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"For the Honor of the Human Spirit"?

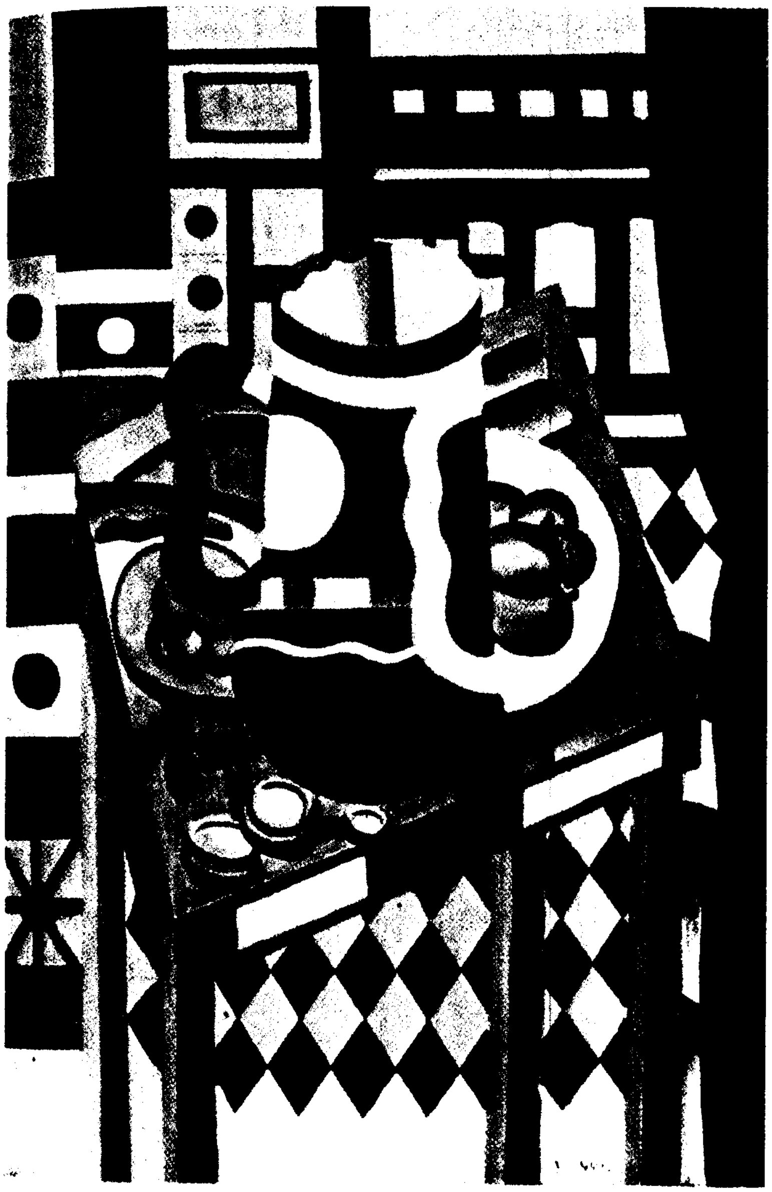
Bourbaki didn't only receive praise. Indeed, it met a certain amount of bitter criticism for its style, its choices, its power, and its lack of interest in anything outside of its vision of pure mathematics. Some of this criticism came from the group's own members.

In a letter to the French mathematician Adrien-Marie Legendre in 1830, the German mathematician Carl Gustav Jacob Jacobi wrote the following memorable words: "[...] Fourier believed that the main goals of mathematics were to serve the common good and to explain natural phenomena. However, a philosopher like him should have known that science is solely for the honor of the human spirit. Therefore a question about numbers is as important as a question about the world." Quoted time and again, Jacobi's eloquent phrase has become an emblem for the defense of pure mathematics. Jean Dieudonné named the book he wrote for the general public *Pour l'honneur de l'esprit humain* after this phrase (although it is published in English as *Mathematics - The Music of Reason*, after a quotation by the nineteenth-century mathematician James Joseph Sylvester) and repeated Jacobi's words in his epigraph. It is not a coincidence that this eminent former member of Bourbaki chose to emphasize Jacobi's point of view. Bourbaki was—and still is—a group of pure mathematicians who are interested mainly in pure mathematics.

Carl Gustav Jacobi, a "pure" mathematician.

The fact that Bourbaki is anchored in pure mathematics—and even a specific type of pure mathematics—did not only create admirers. In fact, many mathematicians detested Bourbaki. The group's style, its choices and omissions, its global vision of mathematics, and its influence all gave rise to harsh criticism. Today this group no longer holds a prominent place in the mathematical world, and the heat of criticism has fallen. But some criticisms remain, and discussing them will help show Bourbaki's place in the world of mathematics.

First, it is not really just to criticize Bourbaki for confining itself to



Fernand Léger's *Still life with beer mug* (1921). In the early the twentieth century, abstraction grew in art as well as in mathematics. In both disciplines, this increased the effort required for understanding a work's features.

pure mathematics. After all, one can't do everything—to each his own tastes and work! Also, the group didn't claim to cover all of mathematics, as it undertook to write a treatise on fundamental subjects that would be useful for all mathematicians. Bourbaki didn't wish to

Adrien-Marie Legendre, initially an "applied" mathematician.

explain theories specific to this or that mathematical domain, so it would also be unjust to criticize Bourbaki for failing to do so. The aspects of Bourbaki's work that do deserve criticism lie elsewhere. The Bourbakis did have to make a choice of approach for each topic they discussed, since there are almost always several different ways of approaching and developing a mathematical subject. However, their choices for *Éléments de mathématique* were sometimes unfortunate. In addition, Bourbaki lacked interest, and sometimes even showed contempt, for any areas not directly connected to pure mathematics. This would not have provoked any anger if the group had not gained considerable intellectual, and even institutional, power. But thanks in part to the group's Seminar (see chapter 7), Bourbaki's influence was so strong that the group's views directed French mathematics a great deal, and some scientific areas suffered because of this. In retrospect, Bourbaki showed a certain narrowness of perspective relative to the evolution of world mathematics, even while the group unquestionably contributed to the rebirth of French mathematics and helped raise it to hold third or fourth place in the world today.

Bourbaki's Choice: Neither Logic nor Applied Math

What was Bourbaki criticized for exactly? The subjects that Bourbaki neglected, and even scorned, included all mathematics having to do with applications. Among these neglected subjects were numerical analysis (which is about numerical calculations and solutions to equations or to other sorts of problems), probability theory, theoretical computer science, game theory, and optimization theory. It is true that most of the topics in applied mathematics lend themselves poorly to the axiomatic method and structural vision that Bourbaki advocated. But Bourbaki's lack of interest in these areas held back their development in France, while applied mathematics in the United States and the Soviet Union benefitted from rapid development starting in the 1940s with the advent of World War II. Even Jean Dieudonné, a zealous member of Bourbaki, admitted in his old age the wrong done to applied mathematics. As he told Marian Schmidt in 1990, "It is possible to say that there was no serious applied mathematics in France for forty years after Poincaré. There was even a snobbery for pure math. When one noticed a talented student, one would tell him 'You should do pure math.' On the other hand, one would advise a mediocre student to do applied math while thinking, 'It's all that he can do!'" In skillful praise of pure mathematics, Dieudonné added that "the truth is actually the reverse. You can't do good work in applied math until you can do good work in pure math."

It is worth discussing the case of probability theory specifically, since Bourbaki's lack of interest in this subject led Bourbaki to choose an approach for one of the books of *Éléments des mathématique* that many today consider unsuitable. Andrei Kolmogorov's work around 1930 made the study of probability into a quite abstract mathematical theory. In particular, this Soviet mathematician developed a set of axioms that form a rigorous foundation for probability theory. This theory is closely linked to measure theory and integration theory, which were founded by Émile Borel and Henri Lebesgue in the early twentieth century (see the box on page 127). Yet Bourbaki decided against using Lebesgue's methods in discussing integration theory. Under the influence of André Weil, Bourbaki instead chose to use the concept of Radon measures on locally compact spaces, which have no use whatsoever in probability theory. As the mathematician and mathematical historian Christian Houzel explained, "For many topics—the study of Brownian motion for example—probability theory relies on a measure defined on sets that are not locally compact. In the end, Bourbaki decided to add a chapter about spaces that are not locally compact to the group's book on integration, but this was akin to admitting that the previous chapters were nearly worthless."

Maybe Bourbaki would have made a different decision about what approach to use if the group had been interested in probability theory, but this in fact was not the case. "Bourbaki stepped away from probability, rejected it, considered it to be unrigorous. The group's considerable influence directed young mathematicians away from probability as well. The Bourbaki group, including myself, is greatly responsible for holding back the development of probability theory in France," Laurent Schwartz writes in his autobiography. Perhaps Bourbaki's contempt for probability is illustrated by what Schwartz writes about a lecture that the eminent American mathematician Joseph Doob gave in Paris some years ago. "The Bourbakis in the audience would interrupt constantly, loudly, and impolitely, saying that the space was not locally compact and that it 'didn't mean anything.' Their attitude was deeply troubling and their lack of courtesy outraging."

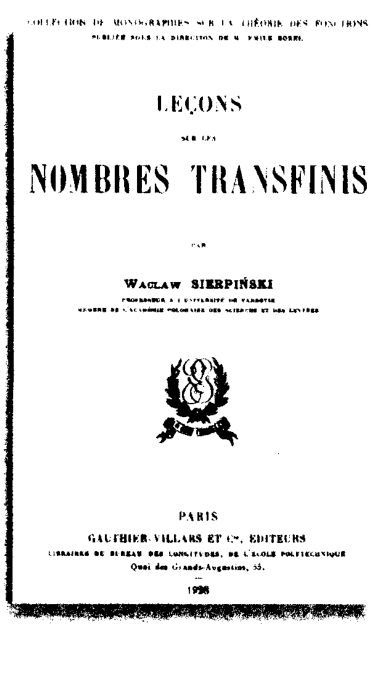
Andrei Kolmogorov (1903–1987), the father of modern probability theory.

Bourbaki's Lack of Interest in the Foundations

Neither was Bourbaki's attitude towards mathematical logic exemplary. At the time, it certainly would have been impossible to criticize the area for being too close to applied mathematics. But for Bourbaki, logic was something outside of mathematics. Jean Dieudonné, who liked to make cutting remarks, writes in the article *Mathématiques vides et mathématiques significatives* ("Empty mathematics and meaningful mathematics") that ninety-five percent of mathematicians "don't care a fig" for mathematical logic, that "when someone comes and talks to us about first and second order logic, recursive functions, and models, which are very nice and very beautiful theories that have yielded remarkable results, we, the mathematicians, don't see any reason people shouldn't study these topics, but on the other hand they don't attract us a bit." This nearsightedness is surprising in a mathematician who had such a broad education in his subject. Now the concepts he sarcastically mentioned play an important role in mathematics, particularly in connection with theoretical computer science. Of course, now these concepts are starting to have applications...

In his 1948 article *L'avenir des mathématiques* ("The future of mathematics"), André Weil expresses his hardly less caricatural point of view: "[...] logic is the mathematician's hygiene, but it does not provide his meals. The big problems constitute the daily bread on which he lives." In other words, Weil believes that none of the problems that logicians study are "big" problems in mathematics. Even the question of whether mathematics is internally coherent leaves Weil as unfeeling as marble. "It would be possible [...] that one day experience would lead us to discover the seed of contradiction in the methods of reasoning we currently use. Then we would need to revise mathematics. It is certain that this won't affect the heart of the subject." Here Weil is probably alluding to Kurt Gödel's proof from around 1930, which showed that it is impossible to prove that a system of axioms at the base of mathematics will never lead to a contradiction, if the proof uses only the axioms in question. Surprisingly, the spectacular results of Gödel and other logicians after him do not bother most mathematicians. They are satisfied with taking a pragmatic stand and not worrying too much about logical subtleties that could affect the foundations of their discipline. This view is understandable, as experience shows that questions about the foundations of mathematics rarely have repercussions in a mathematician's everyday work. That Bourbaki would adopt this point of view is less acceptable, since the group took pride in setting

Sierpinski's *Leçons sur les nombres transfinis.*


LEÇONS

NOMBRES TRANSFINIS

WACLAW SIERPIŃSKI

PARIS

GAUTHIER-VILLARS ET C^ie, ÉDITEURS

Quai des Grands-Augustins, 55.

1928

In la Messuguière in July 1975: Jean-Louis Verdier, Bernard Tessier, André Gramain, and Hyman Bass.

mathematics on the solid base of set theory, the axiomatization of which relies on formal logic.

Furthermore, the Bourbakis' lack of interest in mathematical logic rubbed off on their book on set theory, which starts with a chapter on formal logic. The Bourbakis themselves admit that this book was written "with pain and without pleasure, but we had to do it." Logicians disparaged the result. For example, in his article "The Ignorance of Bourbaki" published in *The Mathematical Intelligencer* in 1992, the British mathematician A.R.D. Mathias describes how Bourbaki's book on set theory shocked him profoundly: "It appeared to be the work of someone who had read *Grundzüge der Mathematik* by Hilbert and Ackermann, and *Leçons sur les nombres transfinis* by Sierpinski, which were both published in 1928, but nothing since." With reason, Mathias emphasizes that when Bourbaki and its members wrote about logic, they usually ignored Gödel and his work, and logicians believe this is unacceptable. A logician from Paris criticized Bourbaki at least as severely as Mathias did by saying that "these chapters should be thrown away [...] The Bourbakis wrote them even though they weren't interested in

A.R.D. Mathias criticized "the ignorance of Bourbaki."

Was Bourbaki too Powerful?

Did Bourbaki monopolize power in the French mathematical community? Bourbaki certainly held considerable intellectual power from the fifties through the seventies. It is also true that many members of Bourbaki held important positions in the mathematical community; they were university presidents, mathematics department chairs at l'École Normale Supérieure, presidents of the French Mathematical Society, presidents of the Centre International de Rencontres Mathématiques (in Marseilles), and members of the French Académie des Sciences. It's less clear, however, that Bourbaki's power exerted itself through these institutional positions. "Everything ascribed to Bourbaki's politics is false. Outside of mathematics, Bourbaki is a much too heterogenous group to be able to define a common position. During my twenty years in Bourbaki, I almost never heard anyone talk about university politics," asserts Michel Demazure. Jean-Pierre Kahane, who was never a member of Bourbaki, also believes that Bourbaki did not try to exert political power, saying that "Bourbaki had intellectual power, not institutional power. Its members didn't hold that many positions of power, and no one criticized people like Delsarte or Dieudonné for their use of power. In fact, they were known for their strict ethics." And Jean-Pierre Bourguignon judges that during the sixties Bourbaki was "a pressure group in the good sense of the word," as its members acted in the interest of the whole mathematical community.

However, these favorable opinions clash with those held by Claude Chevalley, one of the founding members of Bourbaki. In an interview with Denis Guedj published in 1981, Chevalley criticizes the role played by the group in the question of university careers. "From the first conferences until the War," he says, "it was tacitly understood that one should not discuss questions of university careers. Unfortunately, such discussions became pervasive after the War. This was probably because young mathematicians were entering the group, and naturally we wanted to make sure they would find jobs. This was the beginning of a fatal spiral.

Claude Chevalley and François Bruhat at the Bourbaki conference in Amboise, October 1956.

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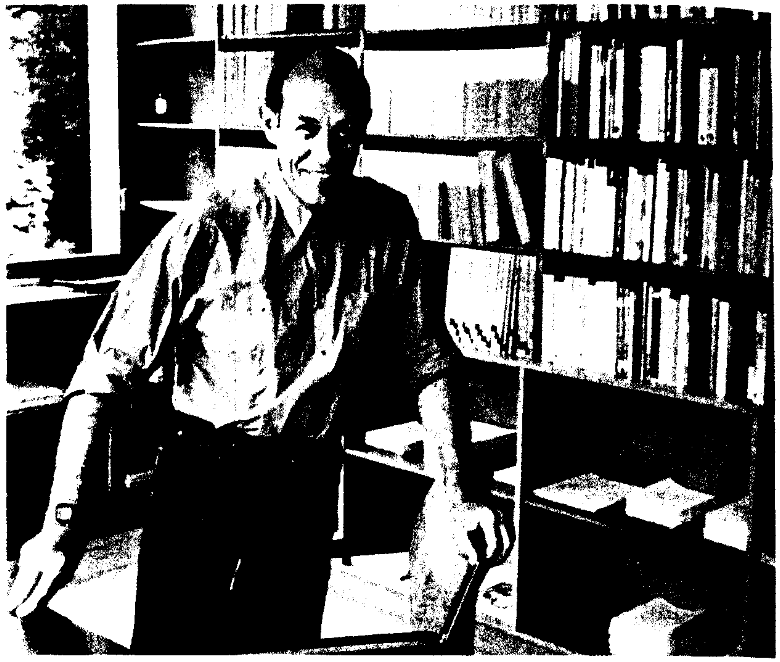
Little by little, we began talking about everyone's careers: it was complete decadence." He also criticizes the change in the group's attitude towards honorific titles. "We had agreed that no member would join the Académie [des Sciences]—I recall the conversations on the subject perfectly. But now they're almost all members."

Everyone agrees that Bourbaki's intellectual influence is strong, or even stifling. Jean-Pierre Bourguignon explained that he felt pressure to focus on algebraic geometry but decided to turn to differential geometry under the influence of Marcel Berger. It was only later, during a visit to the United States, that he realized that the French research in this area was not as poor as he had imagined. Marcel Berger and his students created a French mathematical school that existed alongside the Boubakian school. In addition, Bourguignon explained, an anonymous group similar to Bourbaki was formed around 1974. Named Arthur L. Besse, this group organizes seminars and has published four books on specialized subjects. Incidentally, the pseudonym Arthur L. Besse comes from the fact that the group called its meetings "round tables"—hence the first name Arthur, with L for Lancelot—and because the group held it first conference in Besse-en-Chandesse, the same place as Bourbaki's founding conference! In addition to Arthur L. Besse, several other centers of activity existed separately from Bourbaki, including Gustave Choquet's school of analysis, Jacques-Louis Lions' school of applied mathematics (formed in the early sixties), and André Lichnerowicz's school, which worked differential geometry in problems in theoretical physics. Jean Leray, a great mathematician who had almost no students, also remained separate from Bourbaki. Thus the Bourbaki group was not the only mathematical group in France, although until the seventies it was certainly at the top of the ladder. As Armand Borel, a former Bourbaki, wrote in 1998, "The mathematical atmosphere was not favorable to mathematicians who had a different temperament, a different approach. It's truly regrettable, but we can't hold it against the members of Bourbaki, who didn't force anyone to do research in their way."

the subject and weren't familiar with the work of contemporary logicians. They presented their own system of logic, which turns out to be unusable. The problem is that this book was treated as the main reference for mathematical logic for a long time in France. This caused a lot of damage to the field. The idea that logic is not an interesting subject spread, and traces of this view still remain in France's mathematical community today."

Bourbaki was also strikingly uninterested in physics. Over the course of the previous centuries, physics and mathematics had influenced each other greatly. Mathematics is very useful to physicists, and conversely, problems in physics have inspired many developments and discoveries in mathematics. For example, Fourier introduced his famous trigonometric series while studying the equations of heat propogation. Even the great turn-of-the-century mathematicians David Hilbert and Henri Poincaré were deeply interested in physics and contributed to



David Ruelle, a professor at the Institut des Hautes Études Scientifiques, was one of the pioneers of chaos theory.

clarifying certain topics in this discipline. This was also the case for Bartel van der Waerden and Emmy Noether, who were among the German algebraists so respected by the Bourbaki founders. But this was not the case for the Bourbakis, at least until recently. The example of André Weil is typical. Despite the fact that Göttingen, where Weil lived in 1926, was one of the centers of the quantum physics revolution during the 1920s, Weil didn't notice this development. "As I learned much later," he writes in *Souvenirs d'apprentissage*, "the world of physicists was bubbling with excitement in Göttingen as they were giving birth to quantum mechanics. It's quite remarkable that I didn't notice what was going on around me at all."

In Bourbaki's defense, it must be said that the decades between 1930 and 1960 were not very rich in mathematical physics, the area of physics most inclined to interact with mathematics. According to David Ruelle, a physicist-mathematician working at the Institut des Hautes Études Scientifiques near Paris, "The arrival of quantum mechanics is one of the reasons." Scientists put a massive amount of effort into this new field since there were so many new things to discover, but—at least at that time—it only required moderately elaborate mathematics. Moreover, many physicists showed contempt for the

Bourbaki style of mathematics, which was much more formal than what they were used to and what they thought they needed. These considerations could be what caused some scientists, such as Pierre-Gilles de Gennes (who received the 1991 Nobel Prize in Physics) and Claude Allègre (a geochemist and former French Minister of Education), to criticize mathematics. Meanwhile, mathematicians already had plenty of material to create their own problems, and the internal dynamics of mathematics moved most of the flourishing areas (especially algebraic geometry and algebraic topology) away from the cares of physicists. As Weil said in an interview with the *Gazette des mathématiciens* in October 1991, "Overall, it was a period where most of the big developments in mathematics did not originate in physics."

René Thom, the founder of catastrophe theory.

In short, Bourbaki was mostly impermeable to all subjects outside the heart of pure mathematics. But two remarks should be made to qualify this statement. First, it's important to differentiate Bourbaki from its members. For example, Claude Chevalley, who was the most philosophical of the group, became interested in logic thanks in part to the influence of his friend Jacques Herbrand. It was Chevalley who pressed the group to include formal logic in its treatise. He also wrote a long introduction to Bourbaki's book on set theory. Chevalley was very fond of this text, according to his daughter Catherine, but the group decided against its publication. Similarly, some members, including Laurent Schwartz, Jean-Louis Verdier, and Pierre Cartier, were very open to physics and applied mathematics. Schwartz, whose work affected physics and applied mathematics, gave a course on the mathematical methods of physics for several years. Meanwhile, Cartier, who was interested in the mathematical problems of applied physics, was the first to include lectures on mathematical physics in Bourbaki's seminars.

The second remark is that while Bourbaki sealed itself from neighboring areas of science, the converse was not true. Bourbaki's style and methods influenced researchers in applied mathematics, mathematical physics, and other fields. One example of this influence is the case of Jacques-Louis Lions (1928–2001), who founded the modern school of applied mathematics in France. This former Normalien, who entered l'École Normale in 1947 and studied under Laurent Schwartz, was never a member of Bourbaki. Nevertheless, the group's views show in the somewhat formal nature of his work. Another scientist influenced by Bourbaki is the economist Gérard Debreu, who won a Nobel Prize in 1983 for introducing methods of analysis into economics and for giving a rigorous reformulation of the theory of general equilibrium. Debreu is also a former Normalien, who entered in 1941 and studied mathematics. He turned towards economics after graduation and moved to the United States. Influenced by Bourbaki (he was taught by

Benoît Mandelbrot, the father of fractals.

Alexandre Grothendieck (presenting), with Jean Dieudonné to his left and Claude Chevalley to his right.

Henri Cartan), Debreu introduced the axiomatic method into economics. He thought that economics is not an experimental science like physics and thus that the internal logic of theoretical models in economics must be absolutely coherent. Hence, Lions believed, the axiomatic method is necessary in the theory of economics.

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Measure, Integration, and Probability

The integral is a typical example of a mathematical concept that can be approached from several different points of view. To explain this, let's limit ourselves to the case of a function f defined on an interval $[a, b]$ and taking positive real values. The integral of f over this interval is denoted by $\int_a^b f(x)dx$, whose value corresponds to the area between the x-axis and the curve representing f—one could say that the integral of f "measures" this area. The goal of integration theory is to give a precise meaning to this for as wide a class of functions as possible. The oldest approach is that of Cauchy and Riemann, which basically comes down to defining the integral as the following. The interval $[a, b]$ is subdivided into N intervals of length $\Delta x_1, \Delta x_2, \ldots, \Delta x_N$, and then the sum

$$S_N = f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \cdots + f(x_N)\Delta x_N,$$

where each x_i is a point in the interval Δx_i, is calculated as N approaches infinity. In other words, the area in question is calculated by approximating the region with N rectangles and calculating the sum of the area of these rectangles as N approaches infinity (see box 7 on page 64).

The modern approach, which was developed by Lebesgue, relies on the help of measure theory. In simplified and unrigorous terms, a measure defined on the set **R** of real numbers is a function m that associates to every subset A of **R** a nonnegative number $m(A)$ such that $m(A_1 \cup A_2) = m(A_1) + m(A_2)$ whenever the subsets A_1 and A_2 are disjoint (that is, whenever they have no common elements). These properties of nonnegativity and additivity make measures correspond to the intuitive concepts of length, area, and volume. It can be shown that there is a unique measure m on **R** such that $m([a, b]) = b - a$ for all a and b (in other words, so that the measure of any interval is its length). This measure is called the Lebesgue measure.

Now, how can one use the Lebesgue measure to define the integral of a function f defined on an interval $[a, b]$ and taking real values? Instead of subdividing the interval $[a, b]$ of the x-coordinates as in the case for Riemann integrals, we subdivide an interval $[y_{min}, y_{max}]$ of the y-axis that contains all the values taken by the function f. This yields N consecutive intervals

$$I_1 = [y_{min}, y_1), I_2 = [y_1, y_2), I_3 = [y_2, y_3), \ldots,$$
$$I_{N-1} = [y_{N-2}, y_{N-1}), I_N = [y_{N-1}, y_{max}].$$

For each of these intervals I_p, let A_p be the set of x such that $f(x) \in I_p$. (For example, A_2 is the set of x such that $f(x)$ lies between y_1 and y_2.) In general, A_p is not a simple interval but rather a union of disjoint intervals, or even something more complicated. Now, consider the product $z_p m(A_p)$ for each p, where m is the Lebesgue measure and z_p is a point in I_p, and construct the sum

$$S_N = z_1 m(A_1) + z_2 m(A_2) + \cdots + z_N m(A_N),$$

which approximates the area below the curve representing f and above the x-axis. When the interval $[y_{min}, y_{max}]$ is subdivided into finer and finer intervals, which corresponds to letting N approach infinity, the above sum generally approaches some number. By definition, this sum is the Lebesgue integral of the function f over the interval $[a, b]$.

It would be necessary to delve into the definitions and theorems in detail to show why the Lebesgue integral has numerous advantages over the Riemann integral. Most importantly, however, the Lebesgue integral is more general, since every function that is Riemann integrable is also Lebesgue integrable (with the values of the two integrals being equal), while there are functions that are Lebesgue integrable but not Riemann integrable. For example, the function g defined by $g(x) = 1$ whenever x is rational and $g(x) = 0$ whenever x is irrational is not Riemann integrable but has Lebesgue integral equal to 0 (since the set of rational numbers has measure 0).

The Lebesgue approach—that is, the approach based on measures, which generalize to much more complicated sets than **R**—is also useful in probability theory. Indeed, calculating the probability of an event consists of "measuring" the set of elementary possibilities corresponding to this set. Thus a probability measure is just a special case of the measure defined above: more specifically, a probability measure is a measure p such that $p(\Omega) = 1$, where Ω is the set of all the elementary possibilities. (Since it is guaranteed that one of the possibilities will occur, the probability of the event Ω must be 1.)

But instead of choosing Lebesgue's approach, Bourbaki decided to develop a different one, which we will just briefly explain here. Bourbaki introduced integrals based on linear forms defined on the space of continuous functions. Such a linear form is a function L which associates to each continuous function f a number $L(f)$ such that $L(af_1 + bf_2) = aL(f_1) + bL(f_2)$ for any continuous function f_1 and f_2 and numbers a and b (this property, called linearity, is one of the key properties of integrals). This approach was criticized "because it uses topological concepts—like continuity—even though integration doesn't need topology," explains Jean-Pierre Kahane, a mathematician at the Université de Paris-Sud, adding that "Arnaud Denjoy, who was one of the eldest among the first Bourbakis and who invented an original, but quite complex theory of integration, was disgusted by Bourbaki's treatment of the subject." To be honest, however, it is important to emphasize that Bourbaki's approach did have some useful consequences. For example, it led Laurent Schwartz to create the theory of distributions, which generalizes the concept of a function and involves linear forms defined on a narrower class of functions than that of continuous functions.

Un exemple d'animosité envers Bourbaki est ce double poème d'un goût contestable écrit (en anglais) par Arnold Seiken, mathématicien aux États-Unis, à propos des mathématiques françaises :

Sublime

Pêle-mêle
Henri Poincaré
D'élégants théorèmes
Prouva sans façons,
Classa des variétés
Difféomorphiquement
Utilisant la dualité
Et du génie aussi.

Autre

Pêle-mêle
Nicolas Bourbaki
Ressuscita de parmi les morts,
Pour les mathématiques sauver,
Débita d'assommants traités
Hypercinétiquement,
Prouvant qu'il aurait dû
Dans la tombe rester.

The Mathematical Intelligencer, 1995

(For English translation, see appendix.)

Hyperaxiomatizers Lacking Generalization?

Let's return from these digressions to the criticisms made against the style (in the general sense of the term) of Bourbaki's treatise. One of the first things that struck readers was the formal and abstract nature of the text. The concision of the sentences, the abundance of mathematical symbols, and the extreme dearth of figures clash with the style of textbooks from the beginning of the century. People also accused Bourbaki of excessive formalism and generality. Today, such criticism seems unjustified. A quick visit to a mathematics library at any university would reveal many more formal books than Bourbaki's and books with a much higher number of symbols per square inch. As for generality, Armand Borel, a former member of Bourbaki, explains in an article published in 1998 that "Contrary to my first impressions [...] the book's goal is not to achieve the greatest generality possible. Instead, it is to achieve the most effective generality, the one most able to meet the needs of potential users in various areas. They left out any refinements of theorems that excited specialists without substantially extending the theorem's applicability. Of course, later developments could show that Bourbaki didn't always make the best choices. Nevertheless, this served as a guiding principle." Christian Houzel, who was never a member of Bourbaki, stated a similar opinion: "Bourbaki never tried to obtain maximal generality. On the contrary, the Bourbakis aimed for the minimal generality that could encompass what they knew. This is why the level of generality has evolved with the newer editions of the treatise."

The perception of excessive generality may stem from Bourbaki's style of explanation—progressing from general to specific—and it is certainly not improved by the impersonal tone of the treatise. This aspect of the treatise's style even attracted criticism from Alexandre Grothendieck, who was a "hyperaxiomatizer lacking generalization"—a nickname that, according to Pierre Samuel, the Bourbakis gave to any members who pushed too hard towards abstraction and generalization. In his lengthy *Récoltes et semailles* ("Harvests and Sowings"), which was never published commercially, Grothendieck writes (around 1985) that "the canonical text hardly gave an idea of the atmosphere in which it was written, to say the least. By now I've realized that this is in fact one of the main shortcomings of Bourbaki's writing—that not even an occasional smile is included to suggest that these books were written by people, and people connected by much more than some oath of unconditional fealty to the merciless canons of rigor..."

The criticisms of Gustave Choquet, who was never a member of Bourbaki but who contributed greatly to reforming mathematics education in the universities, are more severe. In an interview with Marian Schmidt published in 1990, he said that "most French mathematicians of my generation acquired much of their mathematical education from Bourbaki. Bourbaki influenced their style and their work and transmitted some of its qualities and flaws. What were these flaws? It seems that any group that works for a long time in isolation is condemned to dogmatism. I think this is the main criticism against Bourbaki: the basic definitions and theorems are presented without any justification or heuristic explanation. They have the desiccation and meagerness of bones whose meat, despite being delicious, has been thrown to the exercises. The reader who neglects to do them all ends up with only a caricature of mathematical activity."

Arnaud Denjoy (1884–1974) was disgusted by the way Bourbaki presented integration in its treatise.

Bourbaki Algebrized Analysis

Bourbaki's penchant for algebraic concepts and methods also has its drawbacks. The group's initial goal was to write a treatise on analysis; instead, the group achieved an algebrization of analysis, which was fruitful in many ways. As Laurent Schwartz writes, "The most clear benefit I think I got out of Bourbaki was to be 'algebrized.' I'm an analysist by nature, and all my work deals with analysis and probability. But I use algebra and algebraic methods as often as possible [...]. I am one of the most algebraic among the analysts. It's Bourbaki who gave me this." However, this algebraist attitude does not suit everyone, and especially not geometers. Marcel Berger, an eminent geometer, was a Bourbaki guinea pig but didn't get interested in the group. Bourbaki's spirit also failed to seduce the differential topologist René Thom, who believes in the importance of geometric intuition. Benoît Mandelbrot, a wholehearted geometer, is one of the most fervent rejecters of Bourbaki. The nephew of Szolem Mandelbrojt (one of the first participants in Bourbaki), Mandelbrot left l'École Normale Supérieure in favor of l'École Polytechnique to escape from the Bourbakis. "Thanks to my uncle, I realized this this group was a militant bunch, that they had strong prejudices against geometry and against every science, and that they were inclined to scorn, or even humiliate, anyone who didn't follow them," he said in an interview published in 1985. He even left France for the United States in 1958 because of the group's "stifling influence" and the predominance of its choices.

Gustave Choquet

But this era is no more. Bourbaki no longer reigns over French mathematics, and even less over world mathematics. Geometry has become fashionable once again, and not only in mathematics. In theoretical physics, for example, the geometry and topology of 10- or 11-dimensional spaces are studied with the goal of constructing a theory that encompasses all the forces and fundamental particles. Geometry has also entered computer science with image processing, realistic visualization of three-dimensional scenes, robotic vision, and so forth. The division between pure and applied mathematics is weakening: even number theory and algebraic geometry have applications in cryptography and information coding. The interactions between physicists and mathematicians have developed in abstract domains like the theory of elementary particles as well as in concrete problems like the turbulence of a flowing fluid. Today, many scientists and engineers study mathematics connected to computer computations, such as numerical calculation, formal calculation, and optimization algorithms. The

Laurent Schwartz at the Bourbaki conference in Pelvoux (Summer 1951).

mathematical atmosphere is no longer that of the 1960s, Bourbaki's golden age. The orderly ranks of mathematicians working "for the honor of the human spirit" have been replaced by plentiful mathematicians working in every direction. The heat raised by the Bourbakian enterprise has dissipated. And, now that this undivided reign of pure mathematics has ended, some are predicting a dictatorship of applied mathematics.

Claude Chevalley (1909–1984)

Claude Chevalley was born in Johannesburg, South Africa, on February 11, 1909. His father, a diplomat, was at the time a consul general. Chevalley's education started at the elementary school of Chançay, a small town near Tours, where his father had acquired a property in 1910. He continued his education at the Lycée Louis-le-Grand in Paris. Although he had been interested in science for a long time, Chevalley did not decide to pursue a career in mathematics until his last year of high school, when he was inspired by an excellent teacher, G. Dufour. In 1926, he entered l'École Normale Supérieure at the age of seventeen. One of his friends there was Jacques Herbrand, who influenced him greatly. (The brilliant Herbrand was interested in mathematical logic, which was an unexplored area at the time in France, and made important contributions to this area before his premature death at the age of twenty-three in a mountaineering accident in 1931.) At l'École Normale, Chevalley also met André Weil, who had just returned from his trip to Italy and Germany. It was with Weil that Chevalley was initiated to the modern aspects of algebraic number theory, which would become one of his main areas of interest. In the academic year 1929–30, a grant allowed Chevalley to carry out his first scientific research, following which he served in the military. From late 1931 until 1936, Claude Chevalley was supported by a grant from the Caisse Nationale des Sciences (the National Science Fund, a precursor of today's Centre National de la Recherche Scientifique) and devoted himself to mathematics, philosophy, and social and political criticism. He spent the academic year 1931–32 in Hamburg, Germany, where he studied under Emil Artin and completed his doctoral thesis. He then spent several months in 1933 in Marburg, where Helmut Hasse was teaching. He married his first cousin Jacqueline in November 1933.

In 1936, Chevalley taught for one semester at the University of Strasbourg as a substitute for Weil, who had been invited to visit the United States. He was an assistant professor at Rennes during the academic year 1937–38 before spending one year at the Institute for Advanced Study in Princeton. At the outbreak of World War II, the French ambassador advised Chevalley to stay in America. Following this advice, Chevalley worked at Princeton University until 1948, when he spent a year in Paris on a Guggenheim Fellowship before moving to Columbia University in New York. Meanwhile, he divorced his wife in 1948 and married Sylvie Bostsarron, a specialist in the history of theater. The couple had one daughter, Catherine Chevalley, who is now a philosopher at the University of Tours. Claude Chevalley returned to France in 1955 to accept an appointment at the Sorbonne, although some mathematicians thought this was an unjust reward for someone who had spent the terrible years under good conditions, and campaigned against his appointment. Chevalley remained at this university until his retirement in 1978. He died on June 28, 1984 after a long illness.

Chevalley's contributions to mathematics are numerous and important. Most are in algebraic number theory (particularly in class field theory), algebraic geometry, and group theory. He also wrote books on these areas, which have now become classics.

Philosophy is the other side of Chevalley's intellectual life. Like Herbrand, he combined rigor in mathematics with very personal experiences of anguish and liberty. He was interested in the epistemology of Émile Meyerson, one of his father's friends, who believed that science reflects the desire to decrease fundamental anguish by discovering stability and lasting truths. For Chevalley, as his daughter explains, mathematics can only be created independently of any exterior purpose. It cannot and should not submit to the reasoning required by, for example, the physicist's reality or the desire to uncover the underlying structure of the world. In other words, Chevalley held the Bourbakian view that one must eliminate reality to do pure mathematics. He was also influenced by another friend, the philosopher Arnaud Dandieu. Along with Robert Aron, Dandieu (who died young in 1933) was an important figure in the Ordre Nouveau movement created in 1930. This was a group of intellectuals of anarchist and European nature who called themselves "personalists" and advocated a new order based on human primacy, direct democracy, an anti-productivist economy, and federalism. Swept into this movement (which has nothing to do with the group of extreme rightists who later took the same name) by Dandieu, Chevalley became one of its key figures until the movement dissolved in 1938. Chevalley concentrated more on mathematics after World War II, but during the sixties he campaigned in a group of intellectual ecologists (with two other Bourbakis, Alexandre Grothendieck and Roger Godement), which published a journal called *Survivre et vivre* ("Survive and Live"). According to those close to him, Chevalley always remained true to the convictions of his youth, and especially to his quest for rigor and liberty. Among the founders of Bourbaki, he, without a doubt, was the most individualistic and the one who best kept a critical eye on their endevour.

10

New Math in the Classroom

In the 1970s the wave of New Math surged into secondary education. Structures inspired by Bourbaki were certainly included, but how much was the group itself involved?

"Definition 4: A subset I of an ordered set E is called an *interval* of E if it satisfies the implication ($x \in I$ and $y \in I$ and $x \leq z \leq y$) $\Rightarrow$ ($z \in I$)."

"Theorem 6: The identity is the only automorphism of the field **R**."

"Theorem: Addition and composition of functions equips the set $L(V)$ of endomorphisms of a vector space B with the structure of a unitary ring."

"Every involutive endomorphism is a symmetry."

"Definition: We say that an orthogonal endomorphism of ϕ of E_3 is a vector rotation to express the fact that the subspace of variables invariant under ϕ has dimension 1 or 3."

"Theorem 2: Let F be a vector function and x_0 be an accumulation point in its domain of definition. Then $\lim_{x \to x_0} \overrightarrow{F(x)} = \overrightarrow{0} \Leftrightarrow \lim_{x \to x_0} \|\overrightarrow{F(x)}\| = 0$."

THEOREM AND DEFINITION

Given: a graduated line (Δ, g).

1. For each pair (a', b') of real numbers such that $a' \neq 0$, the function g' from Δ to **R** defined for any element M of Δ by $g'(M) = a' \cdot g(M) + b'$ is bijective.
2. The family of all bijections thus defined has the following property:
   For any two bijections g' and g'' of this family, there exists a pair (a, b) of real numbers such that $a \neq 0$ and $g''(M) = a' \cdot g'(M) + b$ for every element M of Δ.

Any bijection belonging to this family is called a graduation of Δ. The number $g'(M)$ is called the abscissa of M in the graduation g'.

A property of a graduated line and the definition of a graduation...in eighth grade!

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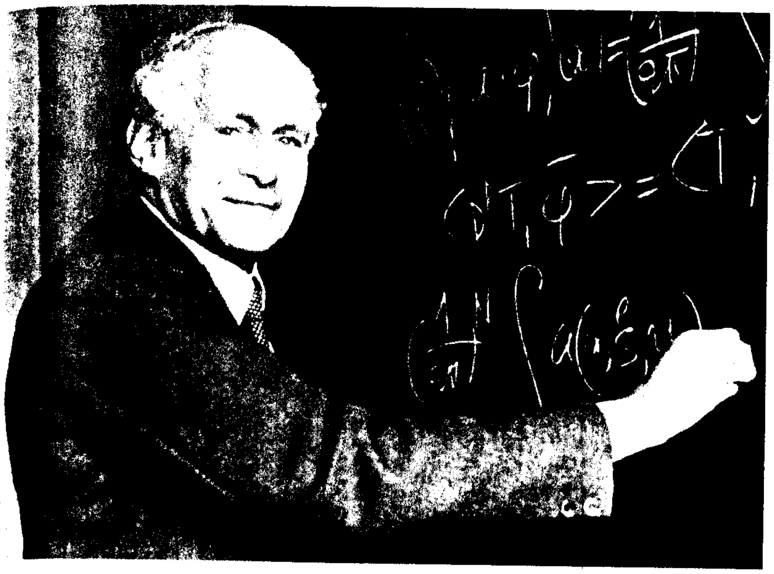

Laurent Schwartz teaching about the theory of distributions. Laurent Schwartz's course revolutionized mathematics teaching at l'École Polytechnique.

Are these statements from Bourbaki's *Éléments de mathématique*? No. Or maybe from mathematical courses for university students or for students preparing to enter the presitigious Grandes Écoles? Once again, no. In fact, they are simply extracts of a French mathematics textbook for high school seniors, published in the 1970s by Hachette in their Aleph$_0$ series. And this textbook consists of six volumes—two for algebra, two for geometry, two for analysis—for a total of over 1500 pages! Its content would leave today's math student stunned.

In France and around the world, this was the era of New Math, an ambitious reform in elementary and secondary math education. However, it was a failed reform whose biggest achievements were wasting gallons of ink and creating heated controversy among parents, teachers, mathematicians, educational psychologists, and others. It was a setback whose results remain today as scars in the educational system. Some people connect Bourbaki with this reform. Why? Are they right?

Bourbaki Attacks Higher Education

Before tackling the subject of seconday education, let's discuss the reform of higher education. At least in France, Bourbaki clearly played an active role in this reform. Indeed, this was the very nature of the group's initial project, the project to write a new treatise on analysis that would replace the obsolete and unsuitable texts used at the time in higher math education. The Bourbaki pioneers, who for the most part were professors at universities outside Paris, started to modernize math education in their respective institutions as soon as the group had been formed. For example, Henri Cartan said that "at Clermont-Ferrand, where the University of Strasbourg had moved in 1939, I taught differential calculus on Banach spaces. This was a real revolution." In this way, students at various universities outside the capital were introduced to slightly more modern mathematics than what had been taught to students just a couple years before. This revolution arrived in Paris somewhat later. While the students of l'École Normale Supérieure didn't have to wait long to benefit from Henri Cartan, who arrived in 1940, students at other Parisian institutions had to wait until the arrival of Gustave Choquet in 1954. According to Henri Cartan, at the end of each academic year the dean would hold a meeting with the math faculty to distribute the teaching of the math classes for the following year. The class on differential and integral calculus had been taught for many years by Georges Valiron, who gave traditional and boring lectures. One year, though, illness prevented Valiron from continuing to teach. At this point, Henri Cartan recalled, "I recommended Choquet, who shared our point of view on teaching." So, Gustave Choquet, who was never a member of Bourbaki, took on the job. As Choquet described to Marian Schmidt, "Until then, the course mostly followed the elementary sections of Goursat's treatise. I modified the focus and content of the course. I thus had the good fortune of being behind the revolution in French mathematical teaching, at first in the classes for advanced university students, and later, as the reforms spread contagiously, for the younger students as well." However, this revolution was not accepted by all, and conservative mathematicians protested bitterly. "For example, I remember that Henri Villat demanded, 'How can you expect students to understand when I don't understand it myself?'" Henri Cartan recalled.

In his autobiographical book *Mathématique: (récit)* (which mentions Bourbaki several times), the author-mathematician Jacques Roubaud describes how students received Choquet's revolution: "Faced with the sudden metamorphosis of mathematics occurring as they watched (and especially as they listened), even the most hardened students, those who had taken classes preparing them for the Grandes Écoles entrance exams and who had survived the slaughter of the exams to earn their certificate in general mathematics, felt their most established convictions waver. Over the course of their studies, these students had developed a stable and familiar view of mathematics, and now this view

Henri Cartan in Bombay, 1960.

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was changing so much that it was becoming locked away, sealed before their eyes. And they didn't find the new face of mathematics to be especially pretty. Those students who had had to repeat a year were the most noticeably distraught, as they found nothing in common between Choquet's classes and Valiron's of the year before. It was as if this science had been replaced by a new one over the summer vacation, a new science that had kept the name of the old one merely out of convenience." Choquet quickly gained reinforcements. By the following year, several professor and lecturer positions had been vacated. According to Martin Andler, a "fierce battle" ended with victory on the part of the revolutionaries. The new faculty members for the 1955 academic year were Chevalley, Ehresmann, Pisot, Zamansky, Godement. All of these, except Zamansky, were at some point members of Bourbaki.

Bourbaki Reaches l'École Polytechnique

The situation at l'École Polytechnique also deserves to be discussed. From World War I until the 1950s, teaching at this elite school—including the teaching of mathematics—was mediocre, despite the presence of Paul Lévy (a great probabilist whose work was long unrecognized) and of Gaston Julia, who at the time was nearing the end of his career. The hiring of Laurent Schwartz (a member of Bourbaki) in 1959, following the retirement of Paul Lévy (Schwartz's father-in-law) gave new life to this institution, which had produced almost no new mathematicians for many years. Schwartz's course on analysis was very succesful, and the changes he introduced—despite resitstance—led to a profound revival of teaching at l'École Polytechnique, especially after 1968. Jean-Pierre Bourguignon described how "there was such a contrast between Schwartz's teaching and the teaching of other subjects that the need to completely renovate the other courses became obvious." According to Bourguignon, the success of this renovation was so great that l'École Polytechnique today produces mathematicians of the highest quality: "At the most recent international congress of mathematicians, the one in Berlin, four of the ten plenary addresses were given by French researchers, and three of those four had graduated from l'École Polytechnique."

Although Bourbaki actively contributed to the renovation of higher mathematics education, this renovation was not entirely the work of Bourbaki. For one thing, this contribution is not the work of Bourbaki (excepting the indirect influence of *Éléments de mathématique*) but rather the work of individual members. It is highly unlikely that

Jean Piaget in 1977.

Bourbaki developed a collective strategy to obtain key positions at universities or the Grandes Écoles (including l'École Normale Supérieure and l'École Polytechnique) or to redefine the content of mathematics courses at these institutions. In addition, the members of Bourbaki were not the only mathematicians to renovate higher mathematics education. Gustave Choquet is the most obvious non-Bourbaki to contribute to this revolution, and others include Jean Leray and André Lichnerowicz, who were appointed as professors at the prestigious Collége de France in 1947 and 1952 respectively.

It is much harder to assess the role of Bourbaki in the New Math movement in secondary education. This famous reform started in the 1950s, when it became apparent that the mathematics taught in secondary schools was worn out and poorly adapted to the modern demands of economics, technology, science and culture. This realization affected many countries, both in France and around the world. But what exactly triggered it?

In the 1950s and 1960s, the West was undergoing a profound transformation that involved economic and industrial growth, scientific and technological progress, and cultural change. Training excellent scientists and engineers was considered necessary for economic development. Furthermore, the competition between the Soviet Union and the West, intensified by the launch of the first Sputnik in 1957, made the West worried about falling behind in technology. A solid knowledge of mathematics was considered necessary for all scientists and engineers.

The evolution of mathematics itself added to this situation. Partly due to the influence of Bourbaki, mathematics had become seen throughout the world as a unified subject based on set theory and built out of general structures (groups, rings, fields, etc.) defined by axioms. This sort of mathematics was only vaguely related to that being taught in elementary and secondary schools, despite the mathematical revolution taking place in universities. Thus mathematicians, as well as young teachers who had taken Bourbaki-influenced classes in college, wished to modernize the teaching of their subject.


André Lichnerowicz (1915–1998) in 1988.

Mathematics Was Everywhere

Linked to this evolution was the idea that mathematics (or the singular noun *mathématique*, as Bourbaki called the subject) was a universal language intended to be used in all disciplines, and in particular in all sciences, including even humanities and social sciences. This went with the modern trend in mathematics of emphasizing the relations

At the 1954 Bourbaki conference in Murols: Roger Godement, Jean Dieudonné, André Weil, Saunders Mac Lane, and Jean-Pierre Serre (from left to right). Roger Godement ardently supported the implementation of Bourbaki's philosophy in university-level math education.

between objects (numbers, functions, geometric figures, and so forth) rather than the objects themselves, whose characteristics were considered to be of little importance. Many declared that "mathematics is everywhere," meaning that it is an essential part of everyone's academic and cultural knowledge. It is likely that this point of view was based partially on the fashion of structuralism affecting philosophy, literature, ethnology, linguistics, and psychology.

A final factor in sparking the New Math revolution was the new trends in pedagogy. Jean Piaget saw an analogy between the mental structures created when a child learns mathematics and the mother-structures (algebraic structures, ordered structures, and topological structures) that Nicolas Bourbaki discussed in its article *L'architecture des mathématiques* ("The Structure of Mathematics"). Furthermore, Piaget and many other educational psychologists emphasized the importance of active learning in a child's intellectual development, which created a movement for teaching based on observations, experiments, and inferences made by the students with the help of their teachers rather than on knowledge obtained directly from the teacher. The Bourbaki style of mathematics seemed better suited to this new sort of teaching than the traditional style did. In addition, mathematics seemed to be more democratic than other subjects, since its emphasis on concepts required no cultural prerequisites. Many argued, erroneously or otherwise, that students would find math more accessible than Latin and Greek, which the schools traditionally used to choose their top students. Such an argument was by no means insignificant in a time when more students were continuing their education through high school and in the social and political atmosphere that led to the May 1968 student uprisings in France.

"Down With Euclid!"

For most countries, the New Math revolution consisted of four phases. The first was a phase of realization and reflection. An important event of this stage was the November 1959 Colloque de Royaumont, a ten-day conference organized by the Organisation for European Economic Cooperation (now called the Organisation for Economic Cooperation and Development) to promote a reform in the content and methodology of secondary math education. It was during this conference that Jean Dieudonné, one of the participants, threw out a provocative declaration in reference to the teaching of geometry: "Down with Euclid!"

The years from 1964 through 1967 marked the second phase, a preparatory one in which committees formed. The third phase saw educational experiments carried out and new curriculums implemented. The fourth and final stage, which occurred during the early 1970s, was the general implementation of the new curriculums. In France, one important step in the New Math reform was the government's creation of the Lichnerowicz Commission in 1967. Chaired by André Lichnerowicz, this committee comprised a total of eighteen university and high school teachers, including Gustave Choquet, the physicist Louis

2.1 CORPS $\mathbb{C}$ DES MATRICES $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

2.1.1 Définition.

On appelle $\mathcal{M}$ l'ensemble des matrices carrées d'ordre 2 à termes réels. Sur le corps $\mathbb{R}$ des nombres réels, on appelle $\mathbb{C}$ le sous-ensemble des matrices $M(a, b)$ de la forme :

$$M(a, b) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

pour tout a et pour tout b.

The definition of complex numbers in terms of matrices, as taught to twelfth-grade students.
(For English translation, see appendix.)

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Néel, Pierre Samuel (a member of Bourbaki at the time), Charles Pisot (a former member of Bourbaki), and André Revuz (one of the main proponents of the reform). As the job of the Lichnerowicz Commission was to propose new curriculums, this committee played a key part in the New Math reform.

It would be tedious to describe the new curricula in detail, but their general principles are simple to explain. The new curricula introduced the principles of mathematical and formal logic and of naive set theory earlier than the old curricula, including an introduction to groups, rings, fields, and vector spaces presented axiomatically; introduced complex numbers (in senior year) and principles of probability theory; and replaced traditional geometry with linear algebra (which deals with linear equations and vector spaces). The new curricula emphasized rigor in definitions, theorems, and proof-writing while deemphasizing all numerical, algebraic, and trigonometric calculations. The reforms in other countries were of the same nature as those in France.

A Revolution and a Counterrevolution

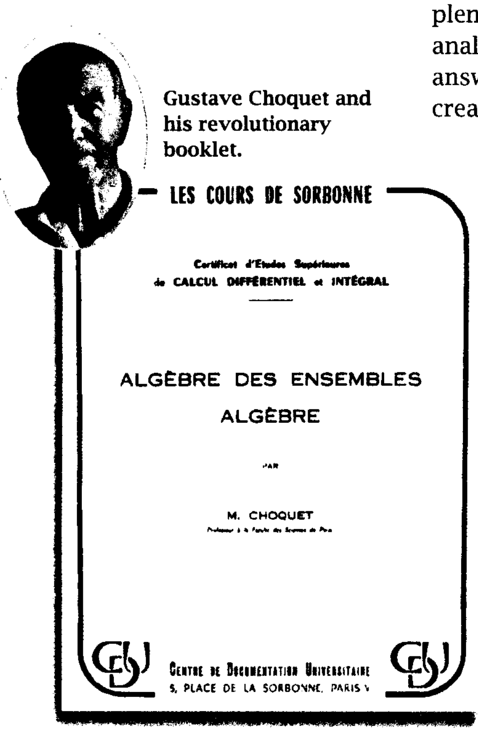
The reform certainly went too far, especially in the implementation of the curricula in classrooms and textbooks. The results of the reform quickly aroused harsh criticism, which the media reported to the general public. The abandonment of geometry, or more precisely its algebrization, was one of the main targets of the complaints. Another was the excessive formalism and abstraction of New Math. "[...] New Math is not in touch with reality," and "[...] Set theoretical algebra does not teach students how to reason" are just two of the many criticisms published by the magazine *Sciences et Vie* in a series of articles (1971–1972) attacking the reform. Eminent mathematicians like Jean Leray and René Thom also charged against New Math. As Leray wrote

> THEOREM AND DEFINITION
>
> For any two pairs (D_1, D_1') and (D_2, D_2') of vector rays in E_2 the relation "There exists a vector rotation f of E_2 such that
>
> $$f(D_1) = D_1' \quad \text{and} \quad f(D_2) = D_2'"$$
>
> is an equivalence relation in DxD, where D is the set of vector rays in E_2.
>
> An equivalence class for this relation is called **an angle of two vector rays in E_2.**

The definition of the angle between two rays, as taught in 1971 to eleventh-grade students...with no diagram!

in the October 1971 issue of *Gazette des mathématiens*, "'New Math' is a series of concepts that are defined without reference to their characteristic properties (axioms) and that fail to lead to statements of any remarkable properties. It's impossible to reason with these concepts or to find interest in them. Learning them is a test of memory that poisons intelligence." Jean Dieudonné denounced it as "a new method of education, a more agressive and stupid method that flies the banner of modernism."

The counterrevolution was in vain, and the New Math revolution continued to be criticized around the world as a global failure, even while this failure was never measured quantitatively by statistics. "There's no doubt about the failure. The teachers complained of being unable to teach according to the curricula and textbooks prescribed by the ministers, the students didn't seem to understand these 'unifying concepts,' and the parents realized that their children could no longer count or solve problems," explained Anna Sierpinska, a Polish woman living in Canada who has served as the vice-president of the International Committe on Mathematical Education. But did New Math fail? To what extent was the new material understood less than the old material? The new curricula did have a certain coherence, so why did their implementation create such extreme problems? An objective and detailed analysis of these questions is lacking even to this day. Some hints at answers do exist in criticisms expressed here and there, but they don't create a clear and global view of the reform's problems. The reasons for the failure remain poorly understood, concealing any lessons that might be learned for the future.



Gustave Choquet and his revolutionary booklet.

On the other hand, we do have the retrospective comments of Laurent Schwartz and Gustave Choquet, two mathematicians who were also renowned as teachers. As Choquet told Marian Schmidt in 1990, "in an abstract and sterilized world, the progress of a hundred years would allow us to elegantly and rigorously present fundamental concepts and theorems *ab ovo*, without referring to experience or geometric intuition. This is what happened, both in France and in other countries, with the drama of the mathematics reform at the end of the 1960s. Jean Dieudonné's famous proclamation 'Down with Euclid!' represents the focus of the ministerial committee in charge of developing new curricula for mathematics education in middle schools and high schools. The motivating idea of the movement was that, since fundamental concepts are necessary for all logical constructions, these concepts (including logic, set theory, algebra, and linear algebra) should be taught first. The result was bound to be catastrophic,

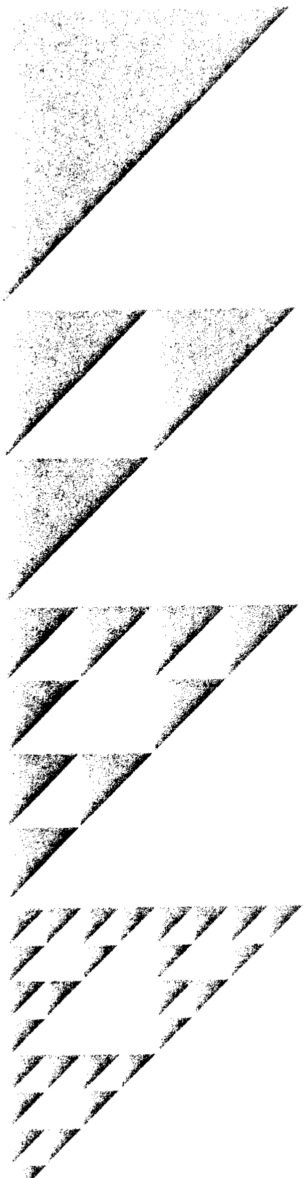
François Bruhat and Michel Demazure in la Messuguière (near Grasse), July 1975.

since all pedagogical considerations, such as the students' motivation and previous knowledge, the teachers' education, and the writing of reasonable textbooks, were swept aside. Also, little effort was made to meet the demands of physicists and engineers." In the official report *La France en mai 1981: l'enseignement et le développement scientifique* ("France in May 1981: Scientific education and development") published in December 1981, Schwartz writes that "[...] the teachers, parents, and children didn't learn modern mathematics but rather just the basic language of an extraordinarily vast and modern subject [...] of which the definitions given in classrooms (around the world!) were only the ABCs of the subject. [...] Little by little, all the richness of the mathematics previously taught in high schools, all the theorems, geometric figures, and connections with other sciences, has been replaced by a plethora of axioms and definitions. Most students find these incomprehensible, and the effect is that the results are *very poor.* Mathematics is rich when it introduces few concepts and structures and many theorems, but the New Math taught in schools introduces an enormous number of concepts and definitions and almost no theorems. This is *very poor* mathematics. [...] The goal of mathematics is *not* to prove rigorously things that everyone knows. Instead, the goal is to find rich results and then, in order to make sure they are true, to prove them."

So what role did Bourbaki play in the introduction of New Math? Dieudonné expressed his personal opinions on education with provocative exclamations, and while he didn't play an official role in developing the new curricula, he did greatly influence them (which is apparent, for example, in the algebraic presentation of geometry). Pierre Samuel participated in the Lichnerowicz Commisision, but so did seventeen other people. In addition, Samuel was not among the most radical. "Lichnerowicz and I were quite moderate, but some of the committee members took things too far," he explained. Cartan and Schwartz gave addresses on contemporary mathematics to the *Association des professeurs de mathématiques*, an association whose members included many high school teachers enthusiastic about the reform. This was basically the extent of the individual contributions by Bourbaki members. As a group, Bourbaki took part neither in the reform nor in the debates surrounding it. According to Pierre Samuel, "Bourbaki held no opinions on high school education. The group didn't even hold any opinions about education in the first years of university, although some members wanted to write a mini-Bourbaki for these earlier years. This idea was abandoned, as we already had a lot to do and there were already good books for this level."

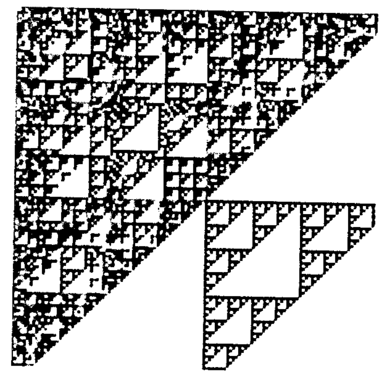
Supicious but Silent

Michel Demazure recalls that the group viewed this reform with much suspicion, and that certain members were strongly against it: "What we all shared was contempt for the pedagogical movement. We were preoccupied with the content of courses, not how to teach them." But although Bourbaki did not participate in the reform, the group did influence it indirectly. According to Anna Sierpinska, "Bourbaki's influence on this movement was particularly evident in the underlying mathematical philosophy of the choices and in the organization of the mathematical content of the new curricula. The goal was to build mathematical knowledge from the earliest classes, starting even in preschool, as an enormous, unified structure built on general concepts like sets, orders, relations, and groups." Thus Bourbaki's vision of mathematics had spread to the world of mathematicians and then to that of higher education; from there, it spread to high school teachers who proposed using it to revolutionize secondary math education, sometimes even explicitly mentioning Bourbaki in their arguments. But Bourbaki, who never claimed that the methods used in the group's treatise could be displaced to secondary education, didn't hold itself responsible for the misinterpretations of its philosophy and didn't try to perfect the extension of its philosophy to high school teaching. This attitude may deserve to be criticized, and indeed Pierre Cartier did: "Bourbaki acted very hypocritically in terms of the reforms in secondary education. The group influenced the movement quite a bit, but it denied any responsibility for the results." Meanwhile, Michel Demazure



Jean-Pierre Kahane believes that the Sierpinski triangle, a fractal structure, is an analogy for the way mathematics curriculums are being drained of their content more and more each year.

saw an element of Jansenism in the attitude held by Bourbaki, whose members were mostly Protestant: "Bourbaki didn't try to justify itself, and the group didn't feel responsible for the misinterpretations others made of its work. The Bourbakis were all against the use of the treatise as a model for teaching, but they held a certain Jansenist philosophy that said, 'Let people believe what they will.'"



Whatever the cause and extent of their failure, the various reforms of the New Math revolution were abandoned towards the end of the 1970s. Following New Math was a counterreform, leading to much less ambitious curricula. Traditional geometry was reintroduced, superfluous formalism was abandoned, calculations were reemphasized. But today's curriculums are also questioned. They seem less coherent, not as well constructed. Each year, more and more topics are removed to make the curricula easier, to the point where Jean-Pierre Kahane, a mathematician who chairs a committee on mathematical education, said that "today's curricula are like Sierpinski's triangle." Vector spaces are no longer included, even though they don't seem to pose any exceptional difficulties and play a considerable role in mathematics and its applications. In addition, number theory has almost completely disappeared. Today's curricula demands little from students' imagination and reasoning abilities and fails to prepare students to create and write proofs. "Students are not really taught to reason," Pierre Samuel complains, while Michel Demazure laments that "now, most of the exercises almost contain their answers in their statements." It seems that abandoning the New Math revolution was throwing the baby out with the bathwater. Today, mathematics inspires in many students as much fear, lack of interest, and even hatred as ever. And what Dieudonné wrote in his 1987 book *Pour l'honneur de l'esprit humain* unfortunately remains true today. "Nothing taught in high school mathematics was discovered after 1800," he writes, although it would be more accurate to replace "nothing" by "almost nothing."