that the manager's accountant needs to tell him. That is, until he gets a tax bill. The Hotel Infinity's accountant has managed to ensure that their tax rate is the lowest possible – lots of intergalactic diversification and artificial tax domiciles – but no matter what the tax rate is, when you apply it to the infinite income the result is infinite.[11] 'How can this have happened?' the manager screams. 'We are ruined. Our tax liability is infinite just like our profit.' The accountant sits him down in a comfortable chair and makes him a nice cup of tea. 'Let me explain,' he says. 'Just go ahead and pay your infinite tax bill. You will find that your profits are quite undiminished. They will still be infinite.'

All does not end well. The Hotel Infinity's long-suffering owners are gradually worn down by the complexities of managing infinitely many guests from infinitely many hotels in infinitely many galaxies. They are stuck in a recession of intergalactic proportions. It is predicted to last for billions of years. They decide to escape by making a *radical* change of business strategy – an infinitely radical change. They decide to rename the chain, rebrand their products, move into a new commercial niche. They decide to become the ultimate in fashionability – the ultimate minimalist hotel. They become the Hotel Zero. Life is simpler. Now there are no rooms, no guests, no staff, no running costs (room temperature is kept at absolute zero), no losses, no problems. There's even canned music in the bar with John Cage's work *4 minutes 33 seconds*[12] continuously playing, blank-canvas modern artworks in the lobby, and a free copy of the author's *Book of Nothing*[13] handed by way of consolation to every hopeful but disappointed guest, of whom there are many, infinitely many. And on the wall the thought for the day reads

> 'If people do not believe that mathematics is simple, it is only because they do not believe how complicated life is.'

chapter four

Infinity Is Not A Big Number

> 'That is infinite, this is infinite, from infinite arises infinite, when infinite is subtracted from infinite, what is left is infinite'
>
> Sanskrit Mantra[1]

AN IMMACULATE MISCONCEPTION

> 'Space is almost infinite. As a matter of fact, we think it *is* infinite'
>
> Dan Quayle[2]

There is an understandable tendency to think of infinity as simply a very big number, just a bit bigger than the biggest number you can think of, always just out of reach, like the end of the rainbow. Yet, to appreciate the subtleties of the infinite it is important to appreciate that infinity is not simply a very big number. It is qualitatively (and not just quantitatively) different from any finite number (like 124,453,567,000,000,000,000,000,000,000,001), no matter how big it is. This idea that infinity is just a very very large number is what most people are likely to think. It is tempting to think that infinity is just a count that keeps on going and so is approximately rather like the biggest number you could ever think of plus a bit more.

ALBERT OF SAXONY'S PARADOX

'And when he had taken the five loaves and the two fishes, he looked up to Heaven, and blessed, and brake the loaves, and gave them to his disciples to set before them; and the two fishes divided he among them all.
And they did all eat, and were filled.
And they took up twelve baskets full of the fragments, and of the fishes.
And they that did eat of the loaves were about five thousand men.

St Mark 6:41–44

Albert Ricmerstop was born in Helmstedt in West Saxony in 1316. He was to become one of the most influential logicians of the Middle Ages, studying in Prague and Paris before becoming first the Rector of the University of Paris and then the founding Rector of the University of Geneva in 1365. Besides producing his large body of work in logic and philosophy, he also played an important role in early political interactions between Church and State by carrying a series of diplomatic missions to the Pope on behalf of the Duke of Austria. As a result, just one year after his appointment in Vienna, he was named Bishop of Halberstadt and he remained in that office until his death in 1390. To later scholars he would be known as Albert of Saxony, or simply 'Albertucius' which means 'little Albert' in order to distinguish him from Albert Magnus ('the Great'), the famous thirteenth-century theologian.

Albert was an acute thinker who played the game of medieval theology in new ways, devising procedures for determining the truth or falsity of statements or 'sophismata'[3] used in teaching and evaluating the

limits of different philosophical systems. The sophismata were sentences which were in some way hard to understand, ambiguous, or paradoxical. The name of the game was to deal with the ones that rival philosophers came up with and create telling examples of your own. A sentence like 'Nothing is something' or 'Only God is infinite' or 'This statement is false' would be simple candidates. Albert was interested in the paradoxes and problems of the infinite and discusses them in his book, *Sophismata*. In the course of his discussions he provided a wonderfully incisive paradox about infinity which was later to form the basis for the definition of an infinite collection and the foundation for a rigorous discussion of actual infinities. This wasn't Albert's intention, of course, but it shows how carefully he thought about the question, and also reveals the influence of the English philosophers of the day whose use of mathematics was enthusiastically taken up and promulgated by Albert.

Albert showed that a single infinite allows you to get something for nothing – indeed, to get as much as you want for nothing. Take an infinitely long beam of wood, with a square cross-section of size 1 unit by 1 unit (see Figure 4.1). Now saw it up into cubes of equal size. You will have an infinite number of these cubes which you can now use as building blocks. Albert argues that you can use them to fill the whole of space by assembling them in a systematic way. Surround the first block by $3^3 - 1 = 26$ blocks so as to make a bigger cube of side equal to 3 units. Now surround that cube with $5^3 - 3^3 = 98$ more blocks to create a new cube of side equal to 5 units. By continuing this process using $7^3 - 5^3$, then $9^3 - 7^3$ then $11^3 - 9^3$, and so on forever, of the original blocks, you would be able to build a single cube of ever-increasing volume. The infinitely long beam that you started with can therefore be cut up and re-assembled to fill the whole of an infinite three-dimensional space!

Albert's clever example shows that even in the fourteenth century there was a clear appreciation of the curious feature of infinity, that it can be put in direct correspondence with a part of itself. The importance of Albert's example was that it destroyed Aristotle's confident dogma that there cannot exist an infinite collection of

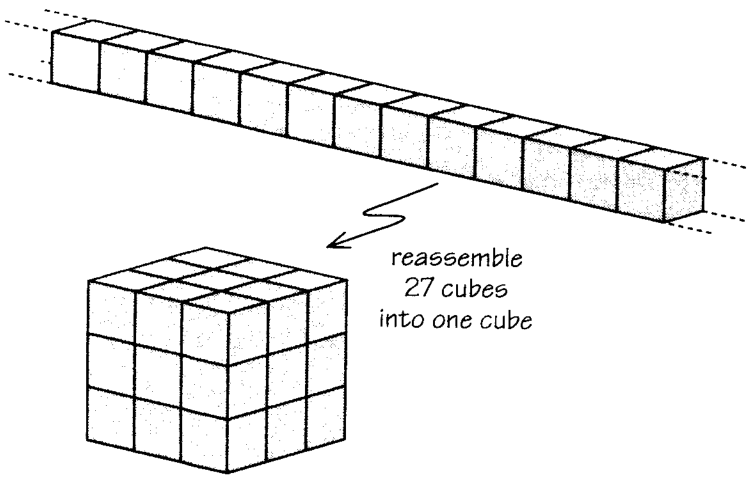


Fig 4.1 *Albert's magic process that shows how to fill the whole of infinite space by cutting up an infinitely long beam of wood that is only 1 centimetre square in cross-section and reorganising the pieces into a cube of ever-increasing size.*

things simply because it would contain a smaller subset that was also infinite and this was absurd. The example shows how such a situation can come about and there is no internal logical contradiction involved. In fact, Albert's example is cleverer than it needed to be in order to make his point, although one can imagine him doing a nice demonstration cutting up a long beam and assembling the first few sets of cubes so that everyone could see what was going to happen if he kept on going forever.

A much simpler example that makes the same point about infinities was suggested by Galileo. It shows his familiarity with the medieval fascination with infinity and sharpens our appreciation of the central paradox. It is interesting that Galileo raises the matter in his book of Dialogues, which was a work of 'popular' science for all literate people to read. It presented important ideas and discoveries in dramatic form using dialogues and argument to bring out the truth of his ideas.

GALILEO'S PARADOX

'You can tell whether a man is clever by his answers. You can tell whether a man is wise by his questions.'

Naguib Mahfouz[4]

Here in the imaginary dialogue that Galileo[5] created we find written down in its simplest form the key paradox about infinite collections that distinguishes them from finite ones. Galileo knows there is something mysterious about infinity, as did Albert of Saxony, but like Albert he makes no attempt to resolve the puzzle. Galileo reveals[6] these mysterious properties one by one. The dialogue is shown overleaf.

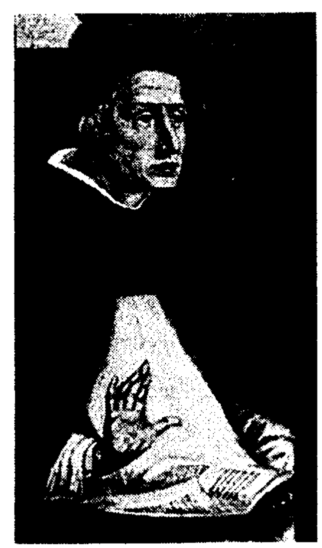


Fig 4.2 *Albert of Saxony (1316–90).*[7]

A Dialogue *between Salviati, Sagredo and Simplicio:*

Sag: I take it for granted that you know which of the numbers are squares and which are not.
Sim: I am quite aware that a squared number is one which results from the multiplication of another number by itself; thus 4, 9, etc., are squared numbers which come from multiplying 2, 3, etc., by themselves.
Salv: Very well; and you also know that just as the products are called squares so the factors are called sides or roots; while on the other hand those numbers which do not consist of two equal factors are not squares. Therefore if I assert that all numbers, including both squares and non-squares, are more than the squares alone, I shall speak the truth, shall I not?
Sim: Most certainly.
Salv: If I should ask further how many squares there are one might reply truly that there are as many as the corresponding number of roots, since every square has its own root and every root its own square, while no square has more than one root and no root more than one square.
Sim: Precisely so.
Salv: But if I inquire how many roots there are, it cannot be denied that there are as many as there are numbers because every number is the root of some square. This being granted we must say that there are as many squares as there are numbers because they are just as numerous as their roots, and all the numbers are roots. Yet at the outset we said there are many more numbers than squares, since the larger portion of them are not squares. Not only so, but the proportionate number of squares diminishes as we pass to larger numbers. Thus up to 100 we have 10 squares, that is the squares constitute 1/10 part of all the numbers; up to 10,000, we find only 1/100 part to be squares; and up to a million only 1/1000 part; on the other hand in an infinite number, if one could conceive of such a thing, we would be forced to admit that there are as many squares as there are numbers all taken together.
Sag: What then must one conclude under these circumstances?
Salv: So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all numbers, nor the latter greater than the former; and finally the attributes 'equal', 'greater', and 'less', are not applicable to infinite, but only to finite, quantities . . . I answer him that one . . . does not contain more or less or just as many points as another, but that each . . . contains an infinite number . . . So much for the first difficulty.
Sag: Pray stop a moment and let me add to what has already been said an idea which just occurs to me. If the preceding be true, it seems to me impossible to say either that one infinite number is greater than another or even that it is greater than a finite number, because if the infinite number were greater than, say, a million it would follow that on passing from the million to higher and higher numbers we would be approaching the infinite; but this is not so; on the contrary, the larger the number to which we pass, the more we recede from [this property] of infinity, because the greater the numbers the fewer [relatively] are the squares contained in them; but the squares in infinity cannot be less than the totality of all the numbers as we have just agreed; hence the approach to greater and greater numbers means a departure from infinity.

First, Galileo points out that if we list all the positive whole numbers

1, 2, 3, 4, 5, 6, 7 . . . and so on

then this list is infinite, because there is no end to it. If you doubt this then name the last number in the sequence (call it B) and I will be able to produce a bigger number (B + 1) by adding 1 to it.

Now, if we square every number in the list by multiplying it by itself, then to each integer there corresponds one square number:

$$1 \times 1 = 1,\ 2 \times 2 = 4,\ 3 \times 3 = 9,\ 4 \times 4 = 16,\ 5 \times 5 = 25, \text{ etc.}$$

The list of squared numbers (1, 4, 9, 16, 25, . . .) is therefore also infinite because it is in one-to-one correspondence with the infinite list of integers. Think of there being a string tied between each number and its square.

number	→	square
1	→	1
2	→	4
3	→	9
4	→	16
5	→	25
6	→	36
7	→	49
8	→	64
9	→	81
10	→	100

. . . *and so on forever* . . .

We now have two lists shown in the columns above.

Now, Galileo asks, which list is bigger? Every entry in the list of squares is tied to one and only one entry in the list of integers so it looks as if they must be equally numerous – the same size. But there is a paradox. Every entry in the list of squares (right-hand column) will also occur somewhere in the left-hand column of integers (the first three are underlined in the table above) so surely the left-hand list must be bigger than the right-hand list because it contains lots of other numbers as well!

Galileo did not resolve this paradox. He concludes only that,

> 'we cannot speak of infinite quantities as being the one greater or less than or equal to another'

In fact, Galileo was making slightly heavy weather of his example. There was no need to make his readers struggle with squared numbers. Just think about the correspondence between all the whole numbers (1, 2, 3, 4, . . .) and all the even numbers (2, 4, 6, 8 . . .) that result from doubling them: so 1 links to 2, 2 links to 4, 3 links to 6, 4 links to 8, and so on. Like cups and saucers in a tea set they are paired one to one. Again, there is a unique one-to-one correspondence between the infinite list of numbers and the infinite list of even numbers. Yet all the even numbers are contained in the first list despite the fact that 'common sense' says there must only be half as many even numbers as there are whole numbers!

number	→	double
1	→	2
2	→	4
3	→	6
4	→	8
5	→	10
6	→	12
7	→	14
8	→	16
9	→	18
10	→	20

. . . *and so on forever* . . .

The important thing to appreciate about these examples is that they reveal something unique to infinite collections. If we had taken *finite* lists of things, then they can only be put in one-to-one linkage with each other if they contain an *equal* number of things. For instance, a finite list of married couples contains an equal number of males and females.

Bob → Jill
Jim → Joyce
Ron → Louise
Roy → Carol

What Galileo's paradox reveals is that infinite collections are not like this: they seem to be able to contain themselves as subsets with plenty left over!

There is a temporal counterpart of these 'getting something for nothing' paradoxes that is usually called the Paradox of Tristram Shandy. It takes Tristram Shandy a whole year to complete an account in his diary of one day in his life. He completes his entry for 1 January, 1760, at midnight on 31 December, 1760; his entry for 2 January, 1760, at midnight on 31 December, 1761, and so on. All the time he is getting further and further behind. If he lives for a finite time he will only have written diary entries for a fraction of the days of his life. But if he lives forever there will be no day of his life for which he has not written a diary entry.

There is also a spatial counterpart, dubbed the 'Map Paradox', which arises when you begin to think of making a map that has a one-to-one scale. We are used to maps that are partial representations of the Earth's surface but, as the American philosopher Josiah Royce first suggested,

> 'suppose that this our resemblance is to be made absolutely exact . . . A map of England, contained within England, . . . One who, with absolute exactness of perception, looked down upon the ideal map thus supposed to be constructed, would see lying upon the surface of England, and at a definite place thereon, a representation of England on as large or small a scale as you please . . . This representation, which would repeat in the outer portions the details of the former, but upon a smaller space, would be seen to contain another England. And this another, and so on without limit.'[8]

This paradox has been frequently revisited by everyone from Lewis Carroll to Jorge Luis Borges. It is essentially a self-reference paradox rather than a paradox of the infinite. To bring the infinite possibilities into play one could instead stand between two plane-parallel mirrors and look at the never-ending line of reflections of reflections (Figure 4.3) that stretch out like the ghosts of Banquo to the crack of doom.

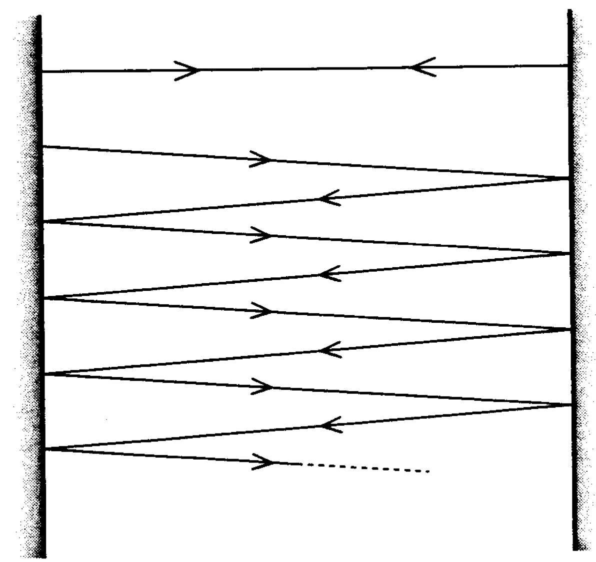


Fig 4.3 *Two parallel mirrors produce a seemingly infinite number of self-reflections. In practice the number is finite because the silvering of the mirrors is not perfect and light is scattering if it moves in a medium that is not a perfect vacuum. Light moves with a finite speed, and so even in perfect conditions an infinite number of images would need an infinite number of reflections to occur and would take an infinite time to produce.*

In reality there are only a finite number of images. The reflectivity is not perfect and the atmosphere scatters light out of the beam's path. Yet, the effects are striking and provide us with one of the simplest and closest snapshots of a potential infinity.

CADMUS AND HARMONIA

'Prove all things.'

St Paul[9]

Mathematicians have long been enchanted by never-ending sequences of numbers. They have beautifully unexpected properties. In 1350 the French mathematician Nicole Oresme proved that the infinite harmonic series of decreasing terms

$$\tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} + \ldots$$

has an infinitely large sum. The proof is very neat. After the first two terms, the next two terms ($\tfrac{1}{3} + \tfrac{1}{4}$) sum to more than $\tfrac{1}{2}$, so do the next four terms, the next eight terms, the next sixteen terms, and so on respectively forever, doubling the number of terms gathered together. The result is that the sum of the series must be bigger than an infinite sum of one halves.[10] And this is obviously infinite![11]

This series appears unexpectedly in all sorts of interesting situations. Suppose you are interested in records for natural phenomena, like record annual rainfalls or high tides.[12] In year 1 of keeping records the rainfall will have to be a record. In year 2 the rainfall has a $\tfrac{1}{2}$ chance of being a record – if it is greater than that of year 1. The expected number of record rainfall years in the first two is therefore $1 + \tfrac{1}{2}$. Carrying on, we see that there is a $\tfrac{1}{3}$ chance that year 3 has

higher rainfall than years 1 and 2. Keep on going and we see that the expected number of record rainfalls in the first N years of record keeping is

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \ldots + \tfrac{1}{N}$$

To find the number of records expected per century if conditions are random just put N = 100 and add up the terms. The answer is 5.19. Certainly in the United Kingdom at the moment there are a lot more record rainfall years – and other climatic records – than the 5 per century that this simple harmonic series predicts. This implies that the weather variations are not random, and that there is a systematic change underlying the observed variations, similar to that expected from so called 'global warming'. Notice that the infinite value of the sum of the series reflects the intuitive fact that there is always a chance of a new record in an infinite sequence of observations.

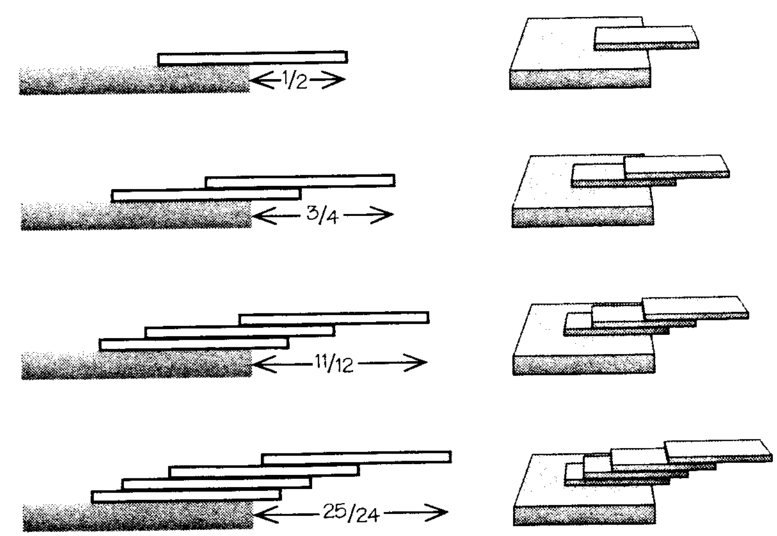


Fig 4.4 *A never-ending stack of books. An infinite number of books can be supported so long as the centre of gravity of the stack never lies beyond the edge of the bottom book. This is possible in principle, not in practice.*

Another nice example of the harmonic series is the book-stacking problem. Pile books on top of one another so that they overhang the side of the table, as shown in Figure 4.4. How far can they protrude over the edge of the table without falling?[13]

They have to be stacked so that the centre of gravity of the stack never lies beyond the edge of the table. Once it does they will start to topple. If each book has size 1 then the maximum possible overhang of N books is just one half of the sum of the harmonic series up to N terms:

$$\text{Maximum Overhang Distance} = \tfrac{1}{2} \times \{1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \ldots + \tfrac{1}{N}\}$$

The amazing thing about this is that the overhang distance can be made as large as you like by making N big enough.[14] To make the overhang greater than 10 times the size of a single book would need a stack of 272,400,600 books. In an ideal world without friction and imperfect surfaces and smallest particles of matter, the overhang could be infinite!

TERMINATOR 0, ½, AND 1

'There may be trouble ahead.'

Irving Berlin[15]

The harmonic series has a clear-cut behaviour which reveals itself very easily after you have picked the right way of looking at things. If an infinite series of terms does not add up to less than a finite number then it is said to be divergent. The ubiquity of the harmonic series[16] seems to endow it with a harmless familiarity. This is a little misleading. A few examples can restore the apoplexy with which divergent series were so long regarded.

Begin with a simple infinite series, which we suppose is equal to S. It consists of alternating plus and minus 1s, so

$$S = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 \ldots$$

We want to evaluate the sum of this never-ending series. If we first group the numbers appearing in S in pairs as shown below in brackets, the sum of the series is 'obviously' zero, because each bracketed pair of +1 and –1 sums to zero:

$$S = (1 - 1) + (1 - 1) + (1 - 1) + (1 - 1) + \ldots$$

$$S = 0 + 0 + 0 + 0 + \ldots$$

And so the sum is S = 0.

But, we could have grouped the terms in the series differently, say by bracketing the next pair along to the right; then

$$S = 1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \ldots$$

Now we can show that S = 1 because each of the bracketed pairs again makes zero, so

$$S = 1 + 0 + 0 + 0 + \ldots$$

So we have proved that S = 0 and S = 1 and so 0 = 1!

But why stop there? We can put the brackets down a third way so that

$$S = 1 - (1 - 1 + 1 - 1 + 1 - \ldots)$$

But the unending series in the brackets is just S again, so we have

$$S = 1 - S$$

and so $2S = 1$ and S must be equal to 1/2 this time. Armed with these results it is not too great a challenge to show that S can be 'proved' to be equal to any number you like. No wonder the great nineteenth-century mathematicians like Niels Abel[17] avoided divergent series like the plague.

Seen like this, infinities seem to provide the basis for no end of financial scams. We are used to badly behaved computer systems being fixed by simply switching the computer off and back on (this action never fixes my car though). Here we seem to be offered the opportunity of doubling our money just by counting it in a different order. Of course, we know that when our series of alternating 'ones' has only a finite number of terms in it there is no problem at all. Its sum must

Fig 4.5 *Georg Cantor (1845–1918) with his wife, Vally.*[18]

either equal 0 or +1. It doesn't matter how we add up the terms that appear, or where we draw in the brackets, the sum is 0 if there are an even number of terms in the series and +1 otherwise. It is only when you are infinitely rich that your assets depend totally on the order in which you count them up.

Not surprisingly, arguments like this made mathematicians very nervous about infinities. It is easy to see why infinity was regarded as a form of logical plague that destroyed the reliability of everything it touched. In the one subject where infinities could be manipulated clearly they led to disaster. As a result, the desire to banish infinities to some quarantined area away from the rest of logical argument, or to regard them as non-existent, was very strong. At most times in the history of human thought there have been mathematicians who wanted to rid their subject of them except as a form of shorthand for sums of things that have no end.

Out of all this ambiguity and confusion, clarity emerged suddenly in the nineteenth century, due to the single-handed efforts of one brilliant man. Georg Cantor (1845–1918) produced a theory that answered all the objections of his predecessors and revealed the unexpected richness hiding in the realm of the infinite (Figure 4.5). Quite suddenly actual infinities became part of mathematics – but not without a struggle.

COUNTABLE INFINITIES

'That action is best which procures the greatest happiness for the greatest numbers.'

Francis Hutcheson[19]

Cantor took the paradoxes that were anathema to mathematicians and used them as the basis for a clear understanding of infinities.

Realising the crucial significance of the strange paradoxes of Albert and Galileo, he changed their status from ill-fitting cast-offs to the central cornerstone of a new theory. Cantor *defined* a countable infinity to be one that can be put into one-to-one correspondence with the list of natural numbers 1, 2, 3, 4, 5, 6, . . . So, for example, the even numbers are countably infinite, so are all the odd numbers. Here is the correspondence for the first nine odd numbers.

1 ⟶ 3
2 ⟶ 5
3 ⟶ 7
4 ⟶ 9
5 ⟶ 11
6 ⟶ 13
7 ⟶ 15
8 ⟶ 17
9 ⟶ 19
10 ⟶ . . . and so on forever

All countably infinite sets therefore have the same 'size' in Cantor's sense. Cantor thought that they were the smallest infinities that could exist and so he denoted them by the first letter of the Hebrew alphabet, the symbol Aleph-nought, $\aleph_0$. Notice how this definition excludes any finite set of objects. Like your tea set of cups and saucers, a finite set can only be put into one-to-one correspondence with another which contains the same number of members (one cup for one saucer).

This leads to some surprising conclusions. Cantor showed that all the fractions formed by dividing one whole number by another (for example $^2/_3$ or $^{11}/_{12}$) are also countably infinite. The trick is to find a system for counting them so that none get missed out. He used a famous diagonal picture to do this. It counts them row by row in the following order:

$^1/_1$,
$^2/_1$, $^1/_2$,
$^1/_3$, $^2/_2$, $^3/_1$,
$^4/_1$, $^3/_2$, $^2/_3$, $^1/_4$,
$^1/_5$, $^2/_4$, $^3/_3$, $^4/_2$, $^5/_1$,
$^6/_1$, $^5/_2$, $^4/_3$, $^3/_4$, $^2/_5$, $^1/_6$, . . .
and so on forever.

The trick is that along each row the numbers on the top and bottom of each fraction add up to give the same number (so in the 4th row down they all add up to 5, i.e. (4+1), (3+2), (2+3), and (1+4)). This creates a definite order for counting all the fractions which will not miss out any one of them. Whereas we might have thought there were vastly more fractions than single numbers, they are equally numerous when counted Cantor's way. All the infinities that were discussed by mathematicians and philosophers in ancient times were countable infinities in Cantor's sense. But are there any others?

UNCOUNTABLE INFINITIES

'Al-Gore-rhythm: a mathematical operation, which if repeated many times, leads to the desired result – especially in Florida.'

Anonymous

Cantor then showed by a new type of mathematical argument that there were bigger, 'uncountable', infinities. The decimals (most of which are never-ending and include the irrational numbers which cannot be written as fractions) could not be counted systematically. They were 'uncountably' infinite. He proved this in a cunning way. Assume that they can be counted. This means that we must be able to draw up a systematic recipe for counting all the unending decimals. The first few in the list might look like these

1 ⟶ 0.23456789 . . .
2 ⟶ 0.575603737 . . .
3 ⟶ 0.463214516 . . .
4 ⟶ 0.846216388 . . .
5 ⟶ 0.562194632 . . .
6 ⟶ 0.466732271 . . .
. . . and so on

Now we are going to create a new decimal by taking the first digit after the decimal point from the first number, the second digit from the second number, and so on forever. I have underlined the digits we are to use as the digits in our new number. The new decimal begins as follows

0.273292 . . .

Now create a new decimal from this one by adding 1 to every one of its infinitely many digits. We get

0.384303 . . .

The remarkable thing about this number is that it *cannot* appear anywhere on the original ordered listing of all the decimals that we had assumed must exist. It must always disagree with every number in the list by at least one of its digits because it was explicitly constructed like that. Therefore the decimals (sometimes called the real numbers or the continuum of numbers) are uncountably infinite. They are infinitely bigger than the natural numbers or the fractions, and in accord with their special status they are denoted by the Hebrew symbol Aleph-one,

$\aleph_1$. Cantor believed that there was no possible infinite collection that was bigger than $\aleph_0$ but smaller than $\aleph_1$, but he was never able to prove it. It turned out to be one of the great problems of mathematics and one that had a most unusual resolution.

This discovery by Cantor – that there are infinities of different sizes and they can be distinguished in a completely unambiguous way – was one of the great discoveries of mathematics. It was also completely counter to the prevailing opinion.

Cantor's predecessor, Bernhard Bolzano (1781–1848), shown in Figure 4.6, began thinking about the paradoxes of the infinite in 1847 when he was sixty-seven years old.

Fig 4.6 *Bernhard Bolzano (1781–1848).*[20]

He came to believe that all infinities were equal. The reason can be seen most simply by looking at another of the 'paradoxes' that Galileo and his medieval predecessors liked to exhibit to challenge the coherence of the idea of the infinite.[21] Take a piece of string and use it to make a semi-circle that has a diameter of one metre. Now imagine an infinitely straight line drawn underneath the semi-circle, parallel to the diameter, see Figure 4.7.

If we draw any straight line from the centre of the semi-circle down to the infinite straight line then it will always cut through the semi-circle at some point on its circumference. The remarkable thing is that the diagram makes it obvious that there is a line like this that links every point on the circumference of the semi-circle to one and only one point on the infinite straight line. So there must be the same number of points on the circumference of the semi-circle as on the line. Moreover, suppose we draw more semi-circles having the same centre but smaller radii. Then the set of all possible straight lines from the centre would pass through every point on the circumference of every circle and each would be in correspondence with every point on the circumference of every other circle. Thus it was argued all these circles contain an infinite number of points on their circumferences and they are all equal in number.

Bolzano concluded that infinite sets are 'equal' because they can

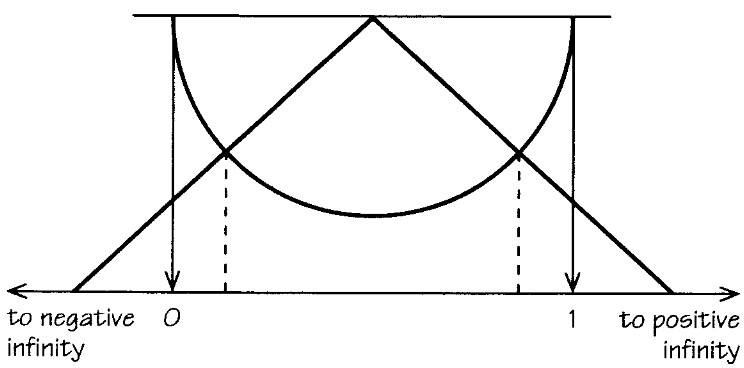


Fig 4.7 *The one-to-one correspondence between a line of one unit in length stretching horizontally from 0 to1 and the entire infinite line from negative infinity* (left) *to positive infinity* (right). *Take any point you choose on the line between negative infinity and positive infinity. Join it by a straight line to the centre of the semi-circle that we have drawn. Where this line cuts the semi-circle we drop a dashed line vertically downwards to pick out a point on the line between 0 and 1. By this process every point on the original line of infinite length ends up at one point on the finite part of the lines between 0 and 1.*

be linked by a correspondence like this. Cantor provided a beautiful example to show that this was not so. Not only were the never-ending decimals – what we called the 'real' numbers – infinitely bigger than the number of whole numbers or fractions, but there could be infinities that were infinitely bigger still.

THE TOWERING INFINITO

'Somebody has to have the last word. Otherwise, every reason can be met with another one and there would be no end to it.'

Albert Camus[22]

Cantor's most dramatic discovery was that infinities are not only uncountable, they are insuperable. He discovered that a never-ending ascending hierarchy of infinities must exist. There is no biggest of all that can contain them all. There is no Universe of universes that we can write down and capture. Before we see how he did that, it is important to say a little about the meaning of the word 'exists' in this context. We are used to using the word on an everyday basis without any ambiguity. 'Cambridge exists', 'inflation exists', seem to be assertions that are clear enough. They are about physical existence. Up until the early nineteenth century, mathematical existence was rather similar. Euclid's geometry existed because it was manifested in the physical world. Indeed, it was believed for thousands of years that there could not be another logically consistent and complete geometrical system. The discovery of non-Euclidean geometries which described the topography of curved surface changed that view. Gradually mathematicians lighted upon a new concept of existence. Mathematical 'existence' meant only logical self-consistency and this neither required nor needed physical existence to complete it. If a mathematician could write down a set of non-contradictory axioms and rules for deducing true statements from them, then those statements would be said to 'exist'. They exist in the same way that positions exist in a game like chess. They are developments according to the rules from the starting position (the axioms). Now it happens that in chess these positions are usually made physical by chess pieces on a fixed board – but this is not necessary. Some experts play in their heads without pieces or a board; others can play by mailing the coordinates of the positions on an imaginary board. So it is with mathematics. Some examples of mathematical existence do have physical existence, but most do not.

When Cantor set about showing that an unending catalogue of mathematical infinities exists, his first aim was to demonstrate mathematical existence: to show that precise definitions of things like infinite sets lead to the conclusions that ever larger ones can be defined. Whether they exist in physical reality is another, quite different, question.

At first you might think that making bigger infinities is child's play. Suppose you have an infinite collection of numbers 1, 2, 3, . . . Just add one more thing to it – say the object *. Isn't that bigger? Unfortunately not; this is just the situation of the Infinite Hotel. Adding one, or two, or even all the whole numbers to a countable infinity still leaves a countable infinity. In Cantor's sense it is the same size. In order to jump up a level to a new order of infinity something different is required, as we saw with the introduction of the never-ending decimals, or 'real' numbers, that are uncountably infinite.

Cantor was able to show that there is no end to the ascending hierarchy of infinities. If you have any infinite set, then you can generate one that is infinitely bigger by considering the set that contains all its subsets. This is called its *power set*. As a finite example consider the set[23] of three objects {A,B,C}. (These could be people and the 'sets' groups of friends, families, or secret societies.) It contains subsets containing the following members (conventionally we include Ø, the empty set which has no members, and the complete set itself in the list of subsets):

{Ø}, {A}, {B}, {C}, {A,B}, {A,C}, {B,C}, {A,B,C}

There are $8 = 2 \times 2 \times 2 = 2^3$ subsets. In general, if the original set has N members then there are $2^N = 2 \times 2 \times 2 \times 2 \times \ldots$ (N times) possible subsets and members of its power set.

Thus from an infinite set like $\aleph_0$ we can create an infinitely larger set (by which we mean one that cannot be put in one-to-one correspondence with it) by forming its power set, $P[\aleph_0]$. Now we

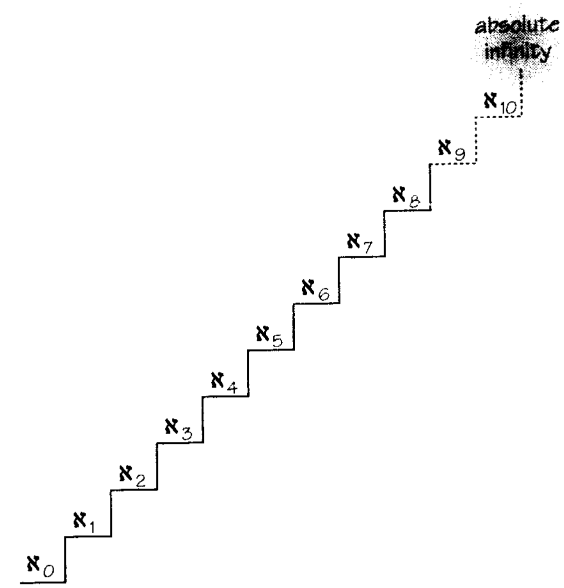


Fig 4.8 *The topless ascending tower of infinities.*

can do the same again by forming the power set of $P[\aleph_0]$. That will be infinitely bigger than $P[\aleph_0]$. And so on, without end.

Thus mathematics gives a never-ending hierarchy of ascending infinities (Figure 4.8). Infinity can never be captured by formulas. This is reminiscent of the ancient attempts to articulate the unreachable nature of God and the Infinite that are found in the great theological writers of the past. It also shows that the number of possible truths is infinite.[24]

These ideas had many theological and philosophical consequences and Cantor found that his ideas about the infinite were well received by scholars in these fields. Alas, within mathematics the story was quite different, as we shall see.

chapter five

The Madness of Georg Cantor

'To be listened to is a nearly unique experience for most people. It is enormously stimulating. Man clamors for the freedom to express himself and for knowing that he counts.'

Robert C. Murphy[1]

CANTOR AND SON

'I continued to do arithmetic with my father, passing proudly through fractions to decimals. I eventually arrived at the point where so many cows ate so much grass, and tanks filled with water in so many hours. I found it quite enthralling.'

Agatha Christie[2]

Cantor & Co. was a successful international wholesale business, and as a result young Georg Cantor was one of six children who grew up in comfortable circumstances, attending good private schools in Frankfurt. Georg had many talents and might well have pursued a career as a musician, as did some of his relatives, or as an artist. Yet in his teenage years he became increasing captivated by mathematics, physics, and astronomy. His father, Georg senior, was strongly supportive of all his studies and also imposed his strong religious beliefs in destiny upon his son; some biographers have wondered whether the paternal support was really just a case of the father's own unfulfilled ambitions being pursued through the life of his eldest son. Yet, for all this, Georg junior seems to have survived his life at home, and he graduated from Darmstadt School in 1862, aged seventeen, with high marks, moving first to study mathematics at the Polytechnical Institute in Zurich and then going on to the famed mathematics course at the University of Berlin, the centre of the mathematical world in the mid-nineteenth century. There he encountered great mathematicians like Karl Weierstrass, Sophie Kowalewski, and Ernst Kummer, who followed in the footsteps of men like Bernhard Riemann and Peter Dirichlet. He was also taught by the influential Leopold Kronecker.

Cantor followed the usual route of a young academic of the day, jumping through the hoops laid out for him by completing his degree and then his doctorate in Berlin, before beginning a form of apprenticeship which involved teaching pupils privately at the university in the city of Halle, a medieval city famous for being the birthplace of the great seventeenth-century composer George Frederick Handel. Halle University was an in-between place for a budding mathematician, geographically half-way between the great universities of Berlin and Göttingen; it was the sort of place that you hoped would be a stepping stone to becoming a professor at one of these two famous mathematical centres.

Unfortunately for Cantor, that call never came and he spent the whole of his career in the minor mathematical department in Halle – where there were few visitors and no mathematicians of Cantor's calibre – living comfortably in a big house with his close family following his marriage in 1875 to his sister's friend, Vally Guttman. Things were to become more exciting for Cantor, but not in ways that he could have wished.

THE CHRONICLE OF KRONECKER

'Logic sometimes makes monsters'

Henri Poincaré[3]

The year 1871 was a watershed in Cantor's career as a mathematician. Until that time, his former professor in Zurich, Leopold Kronecker, had been on good terms with him, sympathetic to his work and helpful in getting him established in Halle. He even provided some important mathematical suggestions which helped Cantor to complete some of his first research papers. Then something changed. Cantor began to work on infinities, and in Kronecker's eyes he had suddenly become 'a corrupter of youth'.[4]

Kronecker was the son of a wealthy Prussian businessman and was in no need of a university salary to support his mathematical career

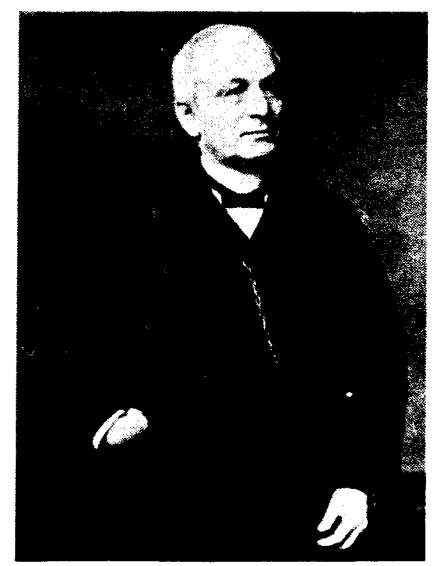


Fig 5.1 *Leopold Kronecker (1823–91).*[5]

(Figure 5.1). He did important work on algebra and number theory in Berlin, but had to spend a period of eleven years away from mathematics while running the family business. Eventually, he returned to become a professor in Berlin in 1882.

The historian of mathematics David Burton writes that

> 'Kronecker was a tiny man, who was increasingly self conscious of his size with age. He took any reference to his height as a slur on his intellectual powers. Making loud voice of his opinions, he was venomous and personal in his attacks on those whose mathematics he disapproved; and his opinions relative to the new theory of infinite sets were ones of ire and indignation . . . Kronecker categorically rejected [Cantor's] ideas [about infinite sets] from the start. He asserted dogmatically, "Definitions must contain the means of reaching a decision in a finite number of steps, and existence proofs must be conducted so that the quantity in question can be calculated with any required degree of accuracy."'[6]

Any discussion of infinite sets was, according to Kronecker, illegitimate since it began with the assumption that infinite sets exist in mathematics.

Kronecker wanted to define mathematics to consist only of those deductions that could be made in a finite number of steps from the natural numbers (1, 2, 3, 4 . . .). This goal is encapsulated in a famous remark he made in a speech: 'God created the natural numbers, and all the rest is the work of man.'

Kronecker was not alone in holding such views, but he was the most influential and vociferous advocate of the mathematical straitjacket called 'finitism'. He believed that we should only do mathematics by building up quantities and arguments in a finite number of steps. Today, this would be classed as the mathematics that a computer could carry out if correctly programmed. We know that this is a small fraction of what is allowed to be mathematics if we do not restrict ourselves to finite step-by-step deductions.

Kronecker would not allow you to assume that something exists if you could not explicitly describe how it could be constructed. Likewise, he would not admit into mathematics those proofs which showed that something must exist without giving the step-by-step recipe for arriving at its construction. In effect, Kronecker believed in a smaller scope for mathematics than did most other mathematicians.

Up until the work of Cantor on infinities, it had been possible to take Gauss's view that infinities in mathematics were always *potential* infinities, and so mention of 'infinity' was just a shorthand for describing a series or a process that had no end: you didn't *do* anything with these infinities. You didn't use them to prove other things were true.

Gauss, the greatest mathematician of the day, had set the tone when he wrote in a letter to his friend Schumacher in 1831 that

> 'I protest against the use of infinite magnitude as something completed, which in mathematics is never permissible. Infinity is merely a façon de parler, the real meaning being a limit which certain ratios approach indefinitely near, while others are permitted to increase without restriction.'

In universities all over the continent of Europe, the division between potential and actual infinities was regarded as crucial, and the general view was that only potential infinities were meaningful.

Despite this current of opinion, most mathematicians held mild views on the issue and rarely encountered a problem where taking a view about finitism really mattered. As a result, most were surprised, and many were irritated, by Kronecker's outspoken finitist views – but the highly-strung, increasingly paranoid Cantor was the most seriously affected by Kronecker's criticisms. All of his work was focused upon defining and manipulating actual infinities and Kronecker characterised this work as a study of things that did not exist, and total 'humbug'![7]

Cantor's hopes of becoming professor of mathematics at the University of Berlin were totally blocked by Kronecker's opposition. Kronecker's influence extended far beyond Berlin, and at Göttingen as well Cantor was repeatedly passed over in favour of seemingly less-distinguished candidates. Kronecker also sat on the editorial boards of journals which delayed or prevented the publication of some of Cantor's work. As a result, Cantor spent his entire professional career, forty-four years, at Halle University, a small college with no mathematical reputation.

Yet Cantor did get his important work published between 1874 and 1884, and it was well known, if occasionally controversial, amongst his young colleagues in Germany at the time – all the more reason for his despair about his lack of advancement. Cantor eventually became so angered by Kronecker's attacks that he wrote directly to the Ministry of Education, hoping to annoy Kronecker by applying for a position vacant in Berlin the following spring. He wrote to his old friend Gösta Mittag-Leffler on 30 December 1883, telling of his desperate measure:

> 'I never thought in the least I would actually come to Berlin . . . since I know that for years Schwarz and Kronecker have intrigued terribly against me, in fear that one day I would come to Berlin, I regarded it as my duty to take the initiative and turn to the Minister himself. I knew precisely the immediate effect this would have: that in fact Kronecker would flare up as if stung by a scorpion, and with his reserve troops would strike up such a howl that Berlin would think it had been transported to the sandy deserts of Africa, with its lions, tigers, and hyenas. It seems that I have actually achieved this goal!'[8]

Kronecker responded the following month by himself writing to Mittag-Leffler (the editor of *Acta Mathematica*) asking if he could publish in his journal a short article setting out his views about certain

mathematical conceptions in which he would show that 'the results of modern . . . set theory [i.e. Cantor's work] are of no real significance'.[9]

Actually, Kronecker had no intention of publishing such a paper, but simply wanted to rattle Cantor into refusing to publish in Mittag-Leffler's journal again in the belief that the editor had betrayed his faith in him by agreeing to publish Kronecker's paper.

At first, however, Cantor was pleased to hear of Kronecker's intention to write a critical article, as it would make Kronecker's opposition public and he would be able to answer it. But then, as Kronecker hoped, Cantor seems to have become suspicious that it would degenerate into personal polemics and told the editor that if the journal published anything critical from Kronecker, he would not support the journal with any of his own work in the future. Kronecker never did send anything to the journal, and the events show something of Cantor's paranoia and despair.

In 1884 Cantor attempted to cool things down by writing directly to Kronecker in a spirit of reconciliation and they had several discussions. However, although Kronecker was outwardly conciliatory, no real peace was made. Cantor concluded there was little hope of success. Indeed, any success Cantor had with others made Kronecker feel even more threatened by Cantor's ideas. Cantor says that, 'It seems to me of no small account that he and his preconceptions have been turned from the offensive *to the defensive* by the success of my work.'[10]

Soon afterwards Mittag-Leffler suggested that one of Cantor's papers should not be published in his journal, saying diplomatically that its insights were 'one hundred years too soon'. This was devastating to Cantor and he never published in the journal again, saying 'I never want to know anything again about *Acta Mathematica*'. (He had also, in 1878, resolved never to publish again in *Crelle's Journal*, another mathematics journal influenced by Kronecker.) As a result, by 1885 he had decided to give up mathematics entirely (Figure 5.2).

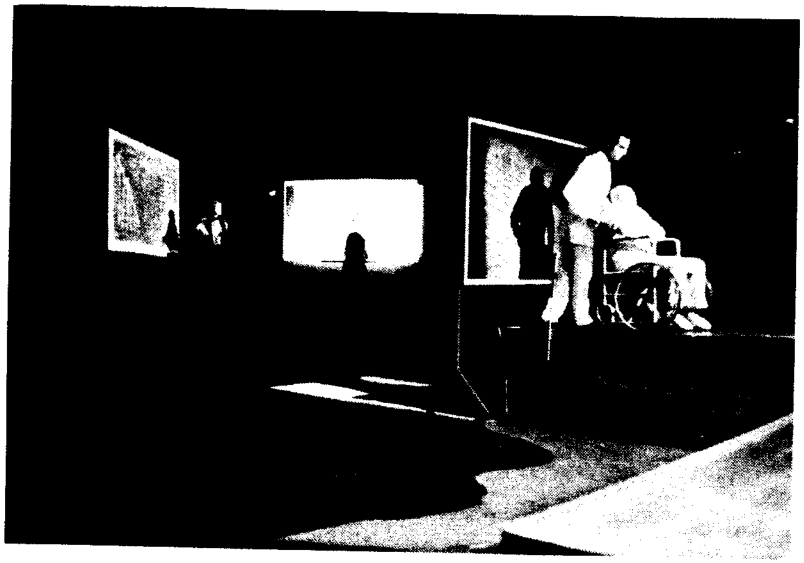
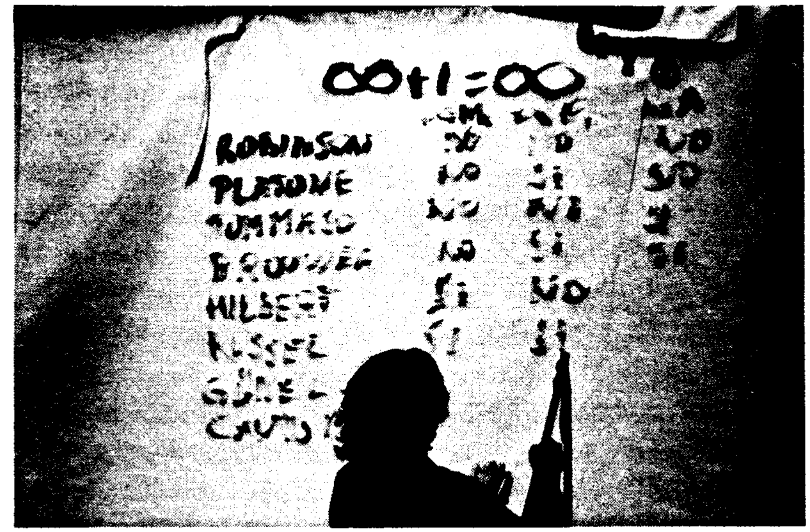


Fig 5.2 *Pictures of Cantor's struggle with mathematics and mathematicians from the Milan production of* Infinities.

Cantor's belief that he was being persecuted by Kronecker because of his mathematical views led to a complete nervous breakdown in 1884. He made a recovery one month later, but subsequently his life was punctuated by bouts of depressive illness which forced him to stay for periods in the clinic in Halle. In the intervals when his mind was clear, he spent a lot of time on studies of the ancient history of counting systems, theology, and history. It was not until the 1900s, when he had finished his research, that Cantor's work started to receive international recognition, with many prizes and honorary degrees being awarded to him. However, this recognition came mainly from outside Germany and Cantor complained, in 1908, of the German mathematicians 'who do not seem to know me, although I have lived and worked among them for fifty-two years'.

Ultimately, as we will see, these events and stresses tipped him into depression and undermined his belief in the worth of his own or any mathematical research. He attempted to transfer from the mathe-



matics department to the philosophy department at his university – a request that was refused. Yet, the university bent over backwards to give him time to rehabilitate himself, and hired temporary lecturers to deputise for him during his periods of illness and absence. To escape these periods of depression Cantor began contemplating the theological implications of his work on the infinite. Its reception by theologians was unexpected.

CANTOR, GOD, AND INFINITY – THE TRINITY WITH AFFINITY

'I entertain no doubts as to the truth of the transfinities, which I have recognised with God's help and which, in their diversity, I have studied for more than twenty years; every year, and almost every day brings me further in this science.'

Georg Cantor[11]

In 1885 Cantor put mathematics to one side and started to correspond with theologians and other intellectuals about infinity. Always someone of strong religious faith, and strongly influenced by his father's forceful beliefs, his attitude towards his work on infinity began to shift in an unusual way. He started to tell his friends that he had not been the inventor of the ideas about infinity that he had published. He was merely a mouthpiece, inspired by God to communicate parts of the mind of God to everyone else. This increased his belief in the truth of his work on infinity, for in his mind it had risen to the elevated status of revealed truth.

Cantor had changed direction at just the right time. The mathematical world in his vicinity may have been under the conservative influence of Kronecker's outspoken views, but when Leo XIII ascended to the papacy in 1878 he brought a liberalisation of the Church's attitudes in many areas. He sought to reconcile science and religion by offering a more enlightened lead from Rome.

This was good news for one Constantin Gutberlet, a priest, philosopher and theologian, and one of Germany's leading neo-Thomists. Gutberlet believed controversially that the human mind could grasp actual infinities and talk meaningfully about them. As a result, he had come under attack from Catholic theologians, but had responded by seizing upon Cantor's mathematical work to argue that it provided clear evidence that the human mind could contemplate the actual infinite. Moreover, if it did so, it would get closer to the true nature of the Divine. The collection of divine thoughts in the mind of an unchanging God, he argued, must comprise a complete and infinite set. This was for him evidence that Cantor's infinities actually existed, and to deny it would require you to give up the infinite and absolute mind of God. High stakes indeed.

Gutberlet's approach is reminiscent of the way in which Euclid's geometry had played an important role in supporting claims that the human mind could have access to matters of ultimate truth. If theologians were challenged by sceptics who argued that ultimate truth was something that transcended the human mind, they could point to Euclid as an example of part of the ultimate truth about the Universe that we have found. In the nineteenth century there would be radical changes to our view of mathematical structures like Euclid's geometry. No longer would it be possible to argue that Euclid's geometry was the one and only logically possible geometry and therefore tells how the world must necessarily be. It was recognised that there can exist other non-Euclidean geometries – infinitely many of them – all logically self-consistent. The fact that they exist mathematically by virtue of being logically self-consistent does not mean that they must exist in physical reality though.

Gutberlet wrote about the vital theological importance of Cantor's work, and entered into correspondence with him over the question of the absolute infinity of God's existence. Cantor was extremely interested in the theological consequences of his ideas, and argued that the higher infinities he had found increased the extent of God's dominion for they had no upper bound: there was no 'biggest' infinity. His never-ending tower of infinities provided a simple answer to the challenge that Gutberlet was facing, that understanding and codifying infinity was reducing the status of God. This might well have been worrying to some, had there been a biggest infinity.

Cantor believed that he could use his knowledge to prevent the Church making grave errors about its doctrines concerning infinity. He thought it was a mission to which he had been called. He declared in a letter to a friend, in 1896, that

> 'From me, Christian philosophy will be offered for the first time the true theory of the infinite.'[12]

He also said,

> 'But now I thank God, the all-wise and the all-good, that He always denied me the fulfilment of this wish [for a position at university either in Göttingen or Berlin], for He

> thereby constrained me, through a deeper penetration into theology, to serve Him and his Holy Roman Catholic Church better than I have been able with my exclusive preoccupation with mathematics.'[13]

Many have felt that Cantor was signalling his despair with all that had gone before and was just turning to a less demanding and controversial activity, away from Kronecker and the rivalries of other mathematicians. However, he interpreted his growing liking for theology and philosophy and his disaffection with mathematics as the work of God. He saw himself as a servant of God who had been given the talent for mathematics in order to be of service to the Church.

He gave up contact with his mathematical friends and was happy about his contacts with Church theologians and philosophers who were interested in his work and thought it significant. Religion renewed his self-confidence and convinced him that his work was important after all, despite the opposition of so many mathematicians. In 1887, Cantor wrote to his colleague Heman of his confidence that he could answer any criticism and overcome any opposition:

> 'My theory stands as firm as a rock; every arrow directed against it will return quickly to its archer. How do I know this? Because I have studied it from all sides for many years; because I have examined all objections which have ever been made against the infinite numbers; and above all, because I have followed its roots, so to speak, to the first infallible cause of all created things.'[14]

Georg Cantor was very interested in how mathematics might reveal the existence of God. In letters to Cardinal Franzelin, he indicated that the infinite, or the 'Absolute', belonged uniquely to God. He believed that it was God who ensured that the hierarchy of transfinite numbers existed, stretching beyond the simplest countable infinities,

increasing without limit. Because the largest of these could never be captured by a single formula – from any infinite set it was always possible to make an infinitely larger one – Cantor regarded the transfinite numbers as ascending directly to the Absolute, to the 'true infinity' whose magnitude was an absolute maximum that was incomprehensible to mere human understanding. The Absolute Infinite was beyond human determination, since once it was determined, the Absolute would no longer be regarded as infinite, because it would then necessarily be finite by definition – once determined it could be added and subtracted and manipulated or infinitely increased, just like the lesser infinities.

Thus Cantor seems to think of Absolute Infinity in the way that Archbishop Anselm thought of God in his famous 'ontological' proof of the existence of God, as being that above which no greater could be conceived.

What did Cantor's colleagues think about his ideas on God and infinity? Constantin Gutberlet had studied under Franzelin. He corresponded with Cantor and took his ideas very seriously. At first he was worried that Cantor's work on mathematical infinity challenged the unique, 'absolute infinity' of God's existence. However, Cantor assured him that instead of diminishing the extent of God's dominion, the transfinite numbers actually made it greater. After talking to Gutberlet, Cantor became even more interested in the theological aspects of his own theory on transfinite numbers.

Furthermore, Gutberlet argued that since the mind of God was unchanging, the collection of Divine thoughts must comprise an absolute, infinite, complete closed set, and offered this as direct evidence for the reality of concepts like Cantor's transfinite numbers. Like Pythagoras and Plato, Cantor believed that the numbers (particularly his transfinite numbers) were externally existing realities in the mind of God. They were discovered. They followed God-given laws, and Cantor believed it was possible to prove their existence from God's perfection and power. Indeed, Cantor said, it would have diminished God's power had God only created finite numbers.

Ironically, Cantor's love of the infinite had a distinctly anti-Pythagorean flavour. Pythagoras believed infinity was the destroyer in the Universe, the malevolent annihilator of worlds. If mathematics were a war, then the struggle was between the finite and the infinite. The Pythagoreans became obsessed with the negative aspects of infinity. They believed that the whole numbers closest to one (and therefore the 'most' finite in some sense of being farthest from the infinite) were the most pure of all numbers.

ALL'S SAD THAT ENDS BAD

'Behold the heaven of heavens cannot contain Thee'

The book of Chronicles[15]

Leopold Kronecker died in 1891 without ever becoming involved in a public criticism of Cantor's work. After 1895, a few of Kronecker's old allies opposed Cantor's ideas but, increasingly, the younger mathematicians supported Cantor and the dispute over finitism just faded away.[16] Cantor, however, never regained his mathematical powers and his decline had a terrible inevitability about it.

As we have seen, he had suffered his first breakdown in May 1884, just after his thirty-ninth birthday. He returned to doing mathematics in the autumn, but his interests had changed. He spent a lot of time working on Elizabethan history (trying to prove that Francis Bacon wrote Shakespeare's plays!), and early theology.

Eventually he suffered further breakdowns, and was in hospital for part of 1899 because of mental instability. He applied for leave of absence from teaching at Halle and wrote to the Ministry of Culture saying he wanted to leave his professorship. If they would pay him the same salary, he would be happy to take a quiet position in a library

somewhere. He wanted to break away from maths and stressed his knowledge of history and theology. He even threatened to apply to join the Russian diplomatic service. All this came to nothing.

In December 1899, while he was out giving a lecture in Leipzig about the Bacon-Shakespeare authorship issue, his youngest son, Rudolf, died suddenly just before his thirteenth birthday. Rudolf, although always frail and in poor health, had been a gifted musician, just as his father had been as a child before he gave up music for mathematics. Despite this cruel blow, Cantor managed to remain of sound mind for three years, but was back in hospital, relieved of his teaching duties again, in the winter of 1902–3. Some of his work was questioned in a public conference in 1904 and this agitated him greatly. He was in hospital during the winter of 1904–5, in 1907–8 and 1911–12. In 1915 an international meeting was planned to celebrate his seventieth birthday, but the war prevented all but a few close German friends from attending. He was admitted to the Halle clinic for the final time on 11 May 1917. He didn't return home. In wartime rationing conditions, food was scarce and he lost weight steadily. He died of heart failure on 6 January 1918, twenty-seven years after Kronecker. At the end of the game, the pawn and the king go back in the same box.