname{ran}\mathcal{P}_2^* \setminus \{0\}$. Then, for a.e. $x \in \mathcal{U}$, there is a discrete set $\mathcal{I}_x \subset (0,\infty)$ with distances between adjacent points admitting an x-independent positive lower bound, such that for each $0 < \delta \ll 1$ one has*

$$\lim_{\substack{t\to\infty \\ t\in(0,\infty)\setminus\mathcal{I}_{x,\delta}}} e^{(\lambda_2+\epsilon)t} |T_t^* f|(x) = \infty, \quad \forall 0 < \epsilon \ll 1,$$

where $\mathcal{I}_{x,\delta}$ is the δ-neighbourhood of $\mathcal{I}_x$ in $(0,\infty)$.

Proof. (1) and (2) are special cases of [19, Corollary V. 3.2] due to Theorem 3.3, the fact $\Re\lambda_{2,i} = \lambda_2$ for all $i \in \{1, \dots, N_*\}$, and the simplicity of the principle eigenvalue λ_1 of $-\mathcal{L}_{\beta_0}^*$. (3) is a simple consequence of (1).

We show (4). Fix $f \in \operatorname{ran}\mathcal{P}_2^* \setminus \{0\}$. We consider three cases.

Case 1. $N_* = 1$. In this case, $\{\lambda \in \sigma(-\mathcal{L}_{\beta_0}^*) : \Re\lambda = \lambda_2\} = \{\lambda_2\}$. Then, f is a generalized eigenfunction of $-\mathcal{L}_{\beta_0}^*$ associated to λ_2, and thus, there exists $\tilde{N} \in \mathbb{N}$ such that $(\mathcal{L}_{\beta_0}^* + \lambda_2)^{\tilde{N}+1} f = 0$ and $(\mathcal{L}_{\beta_0}^* + \lambda_2)^{\tilde{N}} f \neq 0$ in $\mathcal{U}$. By the strong unique continuation principle for elliptic equations (see e.g. [33]), we find $(\mathcal{L}_{\beta_0}^* + \lambda_2)^{\tilde{N}} f \neq 0$ a.e. in $\mathcal{U}$. Since

$$T_t^* f = e^{-\lambda_2 t} \sum_{k=0}^{\tilde{N}} \frac{t^k}{k!} (\mathcal{L}_{\beta_0}^* + \lambda_2)^k f = e^{-\lambda_2 t} \left(\frac{t^{\tilde{N}}}{\tilde{N}!} (\mathcal{L}_{\beta_0}^* + \lambda_2)^{\tilde{N}} f + o(t^{\tilde{N}}) \right) \quad \text{as} \quad t \to \infty,$$

we derive $\lim_{t\to\infty} e^{(\lambda_2+\epsilon)t}|T_t^* f|(x) = \infty$ for a.e. $x \in \mathcal{U}$ and each $0 < \epsilon \ll 1$. The conclusion follows immediately.

Case 2. $N_* = 2K+1$ for some $K \in \mathbb{N}$. Considering the symmetry of the set $\{\lambda \in \sigma(-\mathcal{L}_{\beta_0}^*) : \Re\lambda = \lambda_2\}$ with respect to the real axis, we can re-enumerate it as $\{\lambda_{2,j}\}_{j=-K}^{K}$ such that $\lambda_{2,0} = \lambda_2$ and $\lambda_{2,j} = \overline{\lambda}_{2,-j}$ for $j \in \{1, \dots, K\}$.

Note that $f = \sum_{j=-K}^{K} f_j$, where f_j is the projection of f onto the generalized eigenspace of $\lambda_{2,j}$. Since f is real-valued we must have $f_j = \overline{f}_{-j}$ for all $j \in \{1, \dots, K\}$. We may assume, without loss of generality, that $f_j \neq 0$ for all $j \in \{-K, \dots, K\}$.

Since $\lambda_{2,j}$ is a pole of the resolvent of $-\mathcal{L}_{\beta_0}^*$ with finite order, there exists $\tilde{N}_j \in \mathbb{N}$ such that $(\mathcal{L}_{\beta_0}^* + \lambda_{2,j})^{\tilde{N}_j+1} f_j = 0$ and $(\mathcal{L}_{\beta_0}^* + \lambda_{2,j})^{\tilde{N}_j} f_j \neq 0$. Applying the strong unique continuation principle for elliptic equations (see e.g. [33]), we find

$$(\mathcal{L}_{\beta_0}^* + \lambda_{2,j})^{\tilde{N}_j} f_j \neq 0 \quad \text{a.e. in} \quad \mathcal{U}. \tag{5.5}$$

Clearly, $\tilde{N}_j = \tilde{N}_{-j}$ for all $j \in \{1, \dots, K\}$. Straightforward calculations then give for $t \gg 1$,

$$\begin{aligned} e^{\lambda_2 t} T_t^* f &= \sum_{j=-K}^{K} e^{-\Im\lambda_{2,j} t} \sum_{k=0}^{\tilde{N}_j} \frac{t^k}{k!} (\mathcal{L}_{\beta_0}^* + \lambda_{2,j})^k f_j \\ &= \sum_{j=-K}^{K} e^{-i\Im\lambda_{2,j} t} \left[\frac{t^{\tilde{N}_j}}{\tilde{N}_j!} (\mathcal{L}_{\beta_0}^* + \lambda_{2,j})^{\tilde{N}_j} f_j + o(t^{\tilde{N}_j}) \right] \\ &= \left[\frac{t^{\tilde{N}_0}}{\tilde{N}_0!} (\mathcal{L}_{\beta_0}^* + \lambda_2)^{\tilde{N}_0} f_0 + o(t^{\tilde{N}_0}) \right] \\ &\quad + \sum_{j=1}^{K} \left[\frac{2t^{\tilde{N}_j}}{\tilde{N}_j!} \Re \left(e^{-i\Im\lambda_{2,j} t} (\mathcal{L}_{\beta_0}^* + \lambda_{2,j})^{\tilde{N}_j} f_j \right) + o(t^{\tilde{N}_j}) \right]. \end{aligned} \tag{5.6}$$

Since the asymptotics of $e^{\lambda_2 t} T_t^* f$ as $t \to \infty$ is determined by the terms with the highest degree, we may assume, without loss of generality, that $\tilde{N}_0 = \tilde{N}_1 = \cdots = \tilde{N}_K$.

Set $F_0 := \frac{1}{\tilde{N}_0!}(\mathcal{L}^*_{\beta_0}+\lambda_2)^{\tilde{N}_0} f_0$ and $F_j := \frac{2}{\tilde{N}_j!}(\mathcal{L}^*_{\beta_0}+\lambda_{2,j})^{\tilde{N}_j} f_j$ for $j\in\{1,\dots,K\}$. We rewrite (5.6) as

$$\begin{aligned}\frac{e^{\lambda_2 t}T^*_t f}{t^{\tilde{N}_0}} &= F_0+\sum_{j=1}^K \Re(e^{-i\Im\lambda_{2,j}t}F_j)+o(1)\\ &= F_0+\sum_{j=1}^K (\cos(\Im\lambda_{2,j}t)\Re F_j-\sin(\Im\lambda_{2,j}t)\Im F_j)+o(1) \qquad (5.7)\\ &= F_0+\sum_{j=1}^K |F_j|\sin(\Im\lambda_{2,j}t+\varphi_j)+o(1),\quad \forall t\gg 1,\end{aligned}$$

where $\varphi_j\in[0,2\pi)$ satisfies $\tan\varphi_j=-\frac{\Re F_j}{\Im F_j}$ for $j\in\{1,\dots,K\}$.

Note that (5.5) ensures the set $\mathcal{N}:=\{x\in\mathcal{U}:\exists j\in\{0,1,\dots,K\}\text{ s.t. } |F_j|(x)=0\}$ has zero Lebesgue measure. Fix $x\in\mathcal{U}\setminus\mathcal{N}$. Denote by $\mathcal{I}_x$ the zeros of the function $t\mapsto \sum_{j=1}^K |F_j|(x)\sin(\Im\lambda_{2,j}t+\varphi_j):(0,\infty)\to\mathbb{R}$. It is not hard to see that $\mathcal{I}_x$ is a discrete set with distances between adjacent points admitting a positive lower bound, which is independent of $x\in\mathcal{U}\setminus\mathcal{N}$.

For each $0<\delta\ll 1$, we set $\mathcal{I}_{x,\delta}:=\{t\in(0,\infty):\operatorname{dist}(t,\mathcal{I}_x)<\delta\}$ and conclude from (5.7) that

$$\lim_{\substack{t\to\infty\\ t\in(0,\infty)\setminus\mathcal{I}_{x,\delta}}} e^{(\lambda_2+\epsilon)t}|T^*_t f|(x)=\infty,\quad \forall 0<\epsilon\ll 1.$$

Case 3. $N_*=2K$ for some $K\in\mathbb{N}$. The proof is exactly the same as that in **Case 2** except that f_0 does not appear due to the fact $\lambda_2\notin\{\lambda\in\sigma(-\mathcal{L}^*_{\beta_0}):\Re\lambda=\lambda_2\}$.

This completes the proof. □

Lemma 5.3. *Assume* **(H1)-(H3)**. *The following hold:*

1. $\mathcal{P}^*_1\tilde{f}=\tilde{v}^*_1\int_{\mathcal{U}}\tilde{f}e^{\frac{Q}{2}+\beta_0 U}d\nu_1$ *for all* $\tilde{f}\in L^2(\mathcal{U})$.
2. *For each* $i\in\{1,\dots,N_*\}$ *one has*

$$\mathcal{P}^*_{2,i}\tilde{f}=\sum_{j=1}^{d_i}\tilde{v}^{(*,2)}_{i,j}\langle\tilde{f},\tilde{v}^{(2)}_{i,j}\rangle_{L^2},\quad \forall\tilde{f}\in L^2(\mathcal{U}). \qquad (5.8)$$

In particular, $\mathcal{P}^*_{2,i}$, $i\in\{1,\dots,N_*\}$, *and hence,* $\mathcal{P}^*_2$ *are well-defined on* $\{\tilde{f}:\tilde{f}e^{\frac{Q}{2}+\beta_0 U}\in C_b(\mathcal{U})\}$.

Proof. (1) Note that $\operatorname{ran}(\mathcal{P}^*_1|_{L^2(\mathcal{U})})$ is spanned over $\mathbb{R}$ by $\tilde{v}^*_1$. By the Riesz representation theorem, there exists $h\in L^2(\mathcal{U})$ such that

$$\mathcal{P}^*_1\tilde{f}=\langle\tilde{f},h\rangle_{L^2}\tilde{v}^*_1,\quad\forall\tilde{f}\in L^2(\mathcal{U}). \qquad (5.9)$$

As $\mathcal{P}_1$ and $\mathcal{P}_1^*$ are adjoint to each other it must be true that $\mathcal{P}_1\tilde{v} = \langle \tilde{v}, \tilde{v}_1^* \rangle_{L^2} h$ for all $\tilde{v} \in L^2(\mathcal{U})$. Since $\mathrm{ran}(\mathcal{P}_1|_{L^2(\mathcal{U})})$ is spanned over $\mathbb{R}$ by $\tilde{v}_1$, there exists $C_1 \in \mathbb{R}$ such that $h = C_1\tilde{v}_1$. Thus, the normalization (5.3) gives

$$\tilde{v}_1 = \mathcal{P}_1\tilde{v}_1 = C_1\langle \tilde{v}_1, \tilde{v}_1^* \rangle_{L^2}\tilde{v}_1 = C_1\tilde{v}_1 \int_{\mathcal{U}} \tilde{v}_1 e^{-\frac{Q}{2}-\beta_0 U}\mathrm{d}x,$$

leading to $C_1 = \frac{1}{\int_{\mathcal{U}} \tilde{v}_1 e^{-\frac{Q}{2}-\beta_0 U}\mathrm{d}x}$, and hence, $h = \frac{\tilde{v}_1}{\int_{\mathcal{U}} \tilde{v}_1 e^{-\frac{Q}{2}-\beta_0 U}\mathrm{d}x}$. Inserting this into (5.9) and noting the definition of ν_1 gives rise to the formula for $\mathcal{P}_1^*\tilde{f}$.

(2) The formula for $\mathcal{P}_{2,i}^*\tilde{f}$ can be derived from arguments similar to those in the proof of (1), and the normalization (5.4).

As Lemma 5.1 ensures $C_{ij} := \int_{\mathcal{U}} e^{-\frac{Q}{2}-\beta_0 U}|\tilde{v}_{i,j}^{(2)}|\mathrm{d}x < \infty$, we find for each $\tilde{f}$ satisfying $\tilde{f}e^{\frac{Q}{2}+\beta_0 U} \in C_b(\mathcal{U})$ that $\left|\langle \tilde{f}, \tilde{v}_{i,j}^{(2)} \rangle_{L^2}\right| \leq C_{ij}\sup_{\mathcal{U}}\left|\tilde{f}e^{\frac{Q}{2}+\beta_0 U}\right|$. Thus, using the formula (5.8), we can define $\mathcal{P}_{2,i}^*$, $i \in \{1, \dots, d_i\}$, and hence, $\mathcal{P}_2^*$ on the set $\left\{\tilde{f} : \tilde{f}e^{\frac{Q}{2}+\beta_0 U} \in C_b(\mathcal{U})\right\}$. This completes the proof. □

Lemma 5.4. *Assume* **(H1)**-**(H3)**. *For each $t > 0$, there exists $C = C(t) > 0$ such that*

$$\left\| e^{-\frac{Q}{2}-\beta_0 U}\mathbb{E}^\bullet\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right]\right\|_{L^2} \leq C\|f\|_\infty, \quad \forall f \in C_b(\mathcal{U}).$$

Proof. Fix $f \in C_b(\mathcal{U})$ and set $\tilde{f} := fe^{-\frac{Q}{2}-\beta_0 U}$. Recall that $2_* := \frac{2(d+2)}{d+4} \in (1,2)$ (see Lemma 4.2). Since $e^{-\frac{Q(x)}{2}} = \frac{\left[\prod_{i=1}^d a_i(\xi_i^{-1}(1))\right]^{\frac{1}{4}}}{\left[\prod_{i=1}^d a_i(\xi_i^{-1}(x_i))\right]^{\frac{1}{4}}}$, we find

$$\int_{\mathcal{U}} |\tilde{f}|^{2_*}\mathrm{d}x = \int_{\mathcal{U}} |f|^{2_*}e^{-\frac{2_*Q}{2}-2_*\beta_0 U}\mathrm{d}x \leq \|f\|_\infty^{2_*}\left[\prod_{i=1}^d a_i(\xi_i^{-1}(1))\right]^{\frac{2_*}{4}} \int_{\mathcal{U}} \frac{e^{-2_*\beta_0 U}}{\left[\prod_{i=1}^d a_i(\xi_i^{-1}(x_i))\right]^{\frac{2_*}{4}}}\mathrm{d}x.$$

Arguments as in the proof of Lemma 5.1 yield $\int_{\mathcal{U}} \frac{e^{-2_*\beta_0 U(x)}}{\left[\prod_{i=1}^d a_i(\xi_i^{-1}(x_i))\right]^{\frac{2_*}{4}}}\mathrm{d}x < \infty$. This implies the existence of $C_1 > 0$ (independent of f) such that

$$\|\tilde{f}\|_{L^{2_*}(\mathcal{U})} \leq C_1\|f\|_\infty. \tag{5.10}$$

Let $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ and $\{\tau_n\}_{n\in\mathbb{N}}$ be as in Subsection 2.3. For each $n \in \mathbb{N}$, we recall that $(T_t^{(*,\mathcal{U}_n,2_*)})_{t\geq 0}$ is the positive and analytic semigroup of contractions on $L^{2_*}(\mathcal{U}_n;\mathbb{C})$ generated by $(\mathcal{L}_{\beta_0}^{*,2_*}|_{\mathcal{U}_n}, W^{2,2_*}(\mathcal{U}_n;\mathbb{C}) \cap W_0^{1,2_*}(\mathcal{U}_n;\mathbb{C}))$. Since $\tilde{f} \in C(\overline{\mathcal{U}}_n)$, Proposition 4.1 ensures

$$T_t^{(*,\mathcal{U}_n,2_*)}\tilde{f}|_{\mathcal{U}_n} = e^{-\frac{Q}{2}-\beta_0 U}\mathbb{E}^\bullet\left[f(X_t)\mathbb{1}_{\{t<\tau_n\}}\right], \quad \forall t \in [0,\infty). \tag{5.11}$$

It follows from Lemma 4.2 that for each $t>0$, there is a constant $C_2=C_2(t)>0$ such that $\|T_t^{(*,\mathcal{U}_n,2_*)}\tilde{f}\|_{L^2(\mathcal{U}_n)}\leq C_2\|\tilde{f}\|_{L^{2*}(\mathcal{U}_n)}$ for all $n\in\mathbb{N}$. Letting $n\to\infty$, we derive from Lemma 2.1, (5.11) and Fatou's lemma that

$$\|e^{-\frac{Q}{2}-\beta_0 U}\mathbb{E}^\bullet\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right]\|_{L^2(\mathcal{U})}\leq C_2\|\tilde{f}\|_{L^{2*}(\mathcal{U})}\leq C_1C_2\|f\|_\infty,$$

where we used (5.10) in the second inequality. This completes the proof. □

Lemma 5.5. *Assume* **(H1)-(H3)***. There exists $C>0$ such that for each $f\in C_b(\mathcal{U})$ with $\tilde{f}:=e^{-\frac{Q}{2}-\beta_0 U}f\in L^2(\mathcal{U})$ one has $\|T_t^*\tilde{f}\|_\infty\leq C\|T_{t-1}^*\tilde{f}\|_{L^2}$ for all $t\geq 1$.*

Proof. Fix $f\in C_b(\mathcal{U})$ satisfying $\tilde{f}:=e^{-\frac{Q}{2}-\beta_0 U}f\in L^2(\mathcal{U})$. Theorem 4.1 gives

$$T_t^*\tilde{f}(x)=e^{-\frac{Q(x)}{2}-\beta_0 U(x)}\mathbb{E}^x\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right],\quad \forall (x,t)\in\mathcal{U}\times[0,\infty). \tag{5.12}$$

Let $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ and $\{\tau_n\}_{n\in\mathbb{N}}$ be as in Subsection 2.3. We show the existence of $C>0$ such that

$$\begin{aligned}&\sup_{\mathcal{U}_n} e^{-\frac{Q}{2}-\beta_0 U}\left|\mathbb{E}^\bullet[f(X_t)\mathbb{1}_{\{t<\tau_n\}}\right|\\ &\quad\leq C\left\|e^{-\frac{Q}{2}-\beta_0 U}\mathbb{E}^\bullet\left[f(X_{t-1})\mathbb{1}_{\{t-1<\tau_n\}}\right]\right\|_{L^2(\mathcal{U}_n)},\quad \forall t\geq 1 \text{ and } n\in\mathbb{N}.\end{aligned}\tag{5.13}$$

The lemma then follows immediately from (5.12) and Lemma 2.1.

We show (5.13) by Moser iteration. Recall that for each $n\in\mathbb{N}$ and $N>1$, $(T_t^{(*,\mathcal{U}_n,N)})_{t\geq 0}$ is the positive and analytic semigroup on $L^N(\mathcal{U}_n;\mathbb{C})$ generated by $(\mathcal{L}^{*,N}_{\beta_0}|_{\mathcal{U}_n}, W^{2,N}(\mathcal{U}_n;\mathbb{C})\cap W^{1,N}_0(\mathcal{U}_n;\mathbb{C}))$. Since here for each n we only consider the action of $(T_t^{(*,\mathcal{U}_n,N)})_{t\geq 0}$ on functions in $C(\overline{\mathcal{U}}_n;\mathbb{C})$, we simply write $(T_t^{(n)})_{t\geq 0}$ for all $\left\{(T_t^{(*,\mathcal{U}_n,N)})_{t\geq 0}, N>1\right\}$ in consideration of Proposition 4.1 (5). Obviously, $\tilde{f}_n:=\tilde{f}|_{\mathcal{U}_n}\in C(\overline{\mathcal{U}}_n)$ for all $n\in\mathbb{N}$. It follows from Proposition 4.1 (4) that

$$T_t^{(n)}\tilde{f}_n(x)=e^{-\frac{Q(x)}{2}-\beta_0 U(x)}\mathbb{E}^x\left[f(X_t)\mathbb{1}_{\{t<\tau_n\}}\right],\quad \forall (x,t)\in\mathcal{U}_n\times[0,\infty) \text{ and } n\in\mathbb{N}. \tag{5.14}$$

Set $\tilde{w}_n:=T_\bullet^{(n)}\tilde{f}_n$. It follows from Lemma 4.1 that for all $n\in\mathbb{N}$ and $N\geq 2$,

$$\begin{aligned}&\frac{1}{N}\int_{\mathcal{U}_n}|\tilde{w}_n|^N(\cdot,t_2)\mathrm{d}x+\frac{N-1}{2}\int_{t_1}^{t_2}\int_{\mathcal{U}_n}|\tilde{w}_n|^{N-2}|\nabla\tilde{w}_n|^2\mathrm{d}x\mathrm{d}s\\ &\qquad\leq\frac{1}{N}(1+e^{NM(t_2-t_1)})\int_{\mathcal{U}_n}|\tilde{w}(\cdot,t_1)|^N\mathrm{d}x,\quad \forall t_2>t_1\geq 0,\end{aligned}\tag{5.15}$$

where we recall that $M>0$ is fixed and independent of $n\in\mathbb{N}$ and $N\geq 2$, such that the conclusion in Lemma 3.2 (3) holds. The Sobolev embedding theorem gives

$$\|\tilde{w}_n^{\frac{N}{2}}\|_{L^{2\kappa}(\mathcal{U}_n\times[t_1,t_2])}\leq C_1\left(\sup_{s\in[t_1,t_2]}\|\tilde{w}_n^{\frac{N}{2}}(\cdot,s)\|_{L^2(\mathcal{U}_n)}+\|\nabla\tilde{w}_n^{\frac{N}{2}}\|_{L^2(\mathcal{U}_n\times[t_1,t_2])}\right),$$

where $\kappa := \frac{d+2}{d}$ and $C_1 > 0$ only depends on d. Therefore, (5.15) gives rise to

$$\begin{aligned}
&\left(\int_{t_1}^{t_2}\int_{\mathcal{U}_n} |\tilde{w}_n|^{\kappa N} \mathrm{d}x\mathrm{d}s\right)^{\frac{1}{\kappa}} \\
&\quad \leq 2C_1^2 \left(\sup_{s\in[t_1,t_2]}\int_{\mathcal{U}_n} |\tilde{w}_n(x,s)|^N \mathrm{d}x + \frac{N^2}{4}\int_{t_1}^{t_2}\int_{\mathcal{U}_n} |\tilde{w}_n|^{N-2}|\nabla \tilde{w}_n|^2 \mathrm{d}x\mathrm{d}s\right) \\
&\quad \leq 2C_1^2\left(1+\frac{N}{2(N-1)}\right)(1+e^{NM(t_2-t_1)})\int_{\mathcal{U}_n}|\tilde{w}(\cdot,t_1)|^N\mathrm{d}x \\
&\quad \leq 4C_1^2(1+e^{NM(t_2-t_1)})\int_{\mathcal{U}_n}|\tilde{w}(\cdot,t_1)|^N\mathrm{d}x, \quad \forall t_2>t_1\geq 0,
\end{aligned} \tag{5.16}$$

for all $n\in\mathbb{N}$ and $N\geq 2$. We then deduce from Lemma 4.1 (with κN instead of N) and (5.16) that

$$\begin{aligned}
\frac{1}{\kappa N}\int_{\mathcal{U}_n}|\tilde{w}_n|^{\kappa N}(\cdot,t_3)\mathrm{d}x &\leq \frac{2}{\kappa N(t_2-t_1)}\int_{t_1}^{t_2}\int_{\mathcal{U}_n}|\tilde{w}_n|^{\kappa N}\mathrm{d}x\mathrm{d}s \\
&\leq \frac{2(4C_1^2)^\kappa}{\kappa N(t_2-t_1)}\left(1+e^{NM(t_2-t_1)}\right)^\kappa \|\tilde{w}(\cdot,t_1)\|_{L^N(\mathcal{U}_n)}^{\kappa N}
\end{aligned} \tag{5.17}$$

for all $t_3>t_2>t_1\geq 0$, $n\in\mathbb{N}$ and $N\geq 2$.

Fix $t\geq 1$. For each $\ell\in\mathbb{N}\cup\{0\}$, we set $N=N_\ell:=2\kappa^\ell$, $t_1:=t-2^{-\ell}$, $t_2:=t-\frac{3}{2}2^{-(\ell+1)}$ and $t_3:=t-2^{-(\ell+1)}$ in (5.17) to find

$$\|\tilde{w}_n(\cdot,t-2^{-(\ell+1)})\|_{L^{N_{\ell+1}}(\mathcal{U}_n)} \leq C_2^{\frac{1}{N_{\ell+1}}} 2^{\frac{\ell+2}{N_{\ell+1}}} e^{M2^{-(\ell+1)}} \|\tilde{w}_n(\cdot,t-2^{-\ell})\|_{L^{N_\ell}(\mathcal{U}_n)} \tag{5.18}$$

for all $\ell\in\mathbb{N}\cup\{0\}$ and $n\in\mathbb{N}$, where $C_2>0$ is independent of ℓ and n. Set

$$A_\ell := C_2^{\frac{1}{N_{\ell+1}}} 2^{\frac{\ell+2}{N_{\ell+1}}} e^{M2^{-(\ell+1)}}, \quad \ell\in\mathbb{N}\cup\{0\}.$$

It follows from (5.18) that for each $n\in\mathbb{N}$,

$$\begin{aligned}
\sup_{x\in\mathcal{U}_n}|\tilde{w}_n(x,t)| &= \lim_{k\to\infty}\|\tilde{w}_n(\cdot,1-2^{-(k+1)})\|_{L^{N_{k+1}}} \\
&\leq \lim_{k\to\infty}\left(\prod_{\ell=0}^k A_\ell\right)\times\|\tilde{w}_n(\cdot,t-1)\|_{L^2(\mathcal{U}_n)} = C_3\|\tilde{w}_n(\cdot,t-1)\|_{L^2(\mathcal{U}_n)},
\end{aligned}$$

where $C_3:=\prod_{\ell=0}^\infty A_\ell<\infty$. This, together with (5.14), gives (5.13). This completes the proof. □

Corollary 5.1. *Assume* **(H1)**-**(H3)**. *There exists $C>0$ such that for each $\tilde{f}\in L^2(\mathcal{U})$ one has $\|T_t^*\tilde{f}\|_\infty\leq C\|T_{t-1}^*\tilde{f}\|_{L^2}$ for all $t\geq 1$.*

Proof. By Lemma 5.5, the conclusion holds for all $\tilde{f} \in C_0^\infty(\mathcal{U})$. Thus, the density of $C_0^\infty(\mathcal{U})$ is in $L^2(\mathcal{U})$ and the standard approximation arguments give the desired result. □

Lemma 5.6. *Assume* **(H1)**-**(H3)**. *For each $0 < \epsilon \ll 1$, there exists $C = C(\epsilon) > 0$ such that for each $f \in C_b(\mathcal{U})$ it is true that*

$$\left|\mathbb{E}^\bullet[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}] - e^{\frac{Q}{2}+\beta_0 U}\left(e^{-\lambda_1 t}\tilde{v}_1^*\int_{\mathcal{U}} f \mathrm{d}\nu_1 + T_t^*\mathcal{P}_2^*\tilde{f}\right)\right| \le Ce^{\frac{Q}{2}+\beta_0 U}e^{-(\lambda_2+\epsilon)t}\|f\|_\infty$$

for all $t \ge 2$, where $\tilde{f} := e^{-\frac{Q}{2}-\beta_0 U} f$.

Proof. Fix $f \in C_b(\mathcal{U})$. By the Markov property and homogeneity of X_t,

$$\mathbb{E}^x\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right] = \mathbb{E}^x\left[g(X_{t-1})\mathbb{1}_{\{t-1<S_\Gamma\}}\right], \quad \forall (x,t) \in \mathcal{U}\times[1,\infty), \tag{5.19}$$

where $g := \mathbb{E}^\bullet[f(X_1)\mathbb{1}_{\{1<S_\Gamma\}}] \in C_b(\mathcal{U})$. Lemma 5.4 ensures that $\tilde{g} := e^{-\frac{Q}{2}-\beta_0 U} g \in L^2(\mathcal{U})$ and the existence of $C_1 > 0$ (independent of f) such that

$$\|\tilde{g}\|_{L^2} \le C_1\|f\|_\infty. \tag{5.20}$$

Fix $0 < \epsilon \ll 1$. By Lemma 5.2 (2), there exists $C_2 = C_2(\epsilon) > 0$ such that

$$\left\|T_{t-2}^*\tilde{g} - T_{t-2}^*\mathcal{P}_1^*\tilde{g} - T_{t-2}^*\mathcal{P}_2^*\tilde{g}\right\|_{L^2} \le C_2 e^{-(\lambda_2+\epsilon)(t-2)}\|\tilde{g}\|_{L^2}, \quad \forall t \ge 2.$$

Thanks to Lemma 5.5, there exists $C_3 > 0$ (independent of f) such that

$$\begin{aligned}\left\|T_{t-1}^*\tilde{g} - T_{t-1}^*\mathcal{P}_1^*\tilde{g} - T_{t-1}^*\mathcal{P}_2^*\tilde{g}\right\|_\infty &\le C_3\left\|T_{t-2}^*\tilde{g} - T_{t-2}^*\mathcal{P}_1^*\tilde{g} - T_{t-2}^*\mathcal{P}_2^*\tilde{g}\right\|_{L^2} \\ &\le C_2C_3e^{-(\lambda_2+\epsilon)(t-2)}\|\tilde{g}\|_{L^2} \\ &\le C_1C_2C_3e^{2(\lambda_2+\epsilon)}e^{-(\lambda_2+\epsilon)t}\|f\|_\infty \\ &=: C_4e^{-(\lambda_2+\epsilon)t}\|f\|_\infty, \quad \forall t \ge 2,\end{aligned} \tag{5.21}$$

where we used (5.20) in the third inequality, and $C_4 := C_1C_2C_3e^{2(\lambda_2+\epsilon)}$.

We treat the terms $T_{t-1}^*\tilde{g}$, $T_{t-1}^*\mathcal{P}_1^*\tilde{g}$ and $T_{t-1}^*\mathcal{P}_2^*\tilde{g}$ on the left-hand side of (5.21) to finish the proof. It follows from Theorem 4.1 and (5.19) that

$$T_{t-1}^*\tilde{g} = e^{-\frac{Q}{2}-\beta_0 U}\mathbb{E}^\bullet[g(X_{t-1})\mathbb{1}_{\{t-1<S_\Gamma\}}] = e^{-\frac{Q}{2}-\beta_0 U}\mathbb{E}^\bullet[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}], \quad \forall t \ge 1. \tag{5.22}$$

Noting that Lemma 5.3 (1) and Theorem 5.1 (2) give

$$\mathcal{P}_1^*\tilde{g} = \tilde{v}_1^*\int_{\mathcal{U}}\mathbb{E}^\bullet[f(X_1)\mathbb{1}_{\{1<S_\Gamma\}}]\mathrm{d}\nu_1 = \tilde{v}_1^*e^{-\lambda_1}\int_{\mathcal{U}} f\mathrm{d}\nu_1,$$

we deduce

$$T_{t-1}^*\mathcal{P}_1^*\tilde{g} = T_{t-1}^*\tilde{v}_1^*e^{-\lambda_1}\int_{\mathcal{U}} f\mathrm{d}\nu_1 = e^{-\lambda_1 t}\tilde{v}_1^*\int_{\mathcal{U}} f\mathrm{d}\nu_1, \quad \forall t \ge 1. \tag{5.23}$$

For the term $T^*_{t-1}\mathcal{P}^*_2\tilde{g}$, we show that

$$T^*_{t-1}\mathcal{P}^*_2\tilde{g} = T^*_t\mathcal{P}^*_2\tilde{f}, \quad \forall t \geq 1, \tag{5.24}$$

where $\tilde{f} := e^{-\frac{Q}{2}-\beta_0 U}f$. Due to Lemma 5.2 (1), (5.24) holds if we show $\mathcal{P}^*_{2,i}\tilde{g} = T^*_1\mathcal{P}^*_{2,i}\tilde{f}$ for all $i \in \{0, \dots, N_* - 1\}$.

Recall $\tilde{g} = e^{-\frac{Q}{2}-\beta_0 U}\mathbb{E}^\bullet[f(X_1)\mathbb{1}_{\{1<S_\Gamma\}}]$. Since $\tilde{f}$ does not necessarily belong to $L^2(\mathcal{U})$, we can not directly apply Theorem 4.1 to derive $\tilde{g} = T^*_1\tilde{f}$; otherwise, the conclusion follows immediately from $\mathcal{P}^*_{2,i}T^*_1 = T^*_1\mathcal{P}^*_{2,i}$. We proceed by approximation. Let $\{f_n\}_{n\in\mathbb{N}} \subset C^\infty_0(\mathcal{U})$ be a sequence of functions that satisfy $\|f_n\|_\infty \leq \|f\|_\infty$ for all $n \in \mathbb{N}$ and converge locally uniformly in $\mathcal{U}$ to f. Since $\mathcal{P}^*_{2,i}T^*_1 = T^*_1\mathcal{P}^*_{2,i}$, we derive from Lemma 5.3 (2) that

$$\sum_{j=1}^{d_i}\tilde{v}^{(*,2)}_{i,j}\langle T^*_1\tilde{f}_n, \tilde{v}^{(2)}_{i,j}\rangle_{L^2} = \sum_{j=1}^{d_i}T^*_1\tilde{v}^{(*,2)}_{i,j}\langle\tilde{f}_n, \tilde{v}^{(2)}_{i,j}\rangle_{L^2}, \quad \forall n \in \mathbb{N}, \tag{5.25}$$

where $\tilde{f}_n := e^{-\frac{Q}{2}-\beta_0 U}f_n$. Thanks to Theorem 4.1, we find

$$\langle T^*_1\tilde{f}_n, \tilde{v}^{(2)}_{i,j}\rangle_{L^2} = \int_{\mathcal{U}} e^{-\frac{Q}{2}-\beta_0 U}\mathbb{E}^\bullet[f_n(X_1)\mathbb{1}_{\{1<S_\Gamma\}}]\overline{\tilde{v}^{(2)}_{i,j}}\mathrm{d}x, \quad \forall n \in \mathbb{N}.$$

Since $\tilde{v}^{(2)}_{ij} \in \mathcal{D}^* \subset L^2(\mathcal{U}, \alpha\mathrm{d}x; \mathbb{C})$, Lemma 5.1 ensures $e^{-\frac{Q}{2}-\beta_0 U}\tilde{v}^{(2)}_{i,j} \in L^1(\mathcal{U}; \mathbb{C})$. Letting $n \to \infty$ in the above equality, we conclude from the dominated convergence theorem that $\lim_{n\to\infty}\langle T^*_1\tilde{f}_n, \tilde{v}^{(2)}_{i,j}\rangle_{L^2} = \langle\tilde{g}, \tilde{v}^{(2)}_{i,j}\rangle_{L^2}$. Similarly, $\lim_{n\to\infty}\langle\tilde{f}_n, \tilde{v}^{(2)}_{i,j}\rangle_{L^2} = \langle\tilde{f}, \tilde{v}^{(2)}_{i,j}\rangle_{L^2}$. Letting $n \to \infty$ in (5.25) then yields

$$\sum_{j=1}^{d_i}\tilde{v}^{(*,2)}_{i,j}\langle\tilde{g}, \tilde{v}^{(2)}_{i,j}\rangle_{L^2} = \sum_{j=1}^{d_i}T^*_1\tilde{v}^{(*,2)}_{i,j}\langle\tilde{f}, \tilde{v}^{(2)}_{i,j}\rangle_{L^2},$$

which is the same as $\mathcal{P}^*_{2,i}\tilde{g} = T^*_1\mathcal{P}^*_{2,i}\tilde{f}$ thanks to Lemma 5.3 (2). Hence, (5.24) follows.

Inserting (5.22), (5.23) and (5.24) into (5.21) yields

$$\left\|e^{-\frac{Q}{2}-\beta_0 U}\mathbb{E}^\bullet[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}] - e^{-\lambda_1 t}\tilde{v}^*_1\int_{\mathcal{U}} f\mathrm{d}\nu_1 - T^*_t\mathcal{P}^*_2\tilde{f}\right\|_\infty \leq C_4 e^{-(\lambda_2+\epsilon)t}\|f\|_\infty, \quad \forall t \geq 2.$$

Multiplying both sides by $e^{\frac{Q}{2}+\beta_0 U}$ concludes the proof. □

Lemma 5.7. *Assume* **(H1)**-**(H3)**. *For each $0 < \epsilon \ll 1$, there is $C = C(\epsilon) > 0$ such that for each $f \in C_b(\mathcal{U})$ one has*

$$\|T^*_t\mathcal{P}^*_2\tilde{f}\|_\infty \leq Ce^{(\epsilon-\lambda_2)t}\|f\|_\infty, \quad \forall t \geq 2,$$

where $\tilde{f} := e^{-\frac{Q}{2}-\beta_0 U}f$.

Proof. Fix $f \in C_b(\mathcal{U})$ and set $\tilde{g} := e^{-\frac{Q}{2}-\beta_0 U}\mathbb{E}^\bullet[f(X_1)\mathbb{1}_{\{1<S_\Gamma\}}]$. It is shown in (5.24) that $T_t^*\mathcal{P}_2^*\tilde{f} = T_{t-1}^*\mathcal{P}_2^*\tilde{g}$ for all $t \geq 1$. By Lemma 5.2 (1) and Lemma 5.4, for each $0 < \epsilon \ll 1$, there exists $C_1 = C_1(\epsilon) > 0$ such that

$$\|T_{t-2}^*\mathcal{P}_2^*\tilde{g}\|_{L^2} \leq C_1 e^{(\epsilon-\lambda_2)(t-2)}\|f\|_\infty, \quad \forall t \geq 2. \tag{5.26}$$

Thanks to Lemma 5.5, there exists $C_2 > 0$ such that

$$\|T_t^*\mathcal{P}_2^*\tilde{f}\|_\infty = \|T_{t-1}^*\mathcal{P}_2^*\tilde{g}\|_\infty \leq C_2\|T_{t-2}^*\mathcal{P}_2^*\tilde{g}\|_{L^2}, \quad \forall t \geq 2.$$

This together with (5.26) leads to this lemma. □

We are ready to prove Theorem 5.2.

Proof of Theorem 5.2. Let ν and f be as in the statement. For fixed $0 < \epsilon \ll 1$, we apply Lemma 5.6 to find some $C > 0$ (independent of f) such that

$$\left|\mathbb{E}^\bullet[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}] - e^{\frac{Q}{2}+\beta_0 U}\left(e^{-\lambda_1 t}\tilde{v}_1^*\int_{\mathcal{U}} f\mathrm{d}\nu_1 + T_t^*\mathcal{P}_2^*\tilde{f}\right)\right| \leq Ce^{\frac{Q}{2}+\beta_0 U}e^{-(\lambda_2+\epsilon)t}\|f\|_\infty$$

for all $t \geq 2$, where $\tilde{f} := e^{-\frac{Q}{2}-\beta_0 U}f$.

Since ν is compactly supported in $\mathcal{U}$, integrating the above inequality on $\mathcal{U}$ with respect to ν yields

$$\begin{aligned}\left|\int_{\mathcal{U}}\mathbb{E}^\bullet[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}]\mathrm{d}\nu - e^{-\lambda_1 t}\int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}\tilde{v}_1^*\mathrm{d}\nu\int_{\mathcal{U}} f\mathrm{d}\nu_1 - \int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}T_t^*\mathcal{P}_2^*\tilde{f}\mathrm{d}\nu\right| \\ \leq Ce^{-(\lambda_2+\epsilon)t}\|f\|_\infty\int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}\mathrm{d}\nu, \quad \forall t \geq 2.\end{aligned}$$

In particular, setting $f = \mathbb{1}_{\mathcal{U}}$ yields

$$\begin{aligned}\left|\int_{\mathcal{U}}\mathbb{P}^x[t < S_\Gamma]\mathrm{d}\nu - e^{-\lambda_1 t}\int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}\tilde{v}_1^*\mathrm{d}\nu - \int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}T_t^*\mathcal{P}_2^*\tilde{\mathbb{1}}_{\mathcal{U}}\mathrm{d}\nu\right| \\ \leq Ce^{-(\lambda_2+\epsilon)t}\int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}\mathrm{d}\nu, \quad \forall t \geq 2,\end{aligned}$$

where $\tilde{\mathbb{1}}_{\mathcal{U}} := e^{-\frac{Q}{2}-\beta_0 U}\mathbb{1}_{\mathcal{U}}$.

Since ν is compactly supported in $\mathcal{U}$, we apply Lemma 5.7 to derive

$$\lim_{t\to\infty} e^{(\lambda_1+\epsilon)t}\int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}T_t^*\mathcal{P}_2^*\tilde{f}\mathrm{d}\nu = 0 \quad \text{and} \quad \lim_{t\to\infty} e^{(\lambda_1+\epsilon)t}\int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}T_t^*\mathcal{P}_2^*\tilde{\mathbb{1}}_{\mathcal{U}}\mathrm{d}\nu = 0.$$

It follows that as $t \to \infty$,

$$\begin{aligned}
&\frac{\int_{\mathcal{U}} \mathbb{E}^{\bullet}[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}]\mathrm{d}\nu}{\int_{\mathcal{U}} \mathbb{P}^{\bullet}[t<S_\Gamma]\mathrm{d}\nu} \\
&\quad= \frac{e^{-\lambda_1 t}\int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}\tilde{v}_1^*\mathrm{d}\nu \int_{\mathcal{U}} f\mathrm{d}\nu_1 + \int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U} T_t^*\mathcal{P}_2^*\tilde{f}\mathrm{d}\nu + o(e^{-\lambda_2 t})}{e^{-\lambda_1 t}\int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}\tilde{v}_1^*\mathrm{d}\nu + \int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U} T_t^*\mathcal{P}_2^*\tilde{\mathbb{1}}_{\mathcal{U}}\mathrm{d}\nu + o(e^{-\lambda_2 t})} \\
&\quad= \int_{\mathcal{U}} f\mathrm{d}\nu_1 + \frac{e^{\lambda_1 t}}{\int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}\tilde{v}_1^*\mathrm{d}\nu}\int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U} T_t^*\mathcal{P}_2^*\left(\tilde{f} - \tilde{\mathbb{1}}_{\mathcal{U}}\int_{\mathcal{U}} f\mathrm{d}\nu_1\right)\mathrm{d}\nu + o(e^{-(\lambda_2-\lambda_1)t}),
\end{aligned}$$

which together with Lemma 5.2 (1) leads to the result.

The last part of the lemma follows from Lemma 5.7 and Lemma 5.2 (4). □

Remark 5.2. *Corollary 5.1 implies $\tilde{v}_1^* \in L^\infty(\mathcal{U})$. Lemma 5.7 gives $\sup_{t>1}\|T_t^*\mathcal{P}_2^*\tilde{f}\|_\infty < \infty$. It is then easy to see from the proof of Theorem 5.2 that the conclusions hold for all initial distributions $\nu \in \mathcal{P}(\mathcal{U})$ satisfying $\int_{\mathcal{U}} e^{\frac{Q}{2}+\beta_0 U}\mathrm{d}\nu < \infty$.*

5.3. **Uniqueness and exponential convergence.** In this subsection, we study the uniqueness of QSDs of X_t as well as the conditioned dynamics of X_t for any initial distribution. The result is stated as follows. Recall that ν_1 is the QSD of X_t obtained in Theorem 5.1.

Theorem 5.3. *Assume* **(H1)-(H4)**. *Then, X_t admits a unique QSD, and for each $\nu \in \mathcal{P}(\mathcal{U})$ and $0 < \epsilon \ll 1$, there holds*

$$\lim_{t\to\infty} e^{(\lambda_2-\lambda_1-\epsilon)t}\left|\mathbb{E}^\nu\left[f(X_t)\middle|t<S_\Gamma\right] - \int_{\mathcal{U}} f\mathrm{d}\nu_1\right| = 0, \quad \forall f \in C_b(\mathcal{U}).$$

We need the following result asserting that X_t comes down from infinity under **(H1)-(H4)**.

Lemma 5.8. *Assume* **(H1)-(H4)**. *For each $\lambda > 0$, there are $R = R(\lambda) > 0$ and $C_1 = C_1(\lambda) > 0$ such that $\mathbb{E}^x\left[e^{\lambda S_R}\right] \le C_1$ for all $x \in \mathcal{U}\setminus B_R^+$, where $S_R := \inf\left\{t \ge 0 : X_t \notin \mathcal{U}\setminus B_R^+\right\}$.*

Proof. Recall from (2.4) that $U = V\circ\xi^{-1}$. Set $w := \exp\left\{-\frac{\epsilon}{U^\gamma}\right\}$, where $\epsilon > 0$ is a parameter to be chosen. According to the assumptions on V, we can modify V on a bounded domain to make sure $\inf_{\mathcal{U}} V > 0$, while preserving the other properties. We thus assume without loss of generality that $\inf_{\mathcal{U}} V > 0$. This together with $\lim_{|z|\to\infty} V(z) = \infty$ implies

$$0 < \inf_{\mathcal{U}} w \le \sup_{\mathcal{U}} w \le 1. \tag{5.27}$$

Let C, R_* and γ be as in **(H4)**. Recall $\mathcal{L}^X = \frac{1}{2}\Delta + (p_i - q_i)\partial_i$. Straightforward calculations give

$$\mathcal{L}^X U = \left(\mathcal{L}^Z V\right)\circ \xi^{-1} \leq -CU^{1+\gamma} \quad \text{in} \quad \mathcal{U}\setminus \xi(B^+_{R_*}).$$

It follows that

$$\begin{aligned}\mathcal{L}^X w + \lambda w &= \frac{\epsilon\gamma w \mathcal{L}^X U}{U^{\gamma+1}} + \frac{1}{2}\left(a_i|\partial_{z_i}V|^2\right)\circ\xi^{-1}\left[-\frac{\epsilon\gamma(\gamma+1)}{U^{\gamma+2}} + \frac{\epsilon^2\gamma^2}{U^{2\gamma+2}}\right] w + \lambda w\\ &\leq (-C\epsilon\gamma + \lambda)w + \frac{1}{2}\left(a_i|\partial_{z_i}V|^2\right)\circ\xi^{-1}\left[-\frac{\epsilon\gamma(\gamma+1)}{U^{\gamma+2}} + \frac{\epsilon^2\gamma^2}{U^{2\gamma+2}}\right] w \text{ in } \mathcal{U}\setminus\xi(B^+_{R_*}),\end{aligned}$$

where we used **(H4)** in the inequality.

Set $\epsilon := \frac{3\lambda}{2C\gamma}$. As **(H4)** ensures $\lim_{|z|\to\infty} a_i|\partial_{z_i}V|^2\left[-\frac{\epsilon\gamma(\gamma+1)}{V^{\gamma+2}} + \frac{\epsilon^2\gamma^2}{V^{2\gamma+2}}\right] = 0$, there must exist $R > 0$ such that

$$\mathcal{L}^X w + \lambda w \leq -\frac{\lambda}{3} w \quad \text{in} \quad \mathcal{U}\setminus B^+_R. \tag{5.28}$$

We recall from Remark 2.2 that X_t satisfies the SDE (2.2) before hitting Γ. An application of Itô's formula gives

$$\mathrm{d}e^{\lambda t} w(X_t) = (\mathcal{L}^X w + \lambda w)(X_t)e^{\lambda t}\mathrm{d}t + \partial_i w(X_t)e^{\lambda t}\mathrm{d}W^i_t \quad \text{in} \quad \mathcal{U}.$$

It follows from (5.28) that for each $(x,t)\in(\mathcal{U}\setminus B^+_R)\times[0,\infty)$,

$$\mathbb{E}^x\left[e^{\lambda(t\wedge S_R)} w(X_{t\wedge S_R})\right] = w(x) + \mathbb{E}^x\left[\int_0^{t\wedge S_R}(\mathcal{L}^X w + \lambda w)(X_s)e^{\lambda s}\mathrm{d}s\right] \leq w(x),$$

where S_R is as in the statement of the lemma. Thanks to (5.27), we pass to the limit $t\to\infty$ in the above inequality to conclude $\mathbb{E}^x\left[e^{\lambda S_R}\right] \leq \frac{1}{\inf w}$ for all $x \in \mathcal{U}\setminus B^+_R$. This completes the proof. □

Remark 5.3. *Since $Z_t = \xi^{-1}(X_t)$ and $\xi^{-1}: \overline{\mathcal{U}} \to \overline{\mathcal{U}}$ is a homeomorphism, we find from the above lemma that for each $\lambda > 0$, there exists $R = R(\lambda) > 0$ such that $\sup_{z\in\mathcal{U}\setminus B^+_R}\mathbb{E}^z[e^{\lambda T_R}] < \infty$, where $T_R := \inf\{t \geq 0 : Z_t \notin \mathcal{U}\setminus B^+_R\}$.*

We next prove Theorem 5.3.

Proof of Theorem 5.3. Fix $\nu \in \mathcal{P}(\mathcal{U})$, $f \in C_b(\mathcal{U})$ and $0 < \epsilon \ll 1$. Set $\lambda := \lambda_1 + \lambda_2$. By Lemma 5.8, there exist $R_0 > 0$ and $C_1 > 0$ such that

$$\sup_{(x,t)\in(\mathcal{U}\setminus B^+_{R_0})\times[0,\infty)} e^{\lambda t}\mathbb{P}^x\left[t < S_{R_0}\right] \leq \sup_{x\in\mathcal{U}\setminus B^+_{R_0}} \mathbb{E}^x\left[e^{\lambda S_{R_0}}\right] \leq C_1. \tag{5.29}$$

Clearly, the above inequality holds with $R > R_0$ replacing R_0. Choosing R_0 large enough, we may assume without loss of generality that $\nu(B^+_{R_0}) > 0$. We split

$$\mathbb{E}^\nu\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right] = \int_{B^+_{R_0}}\mathbb{E}^\bullet\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right]\mathrm{d}\nu + \int_{\mathcal{U}\setminus B^+_{R_0}}\mathbb{E}^\bullet\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right]\mathrm{d}\nu, \quad \forall t\geq 0.$$

Applying Lemma 5.6 and Lemma 5.7, we find the existence of $C_2>0$ such that

$$\left|\mathbb{E}^{\bullet}\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right]-e^{\frac{Q}{2}+\beta_0 U}e^{-\lambda_1 t}\tilde{v}_1^*\int_{\mathcal{U}}f\mathrm{d}\nu_1\right|\leq C_2 e^{\frac{Q}{2}+\beta_0 U}e^{(\epsilon-\lambda_2)t}\|f\|_\infty,\quad \forall t\geq 2. \tag{5.30}$$

It follows that

$$\left|\int_{B_{R_0}^+}\mathbb{E}^{\bullet}\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right]\mathrm{d}\nu-A_1e^{-\lambda_1 t}\int_{\mathcal{U}}f\mathrm{d}\nu_1\right|\leq C_2e^{(\epsilon-\lambda_2)t}\|f\|_\infty\int_{B_{R_0}^+}e^{\frac{Q}{2}+\beta_0 U}\mathrm{d}\nu,\quad \forall t\geq 2, \tag{5.31}$$

where $A_1:=\int_{B_{R_0}^+}e^{\frac{Q}{2}+\beta_0 U}\tilde{v}_1^*\mathrm{d}\nu$. Since $\tilde{v}_1^*>0$ in $\mathcal{U}$ and $\nu(B_{R_0}^+)>0$, there holds $A_1>0$.

We claim the existence of a bounded function $A_2:[0,\infty)\to[0,\infty)$ and a $C_3>0$ such that

$$\left|\int_{\mathcal{U}\setminus B_{R_0}^+}\mathbb{E}^{\bullet}\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right]\mathrm{d}\nu-A_2(t)e^{-\lambda_1 t}\int_{\mathcal{U}}f\mathrm{d}\nu_1\right|\leq C_3e^{(\epsilon-\lambda_2)t}\|f\|_\infty,\quad \forall t\gg 1. \tag{5.32}$$

This together with (5.31) leads to the existence of $C_4>0$ such that

$$\left|\mathbb{E}^{\nu}\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right]-(A_1+A_2(t))e^{-\lambda_1 t}\int_{\mathcal{U}}f\mathrm{d}\nu_1\right|\leq C_4e^{(\epsilon-\lambda_2)t}\|f\|_\infty,\quad \forall t\gg 1.$$

In particular, setting $f=\mathbb{1}_{\mathcal{U}}$ yields $\left|\mathbb{P}^{\nu}[t<S_\Gamma]-(A_1+A_2(t))e^{-\lambda_1 t}\right|\leq C_4e^{(\epsilon-\lambda_2)t}$ for all $t\gg 1$. Since $A_1>0$, this implies $\mathbb{P}^{\nu}[t<S_\Gamma]>0$ for $t\gg 1$. Consequently, we deduce

$$\begin{aligned}
&\left|\mathbb{E}^{\nu}\left[f(X_t)\big|t<S_\Gamma\right]-\int_{\mathcal{U}}f\mathrm{d}\nu_1\right|\\
&\quad\leq\left|\frac{\mathbb{E}^{\nu}\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right]}{\mathbb{P}^{\nu}\left[t<S_\Gamma\right]}-\frac{(A_1+A_2(t))e^{-\lambda_1 t}\int_{\mathcal{U}}f\mathrm{d}\nu_1}{\mathbb{P}^{\nu}[t<S_\Gamma]}\right|\\
&\qquad+\left|\frac{(A_1+A_2(t))e^{-\lambda_1 t}\int_{\mathcal{U}}f\mathrm{d}\nu_1}{\mathbb{P}^{\nu}[t<S_\Gamma]}-\int_{\mathcal{U}}f\mathrm{d}\nu_1\right|\\
&\quad=\frac{1}{\mathbb{P}^{\nu}\left[t<S_\Gamma\right]}\left|\mathbb{E}^{\nu}\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right]-(A_1+A_2(t))e^{-\lambda_1 t}\int_{\mathcal{U}}f\mathrm{d}\nu_1\right|\\
&\qquad+\frac{\int_{\mathcal{U}}|f|\mathrm{d}\nu_1}{\mathbb{P}^{\nu}[t<S_\Gamma]}\left|(A_1+A_2(t))e^{-\lambda_1 t}-\mathbb{P}^{\nu}[t<S_\Gamma]\right|\\
&\quad\leq\frac{2C_4e^{(\epsilon-\lambda_2)t}\|f\|_\infty}{\mathbb{P}^{\nu}[t<S_\Gamma]},\quad \forall t\gg 1.
\end{aligned}$$

The theorem follows immediately.

It remains to prove (5.32). To do so, we write for $(x,t)\in(\mathcal{U}\setminus B_{R_0}^+)\times[0,\infty)$,

$$\mathbb{E}^x[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}]=\mathbb{E}^x[f(X_t)\mathbb{1}_{\{t<S_{R_0}\}}]+\mathbb{E}^x[f(X_t)\mathbb{1}_{\{S_{R_0}\le t<S_\Gamma\}}]=:E_1(x,t)+E_2(x,t).$$

It follows from (5.29) that

$$\begin{aligned}\int_{\mathcal{U}\setminus B_{R_0}^+}|E_1(\cdot,t)|\mathrm{d}\nu&\le\|f\|_\infty\sup_{x\in\mathcal{U}\setminus B_{R_0}^+}\mathbb{P}^x\left[t<S_{R_0}\right]\mathrm{d}\nu\\&\le\|f\|_\infty e^{-\lambda t}\sup_{x\in\mathcal{U}\setminus B_{R_0}^+}\mathbb{E}^x[e^{\lambda S_{R_0}}]\le C_1\|f\|_\infty e^{-\lambda t},\quad\forall t\ge0.\end{aligned}\tag{5.33}$$

To treat E_2, we set $h(x,t):=\mathbb{E}^x\left[f(X_t)\mathbb{1}_{\{t<S_\Gamma\}}\right]$ for $(x,t)\in\overline{\mathcal{U}}\times[0,\infty)$. Obviously, $\|h\|_\infty\le\|f\|_\infty$ and $h(x,t)=0$ for $(x,t)\in\Gamma\times[0,\infty)$. The strong Markov property and homogeneity of X_t yield that for each $(x,t)\in(\mathcal{U}\setminus B_{R_0}^+)\times[0,\infty)$,

$$\begin{aligned}E_2(x,t)&=\mathbb{E}^x\left[f(X_t)\mathbb{1}_{\{S_{R_0}\le t<S_\Gamma\}}\right]\\&=\mathbb{E}^x\left[h(X_{S_{R_0}},t-S_{R_0})\mathbb{1}_{\{S_{R_0}\le t\}}\right]\\&=\mathbb{E}^x\left[h(X_{S_{R_0}},t-S_{R_0})\mathbb{1}_{\{S_{R_0}\le t\le S_{R_0}+2\}}\right]+\mathbb{E}^x\left[h(X_{S_{R_0}},t-S_{R_0})\mathbb{1}_{\{t>S_{R_0}+2\}}\right]\\&=:E_{21}(x,t)+E_{22}(x,t).\end{aligned}$$

Note that (5.29) ensures

$$\begin{aligned}\int_{\mathcal{U}\setminus B_{R_0}^+}|E_{21}(\cdot,t)|\mathrm{d}\nu&\le\|h\|_\infty\int_{\mathcal{U}\setminus B_{R_0}^+}\mathbb{P}^x[t<S_{R_0}+2]\mathrm{d}\nu\\&\le\|f\|_\infty e^{-\lambda t}\int_{\mathcal{U}\setminus B_{R_0}^+}\mathbb{E}^x[e^{\lambda(S_{R_0}+2)}]\mathrm{d}\nu\le C_1\|f\|_\infty e^{2\lambda-\lambda t},\quad\forall t\ge0.\end{aligned}\tag{5.34}$$

Fix $0<\epsilon\ll1$. Setting $\Phi:=\exp\left\{\frac{Q(X_{S_{R_0}})}{2}+\beta_0U(X_{S_{R_0}})\right\}$, we see from (5.30) that on the event $\{t\ge S_{R_0}+2\}$ there holds

$$\left|h(X_{S_{R_0}},t-S_{R_0})-\Phi e^{-\lambda_1(t-S_{R_0})}\tilde{v}_1^*(X_{S_{R_0}})\int_{\mathcal{U}}f\mathrm{d}\nu_1\right|\le C_2\Phi e^{(\epsilon-\lambda_2)(t-S_{R_0})}\|f\|_\infty.$$

Since $S_{R_0}\le S_\Gamma$ and $h(X_{S_{R_0}},t-S_{R_0})=0$ if $S_{R_0}=S_\Gamma$, we deduce

$$E_{22}(x,t)=\mathbb{E}^x\left[h(X_{S_{R_0}},t-S_{R_0})\mathbb{1}_{\{S_{R_0}<S_\Gamma\wedge(t-2)\}}\right],\quad\forall(x,t)\in(\mathcal{U}\setminus B_{R_0}^+)\times[0,\infty).$$

As a result, there holds

$$\begin{aligned}
&\left|\int_{\mathcal{U}\setminus B^+_{R_0}} E_{22}(\cdot,t)\mathrm{d}\nu - e^{-\lambda_1 t}\int_{\mathcal{U}\setminus B^+_{R_0}} \mathbb{E}^\bullet\left[\Phi e^{\lambda_1 S_{R_0}}\tilde{v}_1^*(X_{S_{R_0}})\mathbb{1}_{\{S_{R_0}<S_\Gamma\wedge(t-2)\}}\right]\mathrm{d}\nu\int_{\mathcal{U}} f\mathrm{d}\nu_1\right|\\
&\quad\leq C_2 e^{(\epsilon-\lambda_2)t}\int_{\mathcal{U}\setminus B^+_{R_0}}\mathbb{E}^\bullet\left[\Phi e^{-(\epsilon-\lambda_2)S_{R_0}}\mathbb{1}_{\{S_{R_0}<S_\Gamma\wedge(t-2)\}}\right]\mathrm{d}\nu\|f\|_\infty\\
&\quad\leq C_2 e^{(\epsilon-\lambda_2)t}\|f\|_\infty\left(\max_{\mathcal{U}\cap\partial B^+_{R_0}} e^{\frac{Q}{2}+\beta_0 U}\right)\left(\sup_{\mathcal{U}\setminus B^+_{R_0}}\mathbb{E}^\bullet\left[e^{-(\epsilon-\lambda_2)S_{R_0}}\right]\right)\\
&\quad\leq C_5 e^{(\epsilon-\lambda_2)t}\|f\|_\infty,\quad \forall t\geq 0.
\end{aligned}\tag{5.35}$$

where we used (5.29) and the fact $\max_{\mathcal{U}\cap\partial B^+_{R_0}} e^{\frac{Q}{2}+\beta_0 U}<\infty$ to conclude the existence of $C_5>0$ in the last inequality.

Set

$$A_2(t):=\int_{\mathcal{U}\setminus B^+_{R_0}}\mathbb{E}^\bullet\left[\Phi e^{\lambda_1 S_{R_0}}\tilde{v}_1^*(X_{S_{R_0}})\mathbb{1}_{\{S_{R_0}<S_\Gamma\wedge(t-2)\}}\right]\mathrm{d}\nu,\quad\forall t\geq 0.$$

Obviously, A_2 is non-negative and bounded. Since

$$\int_{\mathcal{U}\setminus B^+_{R_0}}\mathbb{E}^\bullet\left[f(X_t)\mathbb{1}_{t<S_\Gamma\}}\right]\mathrm{d}\nu=\int_{\mathcal{U}\setminus B^+_{R_0}}\left[E_1(\cdot,t)+E_{21}(\cdot,t)+E_{22}(\cdot,t)\right]\mathrm{d}\nu,\quad\forall t\geq 0,$$

we deduce from (5.33), (5.34) and (5.35) that

$$\left|\int_{\mathcal{U}\setminus B^+_{R_0}}\mathbb{E}^\bullet\left[f(X_t)\mathbb{1}_{t<S_\Gamma\}}\right]\mathrm{d}\nu - A_2(t)e^{-\lambda_1 t}\int_{\mathcal{U}} f\mathrm{d}\nu_1\right|\leq\left[C_5 e^{(\epsilon-\lambda_2)t}+C_1(1+e^{2\lambda})e^{-\lambda t}\right]\|f\|_\infty$$

for all $t\geq 0$. Since $\lambda=\lambda_1+\lambda_2$ and $0<\epsilon\ll 1$, (5.32) follows. This completes the proof. □

5.4. **Proof of Theorem A and Theorem B.** Because of the fact $X_t=\xi(Z_t)$ and Proposition 2.3, conclusions in Theorem A and Theorem B follow directly from Theorem 5.1, Theorem 5.2 and Theorem 5.3.

6. Applications

In this section, we discuss a series of important applications of Theorem A and Theorem B. We first provide a general result that holds for most ecological models and then show how to apply this result to specific situations, including: stochastic Lotka-Volterra systems of competitive, predator-prey or cooperative type, systems modelled by Holling type functional responses and predator-prey systems modelled by Beddington-DeAngelis functional responses.

Consider the following stochastic system:

$$\mathrm{d}Z_t^i = Z_t^i f_i(Z_t)\mathrm{d}t + \sqrt{\gamma_i Z_t^i}\mathrm{d}W_t^i, \quad i \in \{1,\dots,d\}, \tag{6.1}$$

where $Z_t = (Z_t^i) \in \overline{\mathcal{U}}$, $\{f_i\}_i$ belong to $C^1(\overline{\mathcal{U}})$, $\{\gamma_i\}_i$ are positive constants, and $\{W^i\}_i$ are independent standard one-dimensional Wiener processes on some probability space. We make the following assumption.

(A) There exist $m \geq 0$, $0 \leq n \leq m$, $C_1, C_2, C_3, C_4 > 0$ and $R > 0$ such that

$$-C_1\left(1+\sum_{j=1}^d z_j^m\right) \leq f_i(z) \leq C_2\mathbb{1}_{[0,R]}(z_i) - C_3 z_i^m \mathbb{1}_{(R,\infty)}(z_i) + \delta\sum_{j\neq i} z_j^n, \quad \forall z \in \overline{\mathcal{U}}, \tag{6.2}$$

and

$$|\partial_{z_i} f_i(z)| \leq C_4|z|^{m-1}, \quad \forall z \in \mathcal{U}\setminus B_R^+, \tag{6.3}$$

for all $i \in \{1,\dots,d\}$ and $\delta \geq 0$ if $n < m$ or $\delta \in \left[0, \frac{C_3}{d-1}\right)$ if $n = m$.

Remark 6.1. *Conditions (6.2) and (6.3) say that f_i and $\partial_{z_i} f_i$ are bounded above and below by simple polynomials. Conditions in the case $n < m$ tells us that the intraspecific competition dominates the interactions among species. In the case $n = m$, we can only treat weakly cooperative interactions among species – this is reflected by the smallness of δ. These are natural assumptions that can be applied to many population dynamics models: competitive Lotka-Volterra, weakly cooperative Lotka-Volterra, predator-prey Lotka-Volterra as well as more complex systems modelled by Holling type-II/III functional responses. These assumptions also allow us to use a very simple Lyapunov function $V(z) = |z|^{m+1}$ which satisfies* **(H1)**-**(H3)** *and sometimes* **(H4)**.

Under the assumption **(A)**, the stochastic system (6.1) generates a diffusion process Z_t that has Γ as an absorbing set. Furthermore, Z_t hits Γ in finite time almost surely.

Theorem 6.1. *Assume* **(A)**.

(1) *Z_t admits a QSD μ_1, and there exists $r_1 > 0$ such that*
- *for any $0 < \epsilon \ll 1$ and $\mu \in \mathcal{P}(\mathcal{U})$ with compact support in $\mathcal{U}$ one has*

$$\lim_{t\to\infty} e^{(r_1-\epsilon)t}\left|\mathbb{E}^\mu\left[f(Z_t)\middle|t < T_\Gamma\right] - \int_{\mathcal{U}} f\mathrm{d}\mu_1\right| = 0, \quad \forall f \in C_b(\mathcal{U});$$

- *there exists $f \in C_b(\mathcal{U})$ such that for a.e. $x \in \mathcal{U}$, there is a discrete set $\mathcal{I}_x \subset (0,\infty)$ with distances between adjacent points admitting an x-independent positive lower bound, such that for each $0 < \delta \ll 1$ one has*

$$\lim_{\substack{t\to\infty \\ t\in(0,\infty)\setminus\mathcal{I}_{x,\delta}}} e^{(r_1+\epsilon)t}\left|\mathbb{E}^x[f(X_t)\middle|t < T_\Gamma] - \int_{\mathcal{U}} f\mathrm{d}\mu_1\right| = \infty, \quad \forall 0 < \epsilon \ll 1,$$

where $\mathcal{I}_{x,\delta}$ is the δ-neighbourhood of $\mathcal{I}_x$ in $(0,\infty)$.

(2) *If, in addition,* **(A)** *holds with* $m>0$, *then* Z_t *admits a unique QSD, and for any* $0<\epsilon\ll 1$ *and* $\mu\in\mathcal{P}(\mathcal{U})$, *there holds*

$$\lim_{t\to\infty} e^{(r_1-\epsilon)t}\left|\mathbb{E}^{\mu}\left[f(Z_t)\middle|t<T_\Gamma\right]-\int_{\mathcal{U}} f\mathrm{d}\mu_1\right|=0,\quad \forall f\in C_b(\mathcal{U}).$$

Proof. Let $m,\ C_1,\ C_2,\ C_3,\ C_4,\ R$ and δ be as in **(A)**. Set $V(z):=|z|^{m+1}$ for $z\in\mathcal{U}$. Since $\partial_i V=(m+1)|z|^{m-1}z_i$, we deduce from **(A)** that

$$\begin{aligned}\sum_{i=1}^d z_i f_i\partial_i V&\leq (m+1)|z|^{m-1}\sum_{i=1}^d z_i^2\left(C_2\mathbb{1}_{[0,R]}(z_i)-C_3z_i^m\mathbb{1}_{(R,\infty)}(z_i)+\delta\sum_{j\neq i} z_j^n\right)\\&\leq (m+1)|z|^{m-1}\left[\left(C_2R+C_3R^{m+1}\right)\sum_{i=1}^d z_i-C_3\sum_{i=1}^d z_i^{m+2}+\delta\sum_{i=1}^d\sum_{j\neq i} z_i^2z_j^n\right]\end{aligned}\tag{6.4}$$

for all $z\in\mathcal{U}$.

We apply Young's inequality to find

$$z_i^2z_j^n\leq\begin{cases}\frac{2\alpha}{m+2}z_i^{m+2}+\frac{m\alpha^{-\frac{2}{m}}}{m+2}z_j^{\frac{n(m+2)}{m}}, & \text{if}\quad m>0 \text{ and } n\in[0,m],\\ \frac{2\alpha}{m+2}z_i^{m+2}+\frac{m\alpha^{-\frac{2}{m}}}{m+2}z_j^{m+2}, & \text{if}\quad m=n=0,\end{cases}$$

where $\alpha>0$ is a parameter to be determined. For convenience, we set $\beta(m,n)=\frac{n(m+2)}{m}\in[0,m+2]$ if $m>0$ and $n\in[0,m]$ and $\beta(n,m)=m+2$ if $m=n=0$. Thus, it follows from (6.4) that

$$\begin{aligned}\sum_{i=1}^d z_i f_i\partial_i V&\leq (m+1)(C_2R+C_3R^{m+1})|z|^{m-1}\sum_{i=1}^d z_i-(m+1)C_3|z|^{m-1}\sum_{i=1}^d z_i^{m+2}\\&\quad+\delta(m+1)|z|^{m-1}\sum_{i=1}^d\sum_{j\neq i}\left(\frac{2\alpha}{m+2}z_i^{m+2}+\frac{m\alpha^{-\frac{2}{m}}}{m+2}z_j^{\beta(m,n)}\right)\\&=(m+1)(C_2R+C_3R^{m+1})|z|^{m-1}\sum_{i=1}^d z_i\\&\quad+\frac{\delta m(m+1)\alpha^{-\frac{2}{m}}(d-1)}{m+2}|z|^{m-1}\sum_{i=1}^d z_i^{\beta(m,n)}\\&\quad-(m+1)\left(C_3-\frac{2\delta\alpha(d-1)}{m+2}\right)|z|^{m-1}\sum_{i=1}^d z_i^{m+2}\end{aligned}\tag{6.5}$$

for all $z\in\mathcal{U}$.

Note that $0<\beta(m,n)<m+2$ if $n<m$ and $\beta(n,m)=1$ if $n=m$. We consider the following two cases.

- If $n<m$, we set $\alpha=\frac{(m+2)C_3}{4\delta(d-1)}$ in (6.5) (so that $C_3-\frac{2\delta\alpha(d-1)}{m+2}=\frac{1}{2}C_3>0$) to find the existence of $C_5,R_1>0$ such that
$$\sum_{i=1}^{d} z_i f_i \partial_i V \leq -C_5|z|^{2m+1} \quad \text{in} \quad \mathcal{U}\setminus B^+_{R_1}. \tag{6.6}$$
- If $n=m$, setting $\alpha=1$ in (6.5) leads to
$$\begin{aligned}\sum_{i=1}^{d} z_i f_i \partial_i V &= (m+1)(C_2R+C_3R^{m+1})|z|^{m-1}\sum_{i=1}^{d} z_i \\ &\quad -(m-1)[C_3-\delta(d-1)]|z|^{m-1}\sum_{i=1}^{d} z_i^{m+2}.\end{aligned}$$
Then, it is easy to conclude from $\delta\in[0,\frac{C_3}{d-1})$ the existence of positive constants C_5' and R_1' such that (6.6) holds with C_5 and R_1 replaced by C_5' and R_1', respectively.

As a result, we no longer distinguish the above two cases and assume (6.6) always holds for some $C_5>0$ and $R_1>0$.

Now, we verify **(H1)**-**(H3)**. It is easy to check that **(H1)** and **(H2)** hold. As $V\geq C(d)\sum_{i=1}^d z_i^{m+1}$ in $\mathcal{U}$ for some $C(d)>0$ and $\int_1^\infty \frac{1}{s}\exp\{-\beta s^{m+1}\}\,\mathrm{d}s<\infty$ for any $\beta>0$, **(H3)** (1)(2) follow from (6.6). Since
$$\begin{aligned}\partial_i(z_if_i)&=f_i(z)+z_i\partial_{z_i}f_i(z),\\ \gamma_iz_i\partial^2_{z_iz_i}V&=\gamma_i(m+1)(m-1)|z|^{m-3}z_i^3+\gamma_i(m+1)|z|^{m-1}z_i,\\ \gamma_iz_i|\partial_{z_i}V|^2&=\gamma_i(m+1)^2|z|^{2m-2}z_i^3,\end{aligned}$$
it is straightforward to verify **(H3)** (3)(4) by applying (6.2), (6.3) and (6.6). Hence, an application of Theorem A gives the conclusions in (1).

If $m>0$, **(H4)** holds with $\gamma:=\frac{m}{m+1}$. The conclusion in (2) follows from Theorem B. □

In the following, we apply Theorem 6.1 to various important ecological models.

Example 6.1 (Lotka-Volterra systems)**.** *For each $i\in\{1,\dots,d\}$ let*
$$f_i(z)=r_i-\sum_{j=1}^d c_{ij}z_j, \quad z\in\overline{\mathcal{U}},$$
where $r_i\in\mathbb{R}$, $c_{ii}>0$ and $c_{ij}\in\mathbb{R}$ for $j\neq i$.

Corollary 6.1. *Consider the stochastic system* (6.1) *with* f_i, $i \in \{1, \dots, d\}$ *being as in Example 6.1. Assume*

$$-\min_{i \neq j} c_{ij} < \frac{1}{d-1} \min_i c_{ii}. \tag{6.7}$$

Then, there exists a unique QSD of (6.1) *such that the conclusions of Theorem 6.1 hold.*

Proof. It is straightforward to check that the assumption **(A)** with $m = n = 1$, $C_3 = \min_i c_{ii}$ and $\delta = -\min_{i \neq j} c_{ij}$ is satisfied. The corollary then follows from Theorem 6.1. □

Remark 6.2. *If the system is competitive, namely,* $c_{ij} \geq 0$ *for all* $i \neq j$*, then* (6.7) *is trivially satisfied. If the Lotka-Volterra system has either cooperation or predation, the condition* (6.7) *says that the intraspecific competition terms have to dominate in some sense the cooperative and the predation terms. Note that cooperative systems are known to behave poorly: see* [27, Example 2.3] *for details as to how a two-species stochastic cooperative system can exhibit either blow-up in finite time or have no stationary distributions.*

Example 6.2 (Holling type-II/III functional response)**.** *For each* $i \in \{1, \dots, d\}$,

$$f_i(z) = r_i - \sum_{j=1}^{d} \frac{c_{ij} z_j^k}{1 + z_j^k}, \quad z \in \overline{\mathcal{U}},$$

where $k \in \{1, 2\}$, $r_i \in \mathbb{R}$, $c_{ii} > 0$ *and* $c_{ij} \in \mathbb{R}$ *for* $j \neq i$. *In literature,* $k = 1$ *and* $k = 2$ *correspond to Holling type-*II *and* -III *functional responses, respectively.*

Corollary 6.2. *Consider the stochastic system* (6.1) *with* f_i, $i \in \{1, \dots, d\}$ *being as in Example 6.2. Assume*

$$c_{ii} > r_i, \quad \forall i \in \{1, \dots, d\} \quad \text{and} \quad -\min_{i \neq j} c_{ij} < \frac{1}{d-1} \min_i (c_{ii} - r_i).$$

Then, the conclusions of Theorem 6.1 (1) hold.

Proof. By Theorem 6.1, it suffices to verify the assumption **(A)** with $m = n = 0$. Clearly, the first inequality in (6.2) holds. To check the second inequality in (6.2), we note from the assumptions that there exists $\alpha \in (0, 1)$ such that $-\min_{i \neq j} c_{ij} < \frac{\alpha}{d-1} \min_i \{c_{ii} - r_i\}$. For this $\alpha > 0$, there exists $R > 0$ such that

$$r_i - \frac{c_{ii} z_i^k}{1 + z_i^k} \leq \alpha (r_i - c_{ii}) \leq -\alpha \min_i (c_{ii} - r_i), \quad \forall z_i \in (R, \infty) \text{ and } i \in \{1, \dots, d\},$$

leading to

$$r_i - \sum_{j=1}^{d} \frac{c_{ij} z_j^k}{1+z_j^k} \leq \begin{cases} r_i - (d-1) \times \min_{i\neq j} c_{ij}, & \forall z \in \{z \in \overline{\mathcal{U}} : z_i \in [0,R]\}, \\ -\alpha \min_i (c_{ii} - r_i) - (d-1) \times \min_{i \neq j} c_{ij}, & \forall z \in \{z \in \overline{\mathcal{U}} : z_i \in (R,\infty)\}. \end{cases}$$

This shows the second inequality in (6.2) with $C_2 = \max_i\{r_i\}$, $C_3 = \alpha \min_i(c_{ii} - r_i)$ and $\delta = -\min_{i\neq j} c_{ij}$. Straightforward calculations give (6.3). Hence, the assumption **(A)** with $m = 0$ holds. □

Remark 6.3. *For the stochastic Lotka-Volterra system with Holling type*-II/III *functional response considered in Example 6.2 or Corollary 6.2, the existence of a unique QSD that attracts all initial distributions supported in $\mathcal{U}$ is not expected. This is essentially due to the weak dissipativity of the system. Indeed, in the case $d = 1$, these properties are equivalent to showing that the process comes down from infinity, and therefore, according to* [6, Theorem 7.3 and Proposition 7.5]*, equivalent to Assumption (H5) in* [6]*. However, it is easy to check that (H5) in* [6] *is not satisfied for the Holling type-II/III functional responses.*

The situation in higher dimensions is worse. Even in the competitive case, the dissipativity of the system is weaker than that of the system with $f_i(z) = r_* - c_* \sum_{j=1}^d \frac{z_j^k}{1+z_j^k}$ for all $i \in \{1,\dots,d\}$, where $r_* = \min_{i\in\{1,\dots,d\}} r_i$ and $c_* = \max_{i,j\in\{1,\dots,d\}} c_{ij}$. This latter system does not come down from infinity as it is bounded from below by a decoupled system whose individual components do not come down from infinity. In fact, we have

$$r_* - c_* \sum_{j=1}^{d} \frac{z_j^k}{1+z_j^k} \geq r_* - c_*(d-1) - c_* \frac{z_i^k}{1+z_i^k}, \quad \forall i \in \{1,\dots,d\} \text{ and } z \in \overline{\mathcal{U}}.$$

Hence, the stochastic system in Example 6.2 or Corollary 6.2 does not come down from infinity.

We exhibit below a few more types of functional responses that can be treated by our framework.

Example 6.3. *Consider the functional response*

$$f_i(z) = r_i - c_{ii} z_i - \sum_{j\neq i} \frac{c_{ij} z_j^k}{1+z_j^k}, \quad z \in \overline{\mathcal{U}},$$

where $k \in \{1,2\}$, $r_i \in \mathbb{R}$, $c_{ii} > 0$ and $c_{ij} \in \mathbb{R}$ for $j \neq i$. This is a combination of the regular intraspecific competition of the form $-c_{ii} z_i$ and Holling type functional responses for the interspecific competition/predation.

Corollary 6.3. *Consider the stochastic system* (6.1) *with* f_i*,* $i \in \{1, \dots, d\}$ *being as in Example 6.3. Then, there exists a unique QSD of* (6.1) *such that the conclusions of Theorem 6.1 hold.*

Proof. It is straightforward to check that Assumption **(A)** holds with $m = 1$ and $n = 0$. Then, the application of Theorem 6.1 yields the conclusion. □

Example 6.4. *Consider the extensively used Beddington-DeAngelis predator-prey dynamics. For each* $i \in \{1, \dots, d\}$*, let*

$$f_i(z) = r_i - c_{ii}z_i - \sum_{j \neq i} \frac{c_{ij}z_j}{1 + \sum_{l=1}^d z_l}, \quad z \in \overline{\mathcal{U}},$$

where $r_i \in \mathbb{R}$*,* $c_{ii} > 0$*, and* $c_{ij} \in \mathbb{R}$ *for* $j \neq i$*. This system was first proposed in* [2, 18] *in order to better explain certain predator-prey interactions.*

Corollary 6.4. *Consider the stochastic system* (6.1) *with* f_i*,* $i \in \{1, \dots, d\}$ *being as in Example 6.4. Then, there exists a unique QSD of* (6.1) *such that the conclusions of Theorem 6.1 hold.*

Proof. It is straightforward to check that Assumption **(A)** holds with $m = 1$ and $n = 0$. Then, the application of Theorem 6.1 yields the conclusion. □

Example 6.5. *Let* $d = 2$*. Consider the Crowley-Martin dynamics. Let*

$$f_1(z) = r_1 - c_{11}z_1 - z_2\frac{z_1}{\beta + \alpha z_1 + \alpha_2 z_2 + \alpha_3 z_1 z_2}, \quad z \in \overline{\mathcal{U}},$$

$$f_2(z) = -r_2 - c_{22}z_2 + z_1\frac{z_1}{\beta + \alpha z_1 + \alpha_2 z_2 + \alpha_3 z_1 z_2}, \quad z \in \overline{\mathcal{U}},$$

where $c_{11}, c_{22}, \beta > 0$ *and all the other quantities are nonnegative. This system was first proposed in* [17] *to study dragonflies.*

Corollary 6.5. *Consider the stochastic system* (6.1) *in the case* $d = 2$ *with* f_1 *and* f_2 *being as in Example 6.5. Assume* $\alpha > \frac{2}{3\min\{2c_{11}, c_{22}\}}$*. Then, there exists a unique QSD of* (6.1) *such that the conclusions of Theorem 6.1 hold.*

Proof. Note that $f_1(z) \leq r_1 - c_{11}z_1$ and $f_2(z) \leq -r_2 - c_{22}z_2 + \frac{z_1}{\alpha}$. Following the arguments as in the proof of Theorem 6.1, it is straightforward to see that $V(z) := |z|^2$ for $z \in \mathcal{U}$ is a Lyapunov function satisfying **(H1)**-**(H4)**. From which, the conclusions of Theorem 6.1 hold. □

Appendix A. Proof of technical lemmas

We prove technical lemmas in this appendix.

A.1. **Proof of Lemma 3.2.** We need the following result.

Lemma A.1. *Assume* **(H1)**. *For each* $i \in \{1, \dots, d\}$, *there exists* $C_i > 0$ *such that*

$$\lim_{x_i \to 0} x_i^2 \left[q_i^2(x_i) - q_i'(x_i)\right] = C_i.$$

Proof. Fix $i \in \{1, \dots, d\}$. Recall that $q_i(x_i) = \frac{a_i'(\xi_i^{-1}(x_i))}{4\sqrt{a_i(\xi_i^{-1}(x_i))}}$. It is straightforward to calculate

$$q_i'(x_i) = \frac{1}{4}a_i''(\xi_i^{-1}(x_i)) - \frac{|a_i'|^2(\xi_i^{-1}(x_i))}{8a_i(\xi_i^{-1}(x_i))},$$

which results in

$$(q_i^2 - q_i')(x_i) = \frac{3|a_i'|^2(\xi_i^{-1}(x_i))}{16a_i(\xi_i^{-1}(x_i))} - \frac{1}{4}a_i''(\xi_i^{-1}(x_i)). \tag{A.1}$$

Since $\xi_i^{-1} \in C([0, \infty))$ and $\xi_i^{-1}(0) = 0$, we see from **(H1)** that $\lim_{x_i \to 0} a_i'(\xi_i^{-1}(x_i)) = a_i'(0) > 0$ and $\lim_{x_i \to 0} a_i''(\xi_i^{-1}(x_i)) = a_i''(0)$. Hence,

$$\left(q_i^2 - q_i'\right)(x_i) \sim \frac{3|a_i'|^2(0)}{16a_i(\xi_i^{-1}(x_i))} - \frac{1}{4}a_i''(0) \quad \text{as} \quad x_i \to 0.$$

The conclusion follows if there is $C > 0$ such that

$$a_i(\xi_i^{-1}(x_i)) \sim Cx_i^2 \quad \text{as} \quad x_i \to 0. \tag{A.2}$$

We show that (A.2) holds with $C = \frac{|a_i'(0)|^2}{4}$. The assumption **(H1)** and Taylor's expansion give

$$a_i(z_i) \sim a_i'(0)z_i + o(z_i^2) \quad \text{as} \quad x_i \to 0, \tag{A.3}$$

leading to

$$\xi_i(z_i) = \int_0^{z_i} \frac{\mathrm{d}s}{\sqrt{a_i(s)}} = \int_0^{z_i} \frac{\mathrm{d}s}{\sqrt{a_i'(0)s + o(s^2)}} \sim \frac{2\sqrt{z_i}}{\sqrt{a_i'(0)}} \quad \text{as} \quad z_i \to 0.$$

Thus, $\xi_i^{-1}(x_i) \sim \frac{a_i'(0)x_i^2}{4}$ as $x_i \to 0$. Inserting this into (A.3) yields (A.2) with $C = \frac{|a_i'(0)|^2}{4}$. This completes the proof. □

Remark A.1. *Thanks to* (A.2), *it is straightforward to check from the definition of* Q *given in* (2.5) *that* $Q(x)$ *behaves like* $\sum_{i=1}^d \ln x_i$ *as* x *approaches to* Γ. *Hence,* $e^{-\frac{Q}{2}}$ *is as singular as* $\prod_{i=1}^d \frac{1}{\sqrt{x_i}}$ *near* Γ.

Proof of Lemma 3.2. We first prove (1). Recall that U is given in (2.4). Clearly,

$$\partial_{x_i} U(x) = \partial_{z_i} V(\xi^{-1}(x))\sqrt{a_i(\xi_i^{-1}(x_i))}, \quad \forall x \in \mathcal{U}.$$

We derive from **(H3)** (4) the existence of $C_1 > 0$ and $R_1 > 0$ such that

$$\left(|\nabla U|^2 + |p|^2\right)(x) \le -C_1(b \cdot \nabla_z V)(\xi^{-1}(x)) \le C_1\alpha(x), \quad \forall x \in \mathcal{U} \setminus B_{R_1}^+.$$

Since $\sup_{B^+_{R_1}}(|\nabla U|^2+|p|^2)<\infty$ due to **(H2)** and **(H3)**(1) and $\inf_{\mathcal{U}}\alpha>0$, there must exist some $C_2>0$ such that $(|\nabla U|^2+|p|^2)<C_2\alpha$ in $B^+_{R_1}$. Setting $C:=\min\{C_1,C_2\}$ yields the result.

The rest of the proof is arranged as follows. In **Step 1**, we analyze the asymptotic behaviors of terms in $e_{\beta,N}$ near the boundary Γ and in the vicinity of infinity. Based on these, the asymptotic behaviors of $e_{\beta,N}$ are derived in **Step 2**. The proof of (2) and (3) are respectively given in **Step 3** and **Step 4**. Recall that R_0 and δ_0 are fixed in Subsection 3.1 when defining α.

Step 1. We analyze the asymptotic behaviors of terms in $e_{\beta,N}$.

- For the term $p\cdot\nabla U$, we see from **(H3)** (1) that

$$(p\cdot\nabla U)(x)=(b\cdot\nabla V)(\xi^{-1}(x))\to-\infty\quad\text{as}\quad |x|\to\infty. \tag{A.4}$$

- For the term $\frac{1}{2}\sum_{i=1}^d(q_i^2-q_i')$, Lemma A.1 ensures the existence of $\delta_*\in(0,\delta_0)$ and $C_3,C_4>0$ such that

$$\frac{C_3}{x_i^2}\le\frac{1}{2}(q_i^2-q_i')(x_i)\le\frac{C_4}{x_i^2},\quad \forall x_i\in(0,\delta_*]\text{ and } i\in\{1,\dots,d\}. \tag{A.5}$$

  Since **(H1)** gives $\limsup_{s\to\infty}\left(\frac{|a_i'(s)|^2}{a_i(s)}+a_i''(s)\right)<\infty$, we find from (A.1) and (A.4) that for any $0<\epsilon_1\ll 1$, there exists $R_2=R_2(\epsilon_1)>0$ such that

$$\frac{1}{2}|q_i^2-q_i'|(x_i)\le-\frac{\epsilon_1}{d}(p\cdot\nabla U)(x),\quad \forall x\in\{x\in\mathcal{U}:x_i\in(R_2,\infty)\}\text{ and } i\in\{1,\dots,d\}. \tag{A.6}$$

- For the terms ΔU, $p\cdot q$ and $\nabla\cdot p$, we calculate

$$\partial^2_{x_ix_i}U(x)=\left[\partial^2_{z_iz_i}V(\xi^{-1}(x))a_i(\xi_i^{-1}(x_i))+\frac{1}{2}\partial_{z_i}V(\xi^{-1}(x))a_i'(\xi_i^{-1}(x_i))\right],$$
$$p_i(x)q_i(x_i)=\frac{b_i(\xi^{-1}(x))a_i'(\xi_i^{-1}(x_i))}{4a_i(\xi_i^{-1}(x_i))},$$
$$\partial_{x_i}p_i(x)=\partial_{z_i}b_i(\xi^{-1}(x))-\frac{b_i(\xi^{-1}(x))a_i'(\xi_i^{-1}(x_i))}{2a_i(\xi_i^{-1}(x_i))}.$$

  By **(H1)**-**(H3)**, we have $U\in C^2(\overline{\mathcal{U}})$, and $p\cdot q,\nabla\cdot p\in C(\overline{\mathcal{U}})$. Moreover, **(H3)**(3) and (A.4) guarantee that for any $0<\epsilon_2\ll 1$, there exists $R_3=R_3(\epsilon_2)>0$ such that

$$|\Delta U|+|p\cdot q|+|\nabla\cdot p|\le-\epsilon_2 p\cdot\nabla U\quad\text{in}\quad \mathcal{U}\setminus B^+_{R_3}. \tag{A.7}$$

- For the term $\frac{1}{2}|\nabla U|^2$, we find from $|\nabla U|^2(x)=\sum_{i=1}^d|\partial_{z_i}V|^2(\xi^{-1}(x))a_i(\xi_i^{-1}(x_i))$, the assumption **(H3)**(4) and (A.4) that there are $C_5>0$ and $R_4>0$ such that

$$\frac{1}{2}|\nabla U|^2\leq -C_5(p\cdot\nabla U)\quad\text{in}\quad \mathcal{U}\setminus B^+_{R_4}. \tag{A.8}$$

Step 2. We analyze the asymptotic behaviors of $e_{\beta,N}$ near Γ and in the vicinity of infinity.

Set $R_*:=\max\{R_0,R_2,R_3,R_4\}$ and $C_6:=\frac{1}{2}\max_i\max_{x_i\in[\delta_*,R_*]}|q_i^2-q_i'|(x_i)$. It is obvious that R_* and C_6 depend on ϵ_1 and ϵ_2, which are to be determined in the proof of (3). Since α is piecewise defined, we analyze $e_{\beta,N}$ in four subdomains: $\Gamma_{\delta_*}\bigcap B^+_{R_*}$, $\Gamma_{\delta_*}\bigcap(\mathcal{U}\setminus B^+_{R_*})$, $(\mathcal{U}\setminus\Gamma_{\delta_*})\bigcap B^+_{R_*}$ and $(\mathcal{U}\setminus\Gamma_{\delta_*})\bigcap(\mathcal{U}\setminus B^+_{R_*})$ separately, where we recall that $\Gamma_{\delta_*}:=\{x\in\mathcal{U}:x_i\leq\delta_*\text{ for some }i\in\{1,\dots,d\}\}$.

For notational simplicity, we set

$$\Psi=\frac{\beta}{2}|\Delta U|+\frac{\beta^2}{2}|\nabla U|^2+\beta|p\cdot\nabla U|+|p\cdot q|+|\nabla\cdot p|.$$

(a) In $\Gamma_{\delta_*}\bigcap B^+_{R_*}$. We see from $U\in C^2(\overline{\mathcal{U}})$ and $p\cdot\nabla U,\ p\cdot q,\nabla\cdot p\in C(\overline{\mathcal{U}})$ that $\max_{\Gamma_{\delta_*}\cap B^+_{R_*}}\Psi<\infty$. It follows from (A.5) that

$$\begin{aligned}|e_{\beta,N}|&\leq\sum_{i=1}^d\left(\frac{C_4}{x_i^2}\mathbb{1}_{(0,\delta_*)}(x_i)+C_6\mathbb{1}_{(\delta_*,R_*)}(x_i)\right)+\max_{\Gamma_{\delta_*}\cap B^+_{R_*}}\Psi\\&\leq C_4\sum_{i=1}^d\frac{1}{x_i^2}+dC_6+\max_{\Gamma_{\delta_*}\cap B^+_{R_*}}\Psi\end{aligned}$$

and

$$\begin{aligned}e_{\beta,N}&\geq C_3\sum_{i=1}^d\frac{1}{x_i^2}\mathbb{1}_{(0,\delta_*)}(x_i)-dC_6-\max_{\Gamma_{\delta_*}\cap B^+_{R_*}}\Psi\\&\geq C_3\sum_{i=1}^d\max\left\{\frac{1}{x_i^2},1\right\}-d\left(\frac{C_3}{\delta_*^2}+C_6\right)-\max_{\Gamma_{\delta_*}\cap B^+_{R_*}}\Psi.\end{aligned}$$

(b) In $\Gamma_{\delta_*}\bigcap(\mathcal{U}\setminus B^+_{R_*})$. It follows from (A.5), (A.7) and (A.8) that

$$
\begin{aligned}
|e_{\beta,N}| &\le C_4\sum_{i=1}^d \frac{1}{x_i^2} + dC_6 - \left(\beta+\epsilon_2(1+\frac{\beta}{2})+C_5\beta^2\right)p\cdot\nabla U,\\
e_{\beta,N} &\ge C_3\sum_{i=1}^d \frac{1}{x_i^2}\mathbb{1}_{(0,\delta_*)}(x_i) - dC_6 - \left(\beta-\epsilon_2(1+\frac{\beta}{2})-C_5\beta^2\right)p\cdot\nabla U\\
&\ge C_3\sum_{i=1}^d \max\left\{\frac{1}{x_i^2},1\right\} - d\left(\frac{C_3}{\delta_*^2}+C_6\right) - \left(\beta-\epsilon_2(1+\frac{\beta}{2})-C_5\beta^2\right)p\cdot\nabla U.
\end{aligned}
$$

(c) In $(\mathcal{U}\setminus\Gamma_{\delta_*})\bigcap B^+_{R_*}$. There hold

$$
\begin{aligned}
|e_{\beta,N}| &\le \max_{(\mathcal{U}\setminus\Gamma_{\delta_*})\cap B^+_{R_*}}\left[\Psi+\frac{1}{2}\sum_{i=1}^d |q_i^2-q_i'|\right],\\
e_{\beta,N} &\ge -\max_{(\mathcal{U}\setminus\Gamma_{\delta_*})\cap B^+_{R_*}}\left[\Psi+\frac{1}{2}\sum_{i=1}^d |q_i^2-q_i'|\right].
\end{aligned}
$$

(d) In $(\mathcal{U}\setminus\Gamma_{\delta_*})\bigcap(\mathcal{U}\setminus B^+_{R_*})$. It follows from (A.6), (A.7) and (A.8) that

$$
\begin{aligned}
|e_{\beta,N}| &\le dC_6 - \left(\beta+\epsilon_1+\epsilon_2(1+\frac{\beta}{2})+C_5\beta^2\right)p\cdot\nabla U,\\
e_{\beta,N} &\ge -dC_6 - \left(\beta-\epsilon_1-\epsilon_2(1+\frac{\beta}{2})-C_5\beta^2\right)p\cdot\nabla U.
\end{aligned}
$$

Step 3. We prove (2). As $\alpha\ge\sum_{i=1}^d\max\left\{\frac{1}{x_i^2},1\right\}$ in Γ_{δ_*} and $\inf_{\mathcal{U}}\alpha>0$, we deduce from **Step 2** (a) the existence of $D_1(\beta)>0$ such that $e_{\beta,N}\le D_1(\beta)\alpha$ in $\Gamma_{\delta_*}\cap B^+_{R_*}$ for all $N\ge1$.

Since $\inf_{\mathcal{U}}\alpha>0$ and

$$
\alpha=\begin{cases}\displaystyle\sum_{i=1}^d\max\left\{\frac{1}{x_i^2},1\right\}-p\cdot\nabla U & \text{in}\quad \Gamma_{\delta_*}\bigcap(\mathcal{U}\setminus B^+_{R_*}),\\ -p\cdot\nabla U & \text{in}\quad (\mathcal{U}\setminus\Gamma_{\delta_*})\bigcap(\mathcal{U}\setminus B^+_{R_*}),\end{cases}
$$

we see from **Step 2** (b) and (d) that there exists $D_2(\beta)>0$ such that $|e_{\beta,N}|\le D_2(\beta)\alpha$ in $\mathcal{U}\setminus B^+_{R_*}$ for all $N\ge1$.

Thanks to $\inf_{\mathcal{U}}\alpha>0$, it follows from **Step 2** (c) the existence of $D_3(\beta)>0$ such that $|e_{\beta,N}|\le D_3(\beta)\alpha$ in $(\mathcal{U}\setminus\Gamma_{\delta_*})\cap B^+_{R_*}$ for all $N\ge1$.

Setting $C(\beta):=\max\{D_1(\beta),D_2(\beta),D_3(\beta)\}$ yields (2).

Step 4. We show (3). Setting $\beta_0 := \frac{1}{2C_5}$, $\epsilon_1 := \min\left\{1, \frac{1}{16C_5}\right\}$ and $\epsilon_2 := \min\left\{1, \frac{1}{2+8C_5}\right\}$, we deduce from **Step 2** (b) and (d) that

$$\begin{aligned} e_{\beta_0,N} &\geq C_3 \sum_{i=1}^d \max\left\{\frac{1}{x_i^2}, 1\right\} - d\left(\frac{C_3}{\delta_*^2} + C_6\right) - \left(\beta_0 - \epsilon_2(1+\frac{\beta_0}{2}) - C_5\beta_0^2\right) p\cdot\nabla U \\ &\geq C_3 \sum_{i=1}^d \max\left\{\frac{1}{x_i^2}, 1\right\} - \frac{p\cdot\nabla U}{8C_5} - d\left(\frac{C_3}{\delta_*^2} + C_6\right) \\ &\geq \min\left\{C_3, \frac{1}{8C_5}\right\}\alpha - d\left(\frac{C_3}{\delta_*^2} + C_6\right) \quad \text{in} \quad \Gamma_{\delta_*} \bigcap (\mathcal{U}\setminus B_{R_*}^+) \end{aligned} \tag{A.9}$$

and

$$\begin{aligned} e_{\beta_0,N} &\geq -dC_6 - \left(\beta_0 - \epsilon_1 - \epsilon_2(1+\frac{\beta_0}{2}) - C_5\beta_0^2\right) p\cdot\nabla U \\ &\geq -\frac{p\cdot\nabla U}{16C_5} - dC_6 \geq \frac{1}{16C_5}\alpha - dC_6 \quad \text{in} \quad (\mathcal{U}\setminus\Gamma_{\delta_*}) \bigcap (\mathcal{U}\setminus B_{R_*}^+). \end{aligned} \tag{A.10}$$

Since $\alpha \leq \sum_{i=1}^d \max\left\{\frac{1}{x_i^2}, 1\right\} + \max_{\Gamma_{\delta_*}\bigcap B_{R_*}^+} |p\cdot\nabla U|$ in $\Gamma_{\delta_*}\bigcap B_{R_*}^+$ and $\sup_{(\mathcal{U}\setminus\Gamma_{\delta_*})\bigcap B_{R_*}^+} \alpha < \infty$, we conclude from (a) and (c) the existence of positive constants C_7 and $M > d\left(\frac{C_3}{\delta_*^2} + C_6\right)$ such that

$$e_{\beta_0,N} + M \geq C_7\alpha \quad \text{in} \quad B_{R_*}^+, \quad \forall N \geq 1,$$

which together with (A.9) and (A.10) implies that

$$e_{\beta_0,N} + M \geq C_*\alpha \quad \text{in} \quad \mathcal{U}, \quad \forall N \geq 1,$$

where $C_* := \min\{C_3, \frac{1}{16C_5}, C_7\}$. This proves (3), and completes the proof.

A.2. **Proof of Lemma 4.3.** Suppose $\tilde{w} \in C(\mathcal{U}\times[0,\infty)) \bigcap L^2([0,\infty), \mathcal{H}^1(\mathcal{U}))$ is a weak solution of (4.6). The proof is broken into two steps.

Step 1. We show

$$\frac{1}{2}\int_{\mathcal{U}} \tilde{w}^2(\cdot,t)\mathrm{d}x + \frac{1}{2}\int_0^t\int_{\mathcal{U}} |\nabla\tilde{w}|^2 \mathrm{d}x\mathrm{d}s + \int_0^t\int_{\mathcal{U}} e_{\beta_0,2}\tilde{w}^2\mathrm{d}x\mathrm{d}s = \frac{1}{2}\int_{\mathcal{U}} \tilde{f}^2\mathrm{d}x, \quad \forall t\in[0,\infty). \tag{A.11}$$

The idea of proving (A.11) is based on the classical "energy method". But, we have to deal with the fact that $\tilde{w}$ lacks the differentiability in t. For each $0 < h \ll 1$, we define

$$\tilde{w}_h(x,t) := \frac{1}{h}\int_t^{t+h} \tilde{w}(x,s)\mathrm{d}s, \quad (x,t)\in\mathcal{U}\times[0,\infty).$$

Obviously, $\tilde{w}_h \in C(\mathcal{U}\times[0,\infty))\bigcap L^2([0,\infty),\mathcal{H}^1(\mathcal{U}))$ and $\partial_t\tilde{w}_h \in L^2(\mathcal{U}\times[0,T])$ for each $T>0$. It is easy to verify that $\tilde{w}_h$ is a weak solution of (4.6) with $\tilde{f}$ replaced by $\tilde{f}_h := \tilde{w}_h(\cdot,0) = \frac{1}{h}\int_0^h \tilde{w}_h(\cdot,s)\mathrm{d}s$. Namely, for each $\phi \in C_0^{1,1}(\mathcal{U}\times[0,\infty))$, one has

$$\begin{aligned}\int_{\mathcal{U}} \tilde{w}_h(\cdot,t)\phi(\cdot,t)\mathrm{d}x - \int_{\mathcal{U}} \tilde{f}_h\phi(\cdot,0)\mathrm{d}x - \int_0^t\int_{\mathcal{U}} \tilde{w}_h\partial_t\phi \mathrm{d}x\mathrm{d}s \\ = -\frac{1}{2}\int_0^t\int_{\mathcal{U}} \nabla\tilde{w}_h\cdot\nabla\phi \mathrm{d}x\mathrm{d}s - \int_0^t\int_{\mathcal{U}}(p_i+\beta_0\partial_iU)\tilde{w}_h\partial_i\phi \mathrm{d}x\mathrm{d}s \\ - \int_0^t\int_{\mathcal{U}} e_{\beta_0}\tilde{w}_h\phi\mathrm{d}x\mathrm{d}s, \quad \forall t\in[0,\infty).\end{aligned}\tag{A.12}$$

Let $\{\eta_n\}_{n\in\mathbb{N}} \subset C_0^\infty(\mathcal{U})$ be a sequence of functions taking values in $[0,1]$ and satisfying

$$\eta_n(x) = \begin{cases} 1, & x\in \left(\mathcal{U}\setminus\Gamma_{\frac{2}{n}}\right)\bigcap B^+_{\frac{n}{2}},\\ 0, & x\in \Gamma_{\frac{1}{n}}\bigcup(\mathcal{U}\setminus B_n^+),\end{cases} \quad \text{and} \quad |\nabla\eta_n(x)| \le \begin{cases} 2n, & x\in\Gamma_{\frac{2}{n}}\setminus\Gamma_{\frac{1}{n}},\\ 4, & x\in\left(\mathcal{U}\setminus\Gamma_{\frac{2}{n}}\right)\bigcap\left(B_n^+\setminus B^+_{\frac{n}{2}}\right).\end{cases}$$

By standard approximation arguments, we deduce that (A.12) holds with ϕ replaced by $\eta_n^2\tilde{w}_h$ for each $n\in\mathbb{N}$ and $0<h\ll 1$, namely,

$$\begin{aligned}\int_{\mathcal{U}} \tilde{w}_h^2(\cdot,t)\eta_n^2\mathrm{d}x - \int_{\mathcal{U}}\tilde{f}_h\eta_n^2\tilde{w}_h(\cdot,0)\mathrm{d}x - \int_0^t\int_{\mathcal{U}}\tilde{w}_h\partial_t(\eta_n^2\tilde{w}_h)\mathrm{d}x\mathrm{d}s \\ = -\frac{1}{2}\int_0^t\int_{\mathcal{U}}\nabla\tilde{w}_h\cdot\nabla(\eta_n^2\tilde{w}_h)\mathrm{d}x\mathrm{d}s - \int_0^t\int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot\tilde{w}_h\nabla(\eta_n^2\tilde{w}_h)\mathrm{d}x\mathrm{d}s \\ - \int_0^t\int_{\mathcal{U}} e_{\beta_0}\eta_n^2\tilde{w}_h^2\mathrm{d}x\mathrm{d}s, \quad \forall t\in[0,\infty).\end{aligned}$$

Note that the left hand side of the above equality equals $\frac{1}{2}\int_{\mathcal{U}}(\eta_n^2\tilde{w}_h^2)(\cdot,t)\mathrm{d}x - \frac{1}{2}\int_{\mathcal{U}}\eta_n^2\tilde{f}_h^2\mathrm{d}x$. Thus, for each $t\in[0,\infty)$, $n\in\mathbb{N}$ and $0<h\ll 1$,

$$\begin{aligned}\frac{1}{2}\int_{\mathcal{U}}(\eta_n^2\tilde{w}_h^2)(\cdot,t)\mathrm{d}x - \frac{1}{2}\int_{\mathcal{U}}\eta_n^2\tilde{f}_h^2\mathrm{d}x \\ = -\frac{1}{2}\int_0^t\int_{\mathcal{U}}\nabla\tilde{w}_h\cdot\nabla(\eta_n^2\tilde{w}_h)\mathrm{d}x\mathrm{d}s - \int_0^t\int_{\mathcal{U}}(p_i+\beta_0\partial_iU)\tilde{w}_h\partial_i(\eta_n^2\tilde{w}_h)\mathrm{d}x\mathrm{d}s \\ - \int_0^t\int_{\mathcal{U}} e_{\beta_0}\eta_n^2\tilde{w}_h^2\mathrm{d}x\mathrm{d}s.\end{aligned}\tag{A.13}$$

We claim that passing to the limit $h \to 0$ in (A.13) yields that for each $t \in [0, \infty)$ and $n \in \mathbb{N}$,

$$\begin{aligned}
&\frac{1}{2}\int_{\mathcal{U}}(\eta_n^2\tilde{w}^2)(\cdot,t)\mathrm{d}x - \frac{1}{2}\int_{\mathcal{U}}\eta_n^2\tilde{f}^2\mathrm{d}x \\
&\quad = -\frac{1}{2}\int_0^t\int_{\mathcal{U}}\nabla\tilde{w}\cdot\nabla(\eta_n^2\tilde{w})\mathrm{d}x\mathrm{d}s - \int_0^t\int_{\mathcal{U}}(p_i+\beta_0\partial_i U)\tilde{w}\partial_i(\eta_n^2\tilde{w})\mathrm{d}x\mathrm{d}s \\
&\quad\quad - \int_0^t\int_{\mathcal{U}} e_{\beta_0}\eta_n^2\tilde{w}^2\mathrm{d}x\mathrm{d}s.
\end{aligned} \tag{A.14}$$

Assuming (A.14), we conclude (A.11) from letting $n \to \infty$ in (A.14) and arguments as in the proof of Lemma 3.3 (2).

It remains to justify (A.14). Fix $t \in [0, \infty)$ and $n \in \mathbb{N}$. Note for each $0 < h \ll 1$, there hold

$$\tilde{w}_h(\cdot,t) - \tilde{w}(\cdot,t) = \frac{1}{h}\int_t^{t+h}[\tilde{w}(\cdot,s)-\tilde{w}(\cdot,t)]\mathrm{d}s = \int_0^1[\tilde{w}(\cdot,t+hs)-\tilde{w}(\cdot,t)]\mathrm{d}s,$$

and

$$\tilde{f}_h - \tilde{f} = \frac{1}{h}\int_0^h[\tilde{w}(\cdot,s)-\tilde{f}]\mathrm{d}s = \int_0^1[\tilde{w}(\cdot,hs)-\tilde{f}]\mathrm{d}s.$$

Since $\tilde{w} \in C(\mathcal{U}\times[0,\infty))$, we find for each compact set $K \subset \mathcal{U}$,

$$\sup_{K\times[0,t]}|\tilde{w}_h - \tilde{w}| \to 0 \quad \text{as} \quad h \to 0 \tag{A.15}$$

and

$$\sup_K|\tilde{f}_h - \tilde{f}| \to 0 \quad \text{as} \quad h \to 0.$$

It follows that

$$\lim_{h\to 0}\left[\frac{1}{2}\int_{\mathcal{U}}(\eta_n^2\tilde{w}_h^2)(\cdot,t)\mathrm{d}x - \frac{1}{2}\int_{\mathcal{U}}\eta_n^2\tilde{f}_h^2\mathrm{d}x\right] = \frac{1}{2}\int_{\mathcal{U}}(\eta_n^2\tilde{w}^2)(\cdot,t)\mathrm{d}x - \frac{1}{2}\int_{\mathcal{U}}\eta_n^2\tilde{f}^2\mathrm{d}x, \tag{A.16}$$

and

$$\lim_{h\to 0}\int_0^t\int_{\mathcal{U}}e_{\beta_0}\eta_n^2\tilde{w}_h^2\mathrm{d}x\mathrm{d}s = \int_0^t\int_{\mathcal{U}}e_{\beta_0}\eta_n^2\tilde{w}^2\mathrm{d}x\mathrm{d}s. \tag{A.17}$$

Since

$$\nabla\tilde{w}_h(\cdot,t) - \nabla\tilde{w}(\cdot,t) = \frac{1}{h}\int_t^{t+h}[\nabla\tilde{w}(\cdot,s)-\nabla\tilde{w}(\cdot,t)]\mathrm{d}s = \int_0^1[\nabla\tilde{w}(\cdot,t+hs)-\nabla\tilde{w}(\cdot,t)]\mathrm{d}s,$$

we apply Hölder's inequality and Fubini's theorem to find

$$\begin{aligned}\int_0^t\int_{\mathcal{U}}|\nabla\tilde{w}_h-\nabla\tilde{w}|^2\mathrm{d}x\mathrm{d}t' &\le \int_0^t\int_{\mathcal{U}}\int_0^1|\nabla\tilde{w}(x,t'+hs)-\nabla\tilde{w}(x,t')|^2\mathrm{d}s\mathrm{d}x\mathrm{d}t'\\ &=\int_0^1\int_0^t\int_{\mathcal{U}}|\nabla\tilde{w}(x,t'+hs)-\nabla\tilde{w}(x,t')|^2\mathrm{d}x\mathrm{d}t'\mathrm{d}s \qquad (A.18)\\ &\le \sup_{s\in[0,h]}\int_0^t\int_{\mathcal{U}}|\nabla\tilde{w}(x,t'+s)-\nabla\tilde{w}(x,t')|^2\mathrm{d}x\mathrm{d}t'.\end{aligned}$$

Since $\nabla\tilde{w}\in L^2(\mathcal{U}\times[0,2t])$ and $C_0(\mathcal{U}\times[0,2t])$ is dense in $L^2(\mathcal{U}\times[0,2t])$, for each $\epsilon>0$, we could find some $\Phi\in C_0(\mathcal{U}\times[0,2t])$ such that $\|\Phi-\nabla\tilde{w}\|_{L^2(\mathcal{U}\times[0,2t])}<\epsilon$. Obviously, Φ is uniformly continuous on $\mathcal{U}\times[0,2t]$, resulting in $\sup_{s\in[0,h]}\int_0^t\int_{\mathcal{U}}|\Phi(x,t'+s)-\Phi(x,t')|^2\mathrm{d}x\mathrm{d}t'\to 0$ as $h\to 0$. Therefore,

$$\begin{aligned}&\sup_{s\in[0,h]}\int_0^t\int_{\mathcal{U}}|\nabla\tilde{w}(x,t'+s)-\nabla\tilde{w}(x,t')|^2\mathrm{d}x\mathrm{d}t'\\ &\quad=\sup_{s\in[0,h]}\int_0^t\int_{\mathcal{U}}|\nabla\tilde{w}(x,t'+s)-\Phi(x,t'+s)|^2\mathrm{d}x\mathrm{d}t'\\ &\qquad+\sup_{s\in[0,h]}\int_0^t\int_{\mathcal{U}}|\Phi(x,t'+s)-\Phi(x,t')|^2\mathrm{d}x\mathrm{d}t'\\ &\qquad+\sup_{s\in[0,h]}\int_0^t\int_{\mathcal{U}}|\Phi(x,t')-\nabla\tilde{w}(x,t')|^2\mathrm{d}x\mathrm{d}t'\\ &\quad\le 2\|\Phi-\nabla\tilde{w}\|^2_{L^2(\mathcal{U}\times[0,2t])}+\sup_{s\in[0,h]}\int_0^t\int_{\mathcal{U}}|\Phi(x,t'+s)-\Phi(x,t')|^2\mathrm{d}x\mathrm{d}t'\\ &\quad\le 2\epsilon+\sup_{s\in[0,h]}\int_0^t\int_{\mathcal{U}}|\Phi(x,t'+s)-\Phi(x,t')|^2\mathrm{d}x\mathrm{d}t'.\end{aligned}$$

Letting $h\to 0$ in the above estimates, we find from the arbitrariness of $\epsilon>0$ and (A.18) that

$$\int_0^t\int_{\mathcal{U}}|\nabla\tilde{w}_h-\nabla\tilde{w}|^2\mathrm{d}x\mathrm{d}s\to 0\quad\text{as}\quad h\to 0. \qquad (A.19)$$

Hence, for each $n\in\mathbb{N}$, one has $\lim_{h\to 0}\int_0^t\int_{\mathcal{U}}\eta_n^2|\nabla\tilde{w}_h|^2\mathrm{d}x\mathrm{d}s=\int_0^t\int_{\mathcal{U}}\eta_n^2|\nabla\tilde{w}|^2\mathrm{d}x\mathrm{d}s$.

Since

$$\begin{aligned}&\int_0^t\int_{\mathcal{U}}\eta_n\tilde{w}_h\nabla\tilde{w}_h\cdot\nabla\eta_n\mathrm{d}x\mathrm{d}s-\int_0^t\int_{\mathcal{U}}\eta_n\tilde{w}\nabla\tilde{w}\cdot\nabla\eta_n\mathrm{d}x\mathrm{d}s\\ &\quad=\int_0^t\int_{\mathcal{U}}\eta_n(\tilde{w}_h-\tilde{w})\nabla\tilde{w}_h\cdot\nabla\eta_n\mathrm{d}x\mathrm{d}s+\int_0^t\int_{\mathcal{U}}\eta_n\tilde{w}(\nabla\tilde{w}_h-\nabla\tilde{w})\cdot\nabla\eta_n\mathrm{d}x\mathrm{d}s,\end{aligned}$$

we apply Hölder's inequality to deduce from (A.15) and (A.19) that

$$\lim_{h\to 0}\int_0^t\int_{\mathcal{U}}\eta_n\tilde{w}_h\nabla\tilde{w}_h\cdot\nabla\eta_n\mathrm{d}x\mathrm{d}s=\int_0^t\int_{\mathcal{U}}\eta_n\tilde{w}\nabla\tilde{w}\cdot\nabla\eta_n\mathrm{d}x\mathrm{d}s.$$

Hence,

$$\begin{aligned}&\lim_{h\to 0}\int_0^t\int_{\mathcal{U}}\nabla\tilde{w}_h\cdot\nabla(\eta_n^2\tilde{w}_h)\mathrm{d}x\mathrm{d}s\\&=\lim_{h\to 0}\int_0^t\int_{\mathcal{U}}\eta_n^2|\nabla\tilde{w}_h|^2\mathrm{d}x\mathrm{d}s+2\int_0^t\int_{\mathcal{U}}\eta_n\tilde{w}_h\nabla\tilde{w}_h\cdot\nabla\eta_n\mathrm{d}x\mathrm{d}s\\&=\int_0^t\int_{\mathcal{U}}\eta_n^2|\nabla\tilde{w}|^2\mathrm{d}x\mathrm{d}s+2\int_0^t\int_{\mathcal{U}}\eta_n\tilde{w}\nabla\tilde{w}\cdot\nabla\eta_n\mathrm{d}x\mathrm{d}s\\&=\int_0^t\int_{\mathcal{U}}\nabla\tilde{w}\cdot\nabla(\eta_n^2\tilde{w})\mathrm{d}x\mathrm{d}s.\end{aligned}\tag{A.20}$$

Similar arguments yield

$$\lim_{h\to 0}\int_0^t\int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot\tilde{w}_h\nabla(\eta_n^2\tilde{w}_h)\mathrm{d}x\mathrm{d}s=\int_0^t\int_{\mathcal{U}}(p+\beta_0\nabla U)\cdot\tilde{w}\nabla(\eta_n^2\tilde{w})\mathrm{d}x\mathrm{d}s.\tag{A.21}$$

Consequently, letting $h\to 0$ in (A.13), we conclude (A.14) from (A.16), (A.17), (A.20) and (A.21).

Step 2. We show that $\int_{\mathcal{U}}\tilde{w}^2(\cdot,t)\mathrm{d}x\leq\frac{e^{2Mt}}{M}\int_{\mathcal{U}}\tilde{f}^2\mathrm{d}x$ for all $t\in[0,\infty)$. Hence, $\tilde{w}=0$ if $\tilde{f}=0$. This proves the lemma.

As $e_{\beta_0,2}+M\geq 0$ by Lemma 3.2 (3), we derive from (A.11) that

$$\frac{1}{2}\int_{\mathcal{U}}\tilde{w}^2(\cdot,t)\mathrm{d}x\leq M\int_0^t\int_{\mathcal{U}}\tilde{w}^2\mathrm{d}x\mathrm{d}s+\int_{\mathcal{U}}\tilde{f}^2\mathrm{d}x,\quad\forall t\in[0,\infty).\tag{A.22}$$

Setting $g(t)=\int_0^t\int_{\mathcal{U}}\tilde{w}^2\mathrm{d}x\mathrm{d}s$ for $t\in[0,\infty)$, we arrive at $\frac{1}{2}g'(t)\leq Mg(t)+\int_{\mathcal{U}}\tilde{f}^2\mathrm{d}x$ for all $t\in[0,\infty)$. The conclusion then follows from Gronwall's inequality. □

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