CHAPTER 4

# The Alexandrian School: Euclid

*It is the glory of geometry that from so few principles, fetched from without, it is able to accomplish so much.*
ISAAC NEWTON

## 4.1 Euclid and the *Elements*

### A Center of Learning: The Museum

Toward the end of the fourth century B.C., the scene of mathematical activity shifted from Greece to Egypt. The battle of Chaeronea, won by Philip of Macedon in 338 B.C., saw the extinction of Greek freedom as well as the decay of productive genius on its native soil. Two years later, Philip was murdered by a discontented noble and was succeeded by his 20-year-old son, Alexander the Great. Alexander conquered a great part of the known world within 12 years, from 334 B.C. to his death in 323 B.C., at the age of 33. Because his armies were mainly Greek, he spread Greek culture over wide sections of the Near East. What followed was a new chapter of history, known as the Hellenistic (or Greek-like) Age, which lasted for three centuries, until the Roman Empire was established.

Alexander's great monument in Egypt was the city that still bears his name, Alexandria. Having taken and destroyed the Phoenician seaports in a victorious march down the Eastern Mediterranean, Alexander was quick to see the potential for a new maritime city (a sort of Macedonian Tyre) near the westernmost mouth of the Nile. But he could do little more than lay out the site, because he departed for the conquest of Persia soon afterward. The usual story is that Alexander, with no chalk at hand to mark off the streets, used barley from the commissary instead. This seemed like a good idea until clouds of birds arrived from the delta and ate the grain as fast as it was thrown. Disturbed that this might be a bad omen, Alexander consulted a soothsayer, who concluded that the gods were actually showing that the new city would prosper and give abundant riches.

At Alexander's death, one of his leading generals, Ptolemy, became governor of Egypt and completed the foundation of Alexandria. The city had the advantage of a superb harbor and docking facilities for 1200 ships, so it became with the shortest possible delay the trading center of the world, the commercial junction point of Asia, Africa, and Europe. Alexandria soon outshone and eclipsed Athens, which was reduced to the status of an impoverished provincial town. For nearly a thousand years, it was the center of Hellenistic culture, growing in the later years of the Ptolemaic dynasty to an immense city of a million people. Following its sacking by the Arabs in A.D. 641, the building of Cairo in 969, and the discovery of a shipping route around the Cape of Good Hope, Alexandria withered away, and by the time of the Napoleonic expedition its population had dwindled to a mere 4000.

The early Ptolemies devoted themselves to making Alexandria the center of intellectual life for the whole eastern Mediterranean area. Here they built a great center of learning in the so-called Museum (seat of the Muses), a forerunner of the modern university. The leading scholars of the times—scientists, poets, artists, and writers—came to Alexandria by special invitation of the Ptolemies, who offered them hospitality as long as they wished to stay. At the Museum, they had leisure to pursue their studies, access to the finest libraries, and the opportunity of discussing matters with other resident specialists. Besides free board and exemption from taxes, the members were granted salary stipends, the only demand being that they give regular lectures in return. These fellows of the Museum lived at the king's expense in luxurious conditions, with lecture rooms for their discussions, a colonnaded walkway in which to stroll, and a vast dining hall, where they took their meals together. The poet Theocritus, enjoying the bounty, hailed Ptolemy as "the best paymaster a free man can have." And another sage, Ctesibius of Chalcis, when asked what he gained from philosophy, candidly replied, "Free dinners."

Built as a monument to the splendor of the Ptolemies, the Museum was nonetheless a milestone in the history of science, not to mention royal patronage. It was intended as an institution for research and the pursuit of learning, rather than for education; and for two centuries scholars and scientists flocked to Egypt. At its height, this center must have had several hundred specialists, whose presence subsequently attracted many pupils eager to develop their own talents. Although one poet of the time contemptuously referred to the Museum as a birdcage in which scholars fattened themselves while engaging in trivial argumentation, science and mathematics flourished with remarkable success. Indeed, it is frequently observed that in the history of mathematics there is only one other span of about 200 years that can be compared for productivity to the period 300–100 B.C., namely the period from Kepler to Gauss (1600–1850).

Scholars could not get along without books, so the first need was to collect manuscripts; when these were sufficiently abundant, a building was required to hold them. Established almost simultaneously with the Museum and adjacent to it was the great Alexandrian library, housing the largest collection of Greek works in existence. There had of course been libraries before it, but not one possessed the resources that belonged to the Ptolemies. Manuscripts were officially sought throughout the world, and their acquisition was vigorously pressed by agents who were commissioned to borrow old works for copying if they could not otherwise be obtained; travelers to Alexandria were required to surrender any books that were not already in the library. Many stories are told of the high-handed methods by which the priceless manuscripts were acquired. One legend has it that Ptolemy III borrowed from Athens the rolls kept by the state containing the authorized texts of the writers Aeschylus, Sophocles, and Euripides. Although he had to make a deposit as a guarantee that the precious volumes would be returned, Ptolemy kept the original rolls and sent back the copies (needless to say, he forfeited the deposit). A staff of trained scribes catalogued the books, edited the texts that were not in good condition, and explained those works of the past that were not easily understood by a new generation of Greeks.

The Alexandrian library was not entirely without rivals in the ancient world. The most prominent rival was in Pergamon, a city in western Asia Minor. To prevent Pergamon from acquiring copies of their literary treasures, the jealous Ptolemies, it is said, prohibited the export of papyrus from Egypt. Early writers were careless with numbers and often exaggerated the size of the library. Some accounts speak of the main collection at the library as having grown to 300,000 or even 500,000 scrolls in Caesar's time (48 B.C.), with an additional 200,000 placed in the annex called the Serapeum. The collection had been built partly by the purchase of private libraries, one of which, according to tradition, was Aristotle's. After the death of Aristotle, his personal papers passed into the hands of a collector who, fearing that they would be confiscated for the library at Pergamon, hid all the manuscripts in a cave. The scrolls were badly damaged by insects and moisture, and the Alexandrian copyists made so many errors when restoring the texts that they no longer agreed with the versions of Aristotle's works already housed in the library.

## Euclid's Life and Writings

Before the Museum passed into oblivion in A.D. 641, it produced many distinguished scholars who were to determine the course of mathematics for many centuries: Euclid, Archimedes, Eratosthenes, Apollonius, Pappus, Claudius Ptolemy, and Diophantus. Of these, Euclid (circa 300 B.C.) is in a special class. Posterity has come to know him as the author of the *Elements of Geometry*, the oldest Greek treatise on mathematics to reach us in its entirety. The *Elements* is a compilation of the most important mathematical facts available at that time, organized into 13 parts, or books, as they were called. (Systematic expositions of geometry had appeared in Greece as far back as the fifth century B.C., but none have been preserved, for the obvious reason that all were supplanted by Euclid's *Elements*.) Although much of the material was drawn from earlier sources, the superbly logical arrangement of the theorems and the development of proofs displays the genius of the author. Euclid unified a collection of isolated discoveries into a single deductive system based on a set of initial postulates, definitions, and axioms.

Few books have been more important to the thought and education of the Western world than Euclid's *Elements*. Scarcely any other book save the Bible has been more widely circulated or studied; for 20 centuries, the first six books were the student's usual introduction to geometry. Over a thousand editions of the *Elements* have appeared since the first printed version in 1482; and before that, manuscript copies dominated much of the teaching of mathematics in Europe. Unfortunately, no copy of the work has been found that actually dates from Euclid's own time. Until the 1800s, most of the Latin and English editions were based ultimately on a Greek revision prepared by Theon of Alexandria (circa 365) some 700 years after the original work had been written. But in 1808, it was discovered that a Vatican manuscript that Napoleon had appropriated for Paris represented a more ancient version than Theon's; from this, scholars were able to reconstruct what appears to be the definitive text.

Although the fame of Euclid, both in antiquity and in modern times, rests almost exclusively on the *Elements*, he was the author of at least 10 other works covering a wide variety of topics. The Greek text of his *Data*, a collection of 95 exercises probably intended for students who had completed the *Elements*, is the only other text by Euclid on pure geometry to have survived. A treatise, *Conic Sections*, which formed the foundation of the first four books of Apollonius's work on the same subject, has been irretrievably lost, and so has a three-volume work called *Porisms* (the term *porism* in Greek mathematics means "a corollary"). The latter is the most grievous loss, for it

Euclid
(circa 300 B.C.)

(Smithsonian Institution.)

apparently was a book on advanced geometry, perhaps an ancient counterpart to analytic geometry.

As with the other great mathematicians of ancient Greece, we know remarkably little about the personal life of Euclid. That Euclid founded a school and taught in Alexandria is certain, but nothing more is known save that, the commentator Proclus has told us, he lived during the reign of Ptolemy I. This would indicate that he was active in the first half of the third century B.C. It is probable that he received his own mathematical training in Athens from the pupils of Plato. Two anecdotes that throw some light on the personality of the man have filtered down to us. Proclus, who wrote a commentary to the *Elements*, related that King Ptolemy once asked him if there was not a shorter way to learning geometry than through the *Elements*, to which he replied that there is "no royal road to geometry"—implying thereby that mathematics is no respecter of persons. The other story concerns a youth who began to study geometry with Euclid and inquired, after going through the first theorem, "But what shall I get by learning these things?" After insisting that knowledge was worth acquiring for its own sake, Euclid called his servant and said, "Give this man a coin, since he must make a profit from what he learns." The rebuke was probably adapted from a maxim of the Pythagorean brotherhood that translates roughly as, "A diagram and a step (in knowledge), not a diagram and a coin."

## 4.2 Euclidean Geometry

### Euclid's Foundation for Geometry

For more than two thousand years Euclid has been the honored spokesman of Greek geometry, that most splendid creation of the Greek mind. Since his time, the study of the *Elements*, or parts thereof, has been essential to a liberal education. Generation after generation has regarded this work as the summit and crown of logic, and its study as the best way of developing facility in exact reasoning. Abraham Lincoln at the age of 40, while still a struggling lawyer, mastered the first six books of Euclid, solely as training for his mind. Only within the last hundred years has the *Elements* begun to be supplanted by modern textbooks, which differ from it in logical order, proofs of propositions, and applications, but little in actual content. (The first real pedagogical improvement was by Adrien-Marie Legendre, who in his popular *Eléments de Géométrie*, rearranged and simplified the propositions of Euclid. His book ran from an initial edition in 1794 to a twelfth in 1823.) Nevertheless, Euclid's work largely remains the supreme model of a book in pure mathematics.

Anyone familiar with the intellectual process realizes that the content of the *Elements* could not be the effort of a single individual. Unfortunately, Euclid's achievement has so dimmed our view of those who preceded him that it is not possible to say how far he advanced beyond their preparatory work. Few, if any, of the theorems established in the *Elements* are of his own discovery; Euclid's greatness lies not so much in the contribution of original material as in the consummate skill with which he organized a vast body of independent facts into the definitive treatment of Greek geometry and number theory. The particular choice of axioms, the arrangement of the propositions, and the rigor of demonstration are personally his own. One result follows another in strict logical order, with a minimum of assumptions and very little that is superfluous. So vast was the prestige of the *Elements* in the ancient world that its author was seldom referred by name but rather by the title "The Writer of the *Elements*" or sometimes simply "The Geometer."

Euclid was aware that to avoid circularity and provide a starting point, certain facts about the nature of the subject had to be assumed without proof. These assumed statements, from which all others are to be deduced as logical consequences, are called the "axioms" or "postulates." In the traditional usage, a postulate was viewed as a "self-evident truth"; the current, more skeptical view is that postulates are arbitrary statements, formulated abstractly with no appeal to their "truth" but accepted without further justification as a foundation for reasoning. They are in a sense the "rules of the game" from which all deductions may proceed—the foundation on which the whole body of theorems rests.

Euclid tried to build the whole edifice of Greek geometrical knowledge, amassed since the time of Thales, on five postulates of a specifically geometric nature and five axioms that were meant to hold for all mathematics; the latter he called common notions. (The first three postulates are postulates of construction, which assert what we are permitted to draw.) He then deduced from these 10 assumptions a logical chain of 465 propositions, using them like stepping-stones in an orderly procession from one proved proposition to another. The marvel is that so much could be obtained from so few sagaciously chosen axioms.

Abruptly and without introductory comment, the first book of the *Elements* opens with a list of 23 definitions. These include, for instance, what a point is ("that which has no parts") and what a line is ("being without breadth"). The list of definitions concludes: "Parallel lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction." These would not be taken as definitions in a modern sense of the word but rather as naive descriptions of the notions used in the discourse. Although obscure and unhelpful in some respects, they nevertheless suffice to create certain intuitive pictures. Some technical terms that are used, such as *circumference of a circle*, are not defined at all, whereas other terms, like *rhombus*, are included among the definitions but nowhere used in the work. It is curious that Euclid, having defined parallel lines, did not give a formal definition of *parallelogram*.

Euclid then set forth the 10 principles of reasoning on which the proofs in the *Elements* were based, introducing them in the following way:

## Postulates

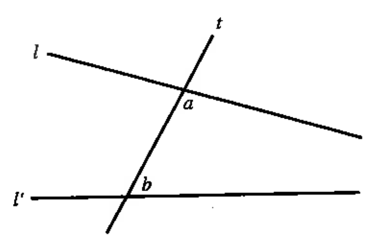
Let the following be postulated:

1. A straight line can be drawn from any point to any other point.
2. A finite straight line can be produced continuously in a line.
3. A circle may be described with any center and distance.
4. All right angles are equal to one another.
5. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, then the two straight lines, if produced indefinitely meet on that side on which the angles are less than two right angles.

## Common Notions

1. Things that are equal to the same thing are also equal to one another.
2. If equals are added to equals, the wholes are equal.
3. If equals are subtracted from equals, the remainders are equal.
4. Things that coincide with one another are equal to one another.
5. The whole is greater than the part.

Postulate 5, better known as Euclid's parallel postulate, has become one of the most famous and controversial statements in mathematical history. It asserts that if two lines $l$ and $l'$ are cut by a transversal $t$ so that the angles $a$ and $b$ add up to less than two right angles, then $l$ and $l'$ will meet on that side of $t$ on which these angles lie. The remarkable feature of this postulate is that it makes a positive statement about the whole extent of a straight line, a region for which we have no experience and that is beyond the reach of possible observation.




Those geometers who were disturbed by the parallel postulate did not question that its content was a mathematical fact. They questioned only that it was not brief, simple, and self-evident, as postulates were supposed to be; its complexity suggested that it should be a theorem instead of an assumption. The parallel postulate is actually the converse of Euclid's Proposition 27, Book I, the thinking ran, so it should be provable. It was thought impossible for a geometric statement not to be provable if its converse was provable. There is even some suggestion that Euclid was not wholly satisfied with his fifth postulate; he delayed its application until he could advance no further without it, though its earlier use would have simplified some proofs.

Almost from the moment the *Elements* appeared and continuing into the nineteenth century, mathematicians have tried to derive the parallel postulate from the first four postulates, believing that these other axioms were adequate for a complete development of Euclidean geometry. All these attempts to change the status of the famous assertion from "postulate" to "theorem" ended in failure, for each attempt rested on some hidden assumption that was equivalent to the postulate itself. Futile so far as the main objective was concerned, these efforts led nevertheless to the discovery of non-Euclidean geometries, in which Euclid's axioms except the parallel postulate all hold and in which Euclid's theorems except those based on the parallel postulate all are true. The mark of Euclid's mathematical genius is that he recognized that the fifth postulate demanded explicit statement as an assumption, without a formal proof.

Detailed scrutiny for over 2000 years has revealed numerous flaws in Euclid's treatment of geometry. Most of his definitions are open to criticism on one ground or another. It is curious that while Euclid recognized the necessity for a set of statements to be assumed at the outset of the discourse, he failed to realize the necessity of undefined terms. A definition, after all, merely gives the meaning of a word in terms of other, simpler words—or words whose meaning is already clear. These words are in their turn defined by even simpler words. Clearly the process of definition in a logical system cannot be continued backward without an end. The only way to avoid the completion of a vicious circle is to allow certain terms to remain undefined.

Euclid mistakenly tried to define the entire technical vocabulary that he used. Inevitably this led him into some curious and unsatisfactory definitions. We are told not what a point and a line are but rather what they are not: "A point is that which has not parts." "A line is without breadth." (What, then, is *part* or *breadth?*) Ideas of "point" and "line" are the most elementary notions in geometry. They can be described and explained but cannot satisfactorily be defined by concepts simpler than themselves. There must be a start somewhere in a self-contained system, so they should be accepted without rigorous definition.

Perhaps the greatest objection that has been raised against the author of the *Elements* is the woeful inadequacy of his axioms. He formally postulated some things, yet omitted any mention of others that are equally necessary for his work. Aside from the obvious failure to state that points and lines exist or that the line segment joining two points is unique, Euclid made certain tacit assumptions that were used later in the deductions but not granted by the postulates and not derivable from them. Quite a few of Euclid's proofs were based on reasoning from diagrams, and he was often misled by visual evidence. This is exemplified by the argument used in his very first proposition (more a problem than a theorem). It involved the familiar construction of an equilateral triangle on a given line segment as base.

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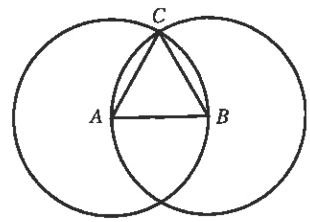
PROPOSITION 1 *For a line segment AB, there is an equilateral triangle having the segment as one of its sides.*

***Proof.*** Using Postulate 3, describe a circle with center $A$ and radius $AB$ passing through point $B$. Now, with center $B$ and radius $AB$, describe a circle passing through $A$. From the point $C$, in which the two circles cut one another, draw the segments $CA$ and $CB$ (Postulate 1 allows this), thereby forming a triangle $ABC$. It is seen that $AC = AB$ and $BC = AB$ because they are radii of the same circle. It then follows from Common Notion 1 that $AB = BC = AC$, and so triangle $ABC$ is equilateral.




There is only one problem with all this. On the basis of spatial intuition, one feels certain that the two circles will intersect at a point $C$ and will not, somehow or other, slip through each other. Yet the purpose of an axiomatic theory is precisely to provide a system of reasoning free of the dependence on intuition. The whole proposition fails if the circles we are told to construct do not intersect, and there is unhappily nothing in Euclid's postulates that guarantees that they do. To remedy this situation, one must add a postulate that will ensure the "continuity" of lines and circles. Later mathematicians satisfactorily filled the gap with the following:

> If a circle or line has one point outside and one point inside another circle, then it has two points in common with the circle.

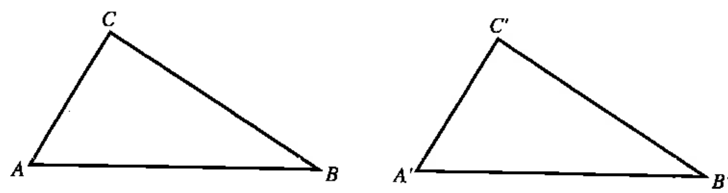
The mere statement of the postulate involves notions of "inside" and "outside" that do not explicitly appear in the *Elements*. If geometry is to fulfill its reputation for logical perfection, considerable attention must be paid to the meaning of such terms and to the axioms governing them.

During the last 25 years of the nineteenth century, many mathematicians attempted to give a complete statement of the postulates needed for proving all the long-familiar theorems of Euclidean geometry. They tried, that is, to supply such additional postulates as would give explicitness and form to the ideas that Euclid left intuitive. By far the most influential treatise on geometry of modern times was the work of the renowned German mathematician David Hilbert (1862–1943). Hilbert, who worked in several areas of mathematics during a long career, published in 1899 his main geometrical work, *Grundlagen der Geometrie* (Foundations of Geometry). In it he rested Euclidean geometry on 21 postulates involving six undefined terms—with which we should contrast Euclid's five postulates and no undefined terms.

## Book I of the *Elements*

The 48 propositions of the first book of the *Elements* deal mainly with the properties of straight lines, triangles, and parallelograms—what today we should call elementary plane geometry. Much of this material is familiar to any student who has had a traditional high-school course in plane and solid geometry. Although we shall not examine all these results in detail, Proposition 4 is one that deserves a close look. This proposition is called the side-angle-side theorem, for it contains the familiar criterion for congruence of triangles, namely, two triangles are congruent if two sides and the included angle of one are congruent to the corresponding sides and included angle of the other. We have used the word *congruent* where Euclid spoke of *equality*. When he referred to two angles (or for that matter, two line segments) as "equal," he meant that they could be made to coincide. For our purposes, it is safe to think of congruent objects as having the same size and shape.

Euclid tried to give a proof of the side-angle-side theorem by picking up one triangle and superimposing it on the other triangle so that the remaining parts of the two triangles fitted. His argument, which was supposedly valid by Common Notion 4, ran substantially as follows: Given $\triangle ABC$ and $\triangle A'B'C'$, where $AB = A'B'$, $\angle A = \angle A'$, and $AC = A'C'$, move $\triangle ABC$ so as to place point $A$ on point $A'$ and side $AB$ on side $A'B'$. Because $AB = A'B'$, point $B$ must fall on point $B'$. Because $\angle A = \angle A'$, the side $AC$ has the same direction as side $A'C'$, and because of the equal lengths of $AC$ and $A'C'$, the points $C$ and $C'$ fall on each other. Now, if $B$ and $B'$ coincide and $C$ and $C'$ coincide, so must the connecting line segments $BC$ and $B'C'$. The two triangles coincide in all respects, so it follows that they are congruent.




Although this "principle of superposition" may seem reasonable enough in dealing with material triangles made of wire or wood, its legitimacy has been questioned for working with conceptual entities whose properties exist only because they have been postulated. Indeed, the prominent British logician Bertrand Russell (1872–1970) spoke of superposition in no uncertain terms as a "tissue of nonsense." The chief criticism is that in assuming that a triangle can be moved about without any alteration in its internal structure, when it is only known that two sides and an included angle remain constant, one is really assuming that these determine the rigidity of the triangle. Thus, in postulating the possibility of movement without change in form or magnitude, congruence itself is actually being postulated. Euclid's proof is therefore a vicious circle of reasoning. It has been conjectured that Euclid felt reluctant to use superposition in proving congruence and did so sparingly in the *Elements* but could not dispense with it entirely, for lack of a better method. Present-day mathematicians avoid the difficulty by taking the side-angle-side theorem as an axiom from which the other congruence theorems are then derived. At any rate, Euclid's approach to the problem of congruence was logically deficient.

Perhaps the most famous of the earlier propositions of Book I is Proposition 5, which states, "In an isosceles triangle, the angles at the base are congruent to one another."