EXAM INFORMATION
& PRACTICE PROBLEMS

History of Math, Duchin

Exam: Wednesday May 10, 12-2pm, Anderson 206

HISTORICAL DISCUSSION

In this section there are two basic types of problems: short answer, and fix-up. For the **short answer** you might be asked to give a brief discussion of a quotation from a reading or a brief explanation of an important concept from the lectures. For the **fix-up**, you will be given a FALSE historical statement, and asked to (a) list a few relevant facts, and (b) fix it. (Many possible fixes will be accepted.)

You will get some choice of what questions to answer.

Examples.

(1) Explain the method of proofs and refutations in a few sentences.

(2) What did Bourbaki mean by "Down with Euclid!" ?

(3) What are some of the contributions of Brahmagupta, and what did the American historian Carl Boyer have to say about them?

(4) Briefly describe two examples where a new notation or technology has a big impact on mathematics.

(5) What does it mean for a mathematical problem to be "undecidable"? Give two examples of problems shown to be undecidable, and give brief historical context for each example.

(6) Fix-ups:

- Emmy Noether was one of the founding members of Bourbaki in 1903.
- Srinivasa Ramanujan was "discovered" by Johann Bernoulli, who verified some of his mathematical theorems by making astronomy-based calculations.
- Leonhard Euler hoarded his results until he could humiliate rival mathematicians with them.
- Mathematics in the early to high middle ages (+500—1200 or so) was developed far from the context of religion.
- Georg Cantor was indifferent to whether his infinite cardinals were treated as numbers, as long as they were given as much respect as infinitesimals.

MATH PROBLEMS

Here are some math topics that you might find represented on the test:

- polyhedra, $V - E + F$, and homeomorphism
- Law of the Excluded Middle and proof by contradiction
- Babylonian numeration and arithmetic
- Egyptian numeration, arithmetic, and fractions
- greedy algorithms
- continued fractions, anthyphairesis, Euclidean algorithm
- *efficiency* vs *accuracy* of continued fraction approximations
- basic modular arithmetic ($\mathbb{Z}/n\mathbb{Z}$ with $+, \cdot$)
- the Chinese Remainder Theorem, Pell's equation

- the partition function
- power series and applications (e.g., Machin-style approx. to π)
- generating functions
- Euclidean geometry and constructible lengths and angles
- the three Greek impossibility problems
- Zeno's paradoxes
- basic ideas of non-Euclidean geometry
- complex numbers and quaternions
- infinitesimals and cardinals
- paradoxes of the infinite; set theory paradoxes
- axiomatic definitions such as group, ring, field (be comfortable with, don't memorize)
- solvability by radicals and relationship to Greek constructibility
- basic examples of groups: cyclic, dihedral, symmetric
- factoring groups: Galois's criterion for solvability

Examples.

(1) Describe Cauchy's heuristic algorithm for computing $V - E + F$, and explain why it only works for spherical polyhedra (homeomorphic to the sphere).

(2) (a) Find 37 times 9 using the Egyptian algorithm.
(b) You're an archaeologist and you find the following calculation on a papyrus fragment; the boxes represent unreadable parts of the artifact. From context, figure out what goes in the boxes.

$$\begin{array}{ll} 1 & 7 \\ \bar{2} & 3\ \bar{2} \\ \bar{4} & 1\,\boxed{}\ ' \\ \boxed{} & \bar{4}\qquad\ ' \end{array}$$

(c) This calculation is most likely what Egyptian fraction expression for what vulgar fraction?
(d) Explain rigorously why there are infinitely many different Egyptian fraction expressions for this same vulgar fraction, and give a second example for the vulgar fraction from your previous answer.

(3) Show the steps to convert $BABA_{16}$ to base 2.

(4) Explain the difference between *accuracy* and *efficiency* for a rational approximation to an irrational number. Consider $\sqrt{2} = [1; 2, 2, 2, \ldots] \approx 1.414213562$. Compute the convergents p_2/q_2 and p_3/q_3. Compare them to 1.414 in terms of accuracy and efficiency.

(5) Define the following: algorithm, greedy algorithm, lexicographic order.

(6) How does Brahmagupta's method for finding solutions to Pell's equation work? Is it an algorithm?

(7) What is the difference between the chord of an angle and its sine?

(8) Explain (generally) how al-Khwarizmi solves a problem like $ax^2 + bx = c$.

(9) If $p(n)$ is the partition function, which is bigger, $p(10)$ or $2 \cdot p(5)$?

(10) Suppose $F(x)$ is the generating function for a sequence (a_n). Suppose $G(x) = x^5 \cdot F(x)$, and suppose that is the generating function for a sequence (b_n). What is the relationship between a_n and b_n?

(11) Given a unit length, explain how to construct $\frac{1+\sqrt{5}}{2}$ in the Euclidean fashion.

(12) How many solutions are there in the quaternions to $x^2 = -1$?

(13) What does it mean for one infinite set to be bigger than another?

(14) What does it mean for a group to be simple? What does it mean for a group to be abelian? Every symmetric group S_n factors into C_2 and A_n, where A_n is something called the *alternating group*. Galois showed that A_n is simple and non-abelian for $n \geq 5$. Why is this a big deal?

(15) Let's use the cycle notation $(abcd)$ to mean $a \mapsto b \mapsto c \mapsto d \mapsto a$, so that (ab) is the transposition swapping a and b. Show that (12) commutes with (34) but not with (13).

(16) Prove that the quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ is not isomorphic to any cyclic group C_n.

(17) What is the largest possible size of a Galois group for a polynomial of degree five?