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EXERCISES

1. Do you think the conditions in Alexandria were more or less favorable for the development of mathematics in the days of Pappus than at the time of Ptolemy? Explain.
2. How did the intellectual conditions at Alexandria compare with those at Rome in the days of Diophantus and Pappus?
3. Would the development of mathematics have been essentially modified if Rome had not fallen in 476? Give reasons for your answer.
4. If you were a mathematician living in the year 500, would you have chosen Alexandria, Rome, Athens, or Constantinople as your home? Give reasons for your answer.
5. Show that the epigram concerning the age of Diophantus leads to the conclusion that he died at the age of eighty-four.
6. Verify that the four numbers listed by Nicomachus as perfect are indeed perfect numbers.
7. Solve the problem of Diophantus in which it is required to find two numbers such that their sum is 10 and the sum of their cubes is 370.
8. Find two rational fractions, other than $\frac{3}{13}$ and $\frac{9}{13}$ satisfying Diophantus' condition that either one when added to the square of the other will produce a perfect square.
9. Prove that the lines OC, BD, and DF in Fig. 11.3 are indeed the arithmetic, the geometric, and the harmonic means, respectively, of AB and BC, as Pappus asserted.
10. Prove Pappus' generalization of the Pythagorean theorem illustrated in Fig. 11.4.
11. Draw carefully a diagram similar to Fig. 11.5 in which AB is 3 inches and BC is 2 inches and find approximately, by measurements, the diameter of the circle C_3 and the distance of its center from the line AC, thus verifying roughly the assertion of Pappus.
12. Solve the problem of the distribution of apples described in the text.
13. Solve the problem of the three pipes described in the text.
14. Show analytically that the Pappus problem for six lines leads to a locus the equation of which is not higher than third degree.

*15. Prove the first Pappus trisection given in the text.
*16. Prove that OT is parallel to AP in Fig. 11.2.
*17. Using the result in Exercise 16, complete the proof of the second Pappus trisection given in the text.
*18. Justify the Pappus theorem on solids of revolution.
*19. Prove the theorem of Proclus on the generation of an ellipse for the case in which the intersecting lines are mutually perpendicular.

CHAPTER XII

China and India

> A mixture of pearl shells and sour dates ... or of costly crystal and common pebbles.
>
> *Al-Biruni's* India

The civilizations of China and India are of far greater antiquity than those 1
of Greece and Rome, although not older than those in the Nile and Mesopotamian valleys. They go back to the Potamic Age, whereas the cultures of Greece and Rome were of the Thalassic Age. Civilizations along the Yangtze and Yellow rivers are comparable in age with those along the Nile or between the Tigris and Euphrates; but chronological accounts in the case of China are less dependable than those for Egypt and Babylonia. Claims that the Chinese made astronomical observations of importance, or described the twelve signs of the zodiac, by the fifteenth millennium B.C. are certainly unfounded, but a tradition that places the first Chinese empire about 2750 B.C. is not unreasonable. More conservative views place the early civilizations of China nearer 1000 B.C. The dating of mathematical documents from China is far from easy, and estimates concerning the *Chou Pei Suan Ching*, generally considered to be the oldest of the mathematical classics, differ by almost a thousand years. The problem of its date is complicated by the fact that it may well have been the work of several men of differing periods. Some consider the *Chou Pei* to be a good record of Chinese mathematics of about 1200 B.C. but others place the work in the first century before our era. A date of about 300 B.C. would appear reasonable, thus placing it in close competition with another treatise, the *Chiu-chang suan-shu*, composed about 250 BC.,[1] that is, shortly before the Han dynasty (202 B.C.). The words "Chou Pei" seem to refer to the use of the gnomon in studying the circular paths of the heavens, and the book of this title is concerned with astronomical calculations, although it includes an introduction on the properties of the right triangle and some work on the use of fractions. The work is cast in the form

[1] Histories of mathematics generally devote little space to Chinese contributions. Exceptional in this respect are D. E. Smith, *History of Mathematics* (1923–1925), and J. E. Hofmann, *Geschichte der Mathematik*, 2nd ed. (Berlin, 1963), Vol. I. An unusually thorough and up-to-date account in the Near and Far East is given in A. P. Juschkewitsch, *Geschichte der Mathematik im Mittelalter* (1964).

of a dialogue between a prince and his minister concerning the calendar; the minister tells his ruler that the art of numbers is derived from the circle and the square, the square pertaining to the earth and the circle belonging to the heavens. The *Chou Pei* indicates that in China, as Herodotus held in Egypt, geometry arose from mensuration; and, as in Babylonia, Chinese geometry was essentially only an exercise in arithmetic or algebra. There seem to be some indications in the *Chou Pei* of the Pythagorean theorem, a theorem treated algebraically by the Chinese.

2 Almost as old as the *Chou Pei*, and perhaps the most influential of all Chinese mathematical books,[2] was the *Chui-chang suan-shu*, or *Nine Chapters on the Mathematical Art*. This book includes 246 problems on surveying, agriculture, partnerships, engineering, taxation, calculation, the solution of equations, and the properties of right triangles. Whereas the Greeks of this period were composing logically ordered and systematically expository treatises, the Chinese were repeating the old custom of the Babylonians and Egyptians of compiling sets of specific problems. The *Nine Chapters* resembles Egyptian mathematics also in its use of the method of "false position," but the invention of this scheme, like the origin of Chinese mathematics in general, seems to have been independent of Western influence.

In Chinese works, as in Egyptian, one is struck by the juxtaposition of accurate and inaccurate, primitive and sophisticated results. Correct rules are used for the areas of triangles, rectangles, and trapezoids. The area of the circle was found by taking three fourths the square on the diameter or one-twelfth the square of the circumference—a correct result if the value three is adopted for π—but for the area of a segment of a circle the *Nine Chapters* uses the approximate results $s(s + c)/2$, where s is the sagitta (that is, the radius minus the apothem) and c the chord or base of the segment. There are problems that are solved by the rule of three; in others square and cube roots are found. Chapter eight of the *Nine Chapters* is significant for its solution of problems in simultaneous linear equations, using both positive and negative numbers. The last problem in the chapter involves four equations in five unknowns, and the topic of indeterminate equations was to remain a favorite among Oriental peoples. The ninth and last chapter includes problems on right-angled triangles, some of which later reappeared in India and Europe. One of these asks for the depth of a pond 10 feet square if a reed growing in the center and extending 1 foot above the water just reaches the surface if drawn to the edge of the pond. Another of these well-known problems is that of the Broken bamboo: There is a bamboo 10 feet high, the

[2] See Joseph Needham, *Science and Civilization in China* (1959), Vol. III, pp. 24–25. For recent mathematical works see Tung-Li Yuan, *Bibliography of Chinese Mathematics 1918–1960* (Washington, D.C., published by the author, 1963).

upper end of which being broken reaches the ground 3 feet from the stem. Find the height of the break.[3]

The Chinese were especially fond of patterns; hence it is not surprising 3
that the first record (of ancient but unknown origin) of a magic square appeared there. The square

4	9	2
3	5	7
8	1	6

was supposedly brought to man by a turtle from the River Lo in the days of the legendary Emperor Yii, reputed to be a hydraulic engineer.[4] The concern for such patterns led the author of the *Nine Chapters* to solve the system of simultaneous linear equations

$$3x + 2y + z = 39$$
$$2x + 3y + z = 34$$
$$x + 2y + 3z = 26$$

by performing column operations on the matrix

1	2	3
2	3	2
3	1	1
26	34	39

to reduce it to

0	0	3
0	5	2
36	1	1
99	24	39

The second form represented the equations $36z = 99$, $5y + z = 24$, and $3x + 2y + z = 39$, from which the values of z, y, and x are successively found with ease.

Had Chinese mathematics enjoyed uninterrupted continuity of tradition, 4
some of the striking anticipations of modern methods might have significantly modified the development of mathematics, but Chinese culture was seriously hampered by abrupt breaks. In 213 B.C., for example, the Chinese

[3] See Yoshio Mikami, *The Development of Mathematics in China and Japan* (1913), p.23.

[4] See D. J. Struik, "On Ancient Chinese Mathematics," *The Mathematics Teacher*, **56** (1963), 424–432.

emperor ordered the burning of books. Some works must obviously have survived, either through the persistence of copies or through oral transmission; and learning did indeed persist, with mathematical emphasis on problems of commerce and the calendar.

There seems to have been contact between India and China, as well as between China and the West, but scholars differ on the extent and direction of borrowing. The temptation to see Babylonian or Greek influence in China, for example, is faced with the problem that the Chinese did not make use of sexagesimal fractions. Chinese numeration remained essentially decimal, with notations rather strikingly different from those in other lands. In China, from early times, two schemes of notation were in use. In one the multiplicative principle predominated, in the other a form of positional notation was used. In the first of these there were distinct ciphers for the digits from one to ten and additional ciphers for the powers of ten, and in the written forms the digits in odd positions (from left to right or from bottom to top) were multiplied by their successor. Thus the number 678 would be written as a six followed by the symbol for one hundred, then a seven followed by the symbol for ten, and finally the symbol for eight.

In the system of "rod numerals" the digits from one to nine appeared as 𝍠 𝍡 𝍢 𝍣 𝍤 𝍥 𝍦 𝍧 𝍨 and the first nine multiples of ten as 𝍩 𝍪 𝍫 𝍬 𝍭 𝍮 𝍯 𝍰 𝍱. By the use of these eighteen symbols alternately in positions from right to left, numbers as large as desired could be represented. The number 56,789, for instance, would appear as 𝍤𝍮𝍦𝍰𝍨. As in Babylonia, a symbol for an empty position appeared only relatively late. In a work of 1247 the number 1,405,536 is written with a round zero symbol as 𝍠𝍬O𝍭𝍤𝍫𝍥. (Occasionally, as in the fourteenth-century form of the arithmetic triangle, the vertical and horizontal rods or strokes were interchanged.)

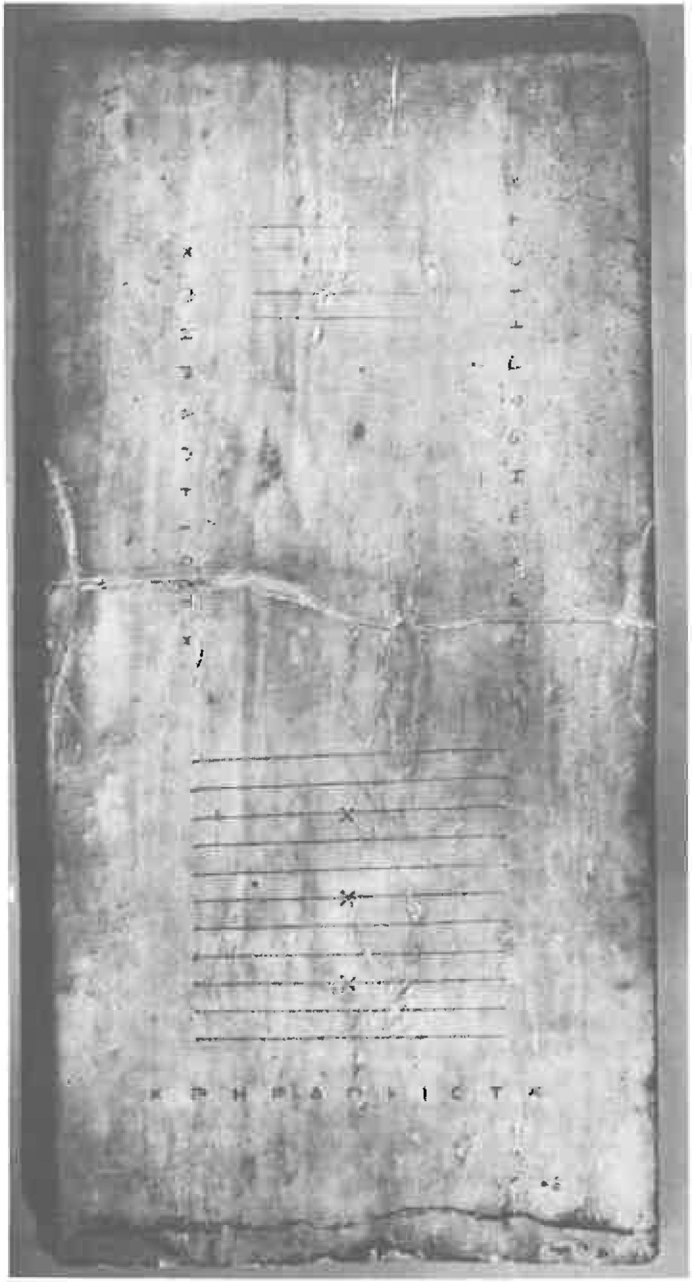
The precise age of the original rod numerals cannot be determined, but they were certainly in use several hundred years before our era, that is, long before the positional notation had been adopted in India. The use of a centesimal, rather than a decimal, positional system in China was convenient for adaptation to computations with the counting board. Distinctive notations for neighboring powers of ten enabled the Chinese to use, without confusion, a counting board with unmarked vertical columns. Before the eighth century the place in which a zero was required was simply left blank. Although in texts older than A.D. 300 the numbers and multiplication tables were written out in words, calculations actually were made with rod numerals on a counting board.

5 The rod numerals of about 300 B.C. were not merely a notation for the written result of a computation. Actual bamboo, ivory, or iron rods were

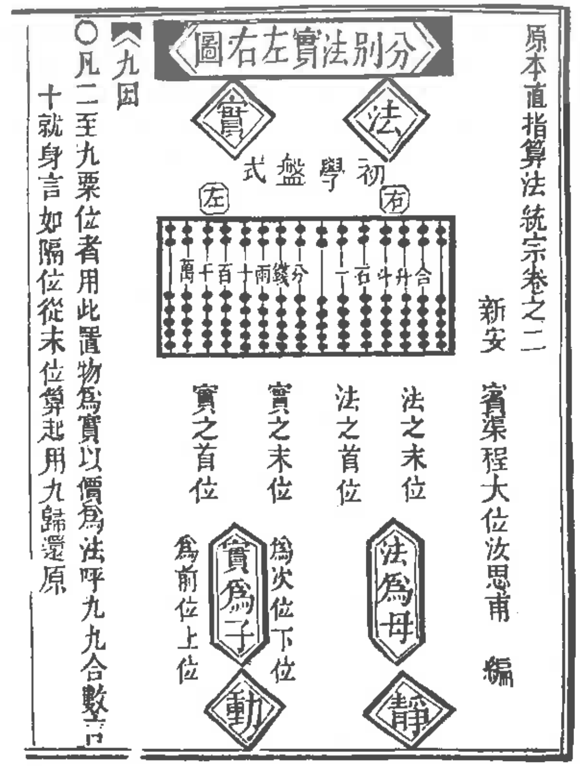
carried about in a bag by administrators and used as a calculating device. Counting rods were manipulated with such dexterity that an eleventh-century writer described them as "flying so quickly that the eye could not follow their movement." Cancellations probably were more rapidly carried out with rods on a counting board than in written calculations. So effective, in fact, was the use of the rods on a counting board that the abacus or rigid counting frame with movable markers on wires was not used so early as has been generally supposed. First clear descriptions of the modern forms, known in China as the *suan phan* and in Japan as the *soroban*, are of the sixteenth century; but anticipations would appear to have been in use perhaps a thousand years earlier. The word *abacus* probably is derived from the Semitic word *abq* or dust, indicating that in other lands, as well as in China, the device grew out of a dust or sand tray used as a counting board. It is possible, but by no means certain, that the use of the counting board in China antedates the European, but clear-cut and reliable dates are not available. In the National Museum in Athens there is a marble slab, dating probably from the fourth century B.C., which appears to be a counting board; and when a century earlier Herodotus wrote, "The Egyptians move their hand from right to left in calculation, while the Greeks move it from left to right," he probably was referring to the use of some sort of counting board. Just when such devices gave way to the abacus proper is difficult to determine; nor can we tell whether or not the appearances of the abacus in China, Arabia, and Europe were independent inventions. The Arabic abacus had ten balls on each wire and no center bar, whereas the Chinese had five lower and two upper counters on each wire, separated by a bar. Each of the upper counters on a wire of the Chinese abacus is equivalent to five on the lower wire; a number is registered by sliding the appropriate counters against the separating bar. (See the accompanying illustration of an abacus.)

No description of Chinese numeration would be complete without reference to the use of fractions. The Chinese were familiar with operations on common fractions, in connection with which they found lowest common denominators. As in other contexts, they saw analogies with the differences in the sexes, referring to the numerator as the "son" and to the denominator as the "mother." Emphasis on *yin* and *yang* (opposites, especially in sex) made it easier to follow the rules for the manipulation of fractions. More important than these, however, was the tendency in China toward decimalization of fractions. As in Mesopotamia a sexagesimal metrology led to sexagesimal numeration, so also in China adherence to the decimal idea in weights and measures resulted in a decimal habit in the treatment of fractions that, it is said, can be traced back as far as the fourteenth century B.C.[5]

[5] See Needham, *op. cit.*, III, 89.



Marble counting board, probably from the fourth century B.C., found on the island of Salamis and now in the National Museum in Athens.



An early printed picture of the abacus, from the *Suan Fa Thung Tsung*, 1593. (Reproduced from Joseph Needham, *Science and Civilization in China*, III, 76.)

Decimal devices in computation sometimes were adopted to lighten manipulations of fractions. In a first-century commentary on the *Nine Chapters*, for example, we find the use of the now familiar rules for square and cube roots, equivalent to $\sqrt{a} = \sqrt{100a}/10$ and $\sqrt[3]{a} = \sqrt[3]{1000a}/10$, which facilitate the decimalization of root extractions.

The idea of negative numbers seems not to have occasioned much difficulty for the Chinese since they were accustomed to calculating with two sets of rods—a red set for positive coefficients or numbers and a black set for negatives. Nevertheless, they did not accept the notion that a negative number might be a solution of an equation.

The earliest Chinese mathematics is so different from that of comparable periods in other parts of the world that the assumption of independent development would appear to be justified. At all events, it seems safe to say 6 that if there was some intercommunication before 400, then more mathematics came out of China than went in. For later periods the question becomes more difficult. The use of the value three for π in early Chinese mathematics is scarcely an argument for dependence on Mesopotamia, especially since the search for more accurate values, from the first centuries of the Christian era, was more persistent in China than elsewhere. Values such as 3.1547, $\sqrt{10}$, 92/29, and 142/45 are found; and in the third century Liu Hui, an important commentator on the *Nine Chapters*, derived the figure 3.14 by use of a regular polygon of 96 sides and the approximation 3.14159 by considering a polygon of 3072 sides. In Liu Hui's reworking of the *Nine Chapters* there are many problems in mensuration, including the correct determination of the volume of a frustum of a square pyramid. For a frustum of a circular cone a similar formula was applied, but with a value of three for π. Unusual is the rule that the volume of a tetrahedron with two opposite edges perpendicular to each other is one-sixth the product of these two edges and their common perpendicular. The method of false position is used in solving linear equations, but there are also more sophisticated results, such as the solution, through a matrix pattern, of a Diophantine problem involving four equations in five unknown quantities. The approximate solution of equations of higher degree seems to have been carried out by a device similar to what we know as "Horner's method." Liu Hui also included, in his work on the *Nine Chapters*, numerous problems involving inaccessible towers and trees on hillsides.[6]

The Chinese fascination with the value of π reached its high point in the work of Tsu Ch'ung-chih (430–501). One of his values was the familiar Archimedean 22/7, described by Tsu Ch'ung-chih as "inexact"; his "accurate" value was 355/113. If one persists in seeking possible Western influence, one can explain away this remarkably good approximation, not equaled anywhere until the fifteenth century, by subtracting the numerator and denominator, respectively, of the Archimedean value from the numerator and denominator of the Ptolemaic value 377/120. However, Tsu Ch'ung-chih went even further in his calculations, for he gave 3.1415927 as an "excess" value and 3.1415926 as a "deficit value."[7] The calculations by which he arrived at these bounds, apparently aided by his son Tsu Cheng-chih, were probably contained in one of his books, since lost. In any case, his results were remarkable for that age, and it is fitting that today a landmark on the moon bears his name.

We should bear in mind that accuracy in the value of π is more a matter

[6] See the excellent article on Liu Hui, written by Ho Peng-Yoke, to appear in the forthcoming volumes of the *Dictionary of Scientific Biography*.

[7] See the article cited in footnote 6. There seems to be some confusion in the citation of this value by Mikami, *op. cit.*, p. 50, by Smith, *op. cit.*, II, 309, and Hofmann, *op. cit.*, I, 76.

of computational stamina than of theoretical insight. The Pythagorean theorem alone suffices to give as accurate an approximation as may be desired. Starting with the known perimeter of a regular polygon of n sides inscribed in a circle, the perimeter of the inscribed regular polygon of $2n$ sides can be calculated by two applications of the Pythagorean theorem. Let C be a circle with center O and radius r (Fig. 12.1) and let $PQ = s$ be a

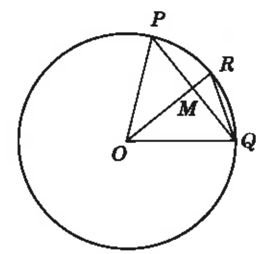


FIG. 12.1

side of a regular inscribed polygon of n sides having a known perimeter. Then the apothem $OM = u$ is given by $u = \sqrt{r^2 - (s/2)^2}$; hence the sagitta $MR = v = r - u$ is known. Then the side $RQ = w$ of the inscribed regular polygon of $2n$ sides is found from $w = \sqrt{v^2 + (s/2)^2}$; hence the perimeter of this polygon is known. The calculation, as Liu Hui saw, can be shortened by noting that $w^2 = 2rv$. An iteration of the procedure will result in an ever closer approximation to the perimeter of the circle, in terms of which π is defined.

Chinese mathematical problems often appear to be more picturesque 7
than practical, and yet Chinese civilization was responsible for a surprising number of technological innovations. The use of printing and gunpowder (eighth century) and of paper and the mariner's compass (eleventh century) was earlier in China than elsewhere, and earlier also than the high-water mark in Chinese mathematics that occurred in the thirteenth century, during the latter part of the Sung period. At that time there were mathematicians working in various parts of China; but relations between them seem to have been remote, and, as in the case of Greek mathematics, we evidently have relatively few of the treatises that once were available. The last and greatest of the Sung mathematicians was Chu Shih-chieh (fl. 1280–1303), yet we know little about him—not even when he was born or when he died. He was a resident of Yen-shan, near modern Peking, but he seems to have spent some twenty years as a wandering scholar who earned his living by teaching mathematics, even though he had the opportunity to write two treatises. The

first of these, written in 1299, was the *Suan-hsüeh ch'i-meng* ("Introduction to Mathematical Studies"), a relatively elementary work that strongly influenced Korea and Japan, although in China it was lost until it reappeared in the nineteenth century.[8] Of greater historical and mathematical interest is the *Ssu-yüan yü-chien* ("Precious Mirror of the Four Elements") of 1303. In the eighteenth century this too disappeared in China, only to be rediscovered in the next century. The four elements, called heaven, earth, man, and matter, are the representations of four unknown quantities in the same equation. The book marks the peak in the development of Chinese algebra, for it deals with simultaneous equations and with equations of degrees as high as fourteen. In it the author describes a transformation method that he calls *fan fa*, the elements of which seem to have arisen long before in China, but which generally bears the name of Horner, who lived half a millennium later. In solving the equation $x^2 + 252x - 5292 = 0$, for example, Chu Shih-chieh first obtained $x = 19$ as an approximation (a root lies between $x = 19$ and $x = 20$) and then used the *fan-fa*, in this case the transformation $y = x - 19$, to obtain the equation $y^2 + 290y - 143 = 0$ (with a root between $y = 0$ and $y = 1$). He then gave the root of the latter as (approximately) $y = 143/(1 + 290)$; hence the corresponding value of x is $19\frac{143}{291}$. For the equation $x^3 - 574 = 0$ he used $y = x - 8$ to obtain $y^3 + 24y^2 + 192y - 62 = 0$, and he gave the root as $x = 8 + 62/(1 + 24 + 192)$ or $x = 8\frac{2}{7}$. In some cases he found decimal approximations.

8 That the so-called Horner method was a commonplace in China is indicated by the fact that at least three other mathematicians of the later Sung period made use of similar devices. One of these was Li Chih (or Li Yeh, 1192–1279), a mathematician of Peking who was offered a government post by Khublai Khan in 1260, but politely found an excuse to decline it. His *Ts'e-yuan hai-ching* ("Sea-Mirror of the Circle Measurements") includes 170 problems dealing with circles inscribed within, or escribed without, a right triangle and with determining the relationships between the sides and the radii, some of the problems leading to equations of fourth degree. Although he did not describe his method of solution of equations, including some of sixth degree, it appears that it was not very different from that used by Chu Shih-chieh and Horner.[9] Others who used the Horner method were Ch'in Chiu-shao (ca. 1202–ca. 1261) and Yang Hui (fl. ca. 1261–1275). The former was an unprincipled governor and minister who acquired immense wealth within a hundred days of assuming office. His *Shu-shu chiu-chang* ("Mathematical Treatise in Nine Sections") marks the high point in Chinese indeterminate analysis, with the invention of routines for solving simultaneous congruences. In this work also he found the square root of 71,824 by steps paralleling those in the Horner method. With 200 as the first approximation to a root of $x^2 - 71{,}824 = 0$, he diminished the roots of this by 200 to obtain $y^2 + 400y - 31{,}824 = 0$. For the latter equation he found 60 as an approximation, and diminished the roots by 60, arriving at a third equation, $z^2 + 520z - 4224 = 0$, of which 8 is a root. Hence the value of x is 268. In a similar way he solved cubic and quartic equations. The same "Horner" device was used by Yang Hui, about whose life almost nothing is known and whose work has survived only in part. Among his contributions that are extant are the earliest Chinese magic squares of order greater than three, including two each of orders four through eight and one each of orders nine and ten.[10]

[8] See the extensive forthcoming article on Chu Shih-chieh by Ho Peng-Yoke to appear in the *Dictionary of Scientific Biography*. See also Needham, *op. cit.*, III, 38–53.

[9] See the article on Li Chih by Ho Peng-Yoke to appear in *Dictionary of Scientific Biography*.

Yang Hui's works included also results in the summation of series and the so-called Pascal triangle, things that were published and better known through the *Precious Mirror* of Chu Shih-chieh, with which the Golden Age of Chinese mathematics closed. A few of the many summations of series found in the *Precious Mirror* are the following: 9

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = n(n + 1)(2n + 1)/3!$$

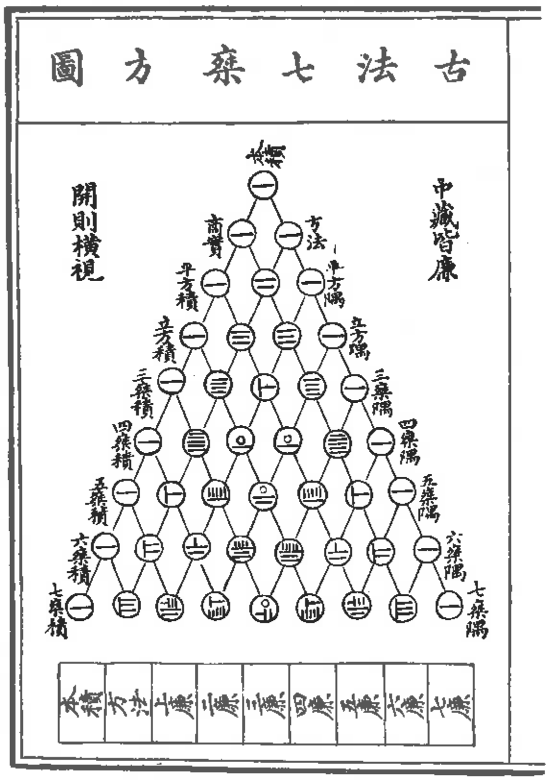
$$1 + 8 + 30 + 80 + \cdots + n^2(n + 1)(n + 2)/3! = n(n + 1)(n + 2)(n + 3) \times (4n + 1)/5!$$

However, no proofs are given, nor does the topic seem to have been continued again in China until about the nineteenth century. Chu Shih-chieh handled his summations through the method of finite differences, some elements of which seem to date in China from the seventh century; but shortly after his work the method disappeared for many centuries.

The *Precious Mirror* opens with a diagram of the arithmetic triangle, inappropriately known in the West as "Pascal's triangle." In Chu's arrangement we have the coefficients of binomial expansions through the eighth power, clearly given in rod numerals and a round zero symbol. Chu disclaims credit for the triangle, referring to it as a "diagram of the old method for finding eighth and lower powers." A similar arrangement of coefficients through the sixth power had appeared in the work of Yang Hui, but without the round zero symbol. There are references in Chinese works of about 1100 to tabulation systems for binomial coefficients, and it is likely that the arithmetic triangle originated in China by about that date. It is interesting to note that the Chinese discovery of the binomial theorem for integral

[10] Excellent articles, including much more on the work of Ch'in Chiu-shao and Yang Hui, written by Ho Peng-Yoke, will appear in the forthcoming *Dictionary of Scientific Biography*.

powers was associated in its origin with root extractions, rather than with powers. The equivalent of the theorem apparently was known to Omar Khayyam at about the time that it was being used in China, but the earliest extant Arabic work containing it is by Al-Kashi in the fifteenth century. By that time Chinese mathematics had failed to match achievements in Europe



The "Pascal" Triangle as depicted in 1303 at the front of Chu Shih-Chieh's *Ssu Yuan Yü Chien*. It is entitled "The Old Method Chart of the Seven Multiplying Squares" and tabulates the binomial coefficients up to the eighth power. (Reproduced from Joseph Needham, *Science and Civilization in China*, III, 135.)

and the Near East, and it is likely that by then more mathematics went into China than came out. Still to be answered is the thorny problem of determining the relative influences of China and India on each other during the first millennium of our era.

Archeological excavations at Mohenjo Daro give evidence of an old and 10
highly cultured civilization in India during the era of the Egyptian pyramid builders, but we have no Indian mathematical documents from that age. Later the country was occupied by Aryan invaders who introduced the caste system and developed the Sanskrit literature. The great religious teacher, Buddha, was active in India at about the time that Pythagoras is said to have visited there, and it sometimes is suggested that Pythagoras learned his theorem from the Hindus. Recent studies make this highly unlikely in view of Babylonian familiarity with the theorem at least a thousand years earlier.

The fall of the Western Roman Empire traditionally is placed in the year 476; it was in this year that Aryabhata, author of one of the oldest Indian mathematical texts, was born. It is clear, however, that there had been mathematical activity in India long before this time—probably even before the mythical founding of Rome in 753 B.C. India, like Egypt, had its "rope-stretchers"; and the primitive geometrical lore acquired in connection with the laying out of temples and the measurement and construction of altars took the form of a body of knowledge known as the *Sulvasūtras* or "rules of the cord." *Sulva* (or *sulba*) refers to cords used for measurements, and *sūtra* means a book of rules or aphorisms relating to a ritual or science. The stretching of ropes is strikingly reminiscent of the origin of Egyptian geometry, and its association with temple functions reminds one of the possible ritual origin of mathematics. However, the difficulty of dating the rules is matched also by doubt concerning the influence they had on later Hindu mathematicians. Even more so than in the case of China, there is a striking lack of continuity of tradition in the mathematics of India; significant contributions are episodic events separated by intervals without achievement.[11]

Three versions, all in verse, of the work referred to as the *Sulvasūtras* are 11
extant, the best-known being that bearing the name of Apastamba. In this primitive account, dating back perhaps as far as the time of Pythagoras, we find rules for the construction of right angles by means of triples of cords the lengths of which form Pythagorean triads, such as 3, 4, and 5, or 5, 12, and 13, or 8, 15, and 17, or 12, 35, and 37. However, all of these triads are easily derived from the old Babylonian rule; hence Mesopotamian influence in the *Sulvasūtras* is not unlikely. Apastamba knew that the square on the diagonal of a rectangle is equal to the sum of the squares on the two adjacent

[11] The reader should be forewarned that there are a number of books in which the contributions from India are grossly overrated. One such instance is B. K. Sarkar, *Hindu Achievements in Exact Science* (New York, 1918). The two-volume *History of Hindu Mathematics* by B. Datta and A. N. Singh (1935–1938) is much more reliable, but even this must be qualified along the lines indicated by Solomon Gandz when he reviewed Volume I in *Isis*, **25** (1936), 478–488.

sides, but this form of the Pythagorean theorem also may have been derived from Mesopotamia. Less easily explained is another rule given by Apastamba —one that strongly resembles some of the geometrical algebra in Book II of Euclid's *Elements*. To construct a square equal in area to the rectangle $ABCD$ (Fig. 12.2), lay off the shorter sides on the longer, so that $AF = AB = BE = CD$, and draw HG bisecting segments CE and DF; extend EF to K, GH to L, and AB to M so that $FK = HL = FH = AM$, and draw LKM.

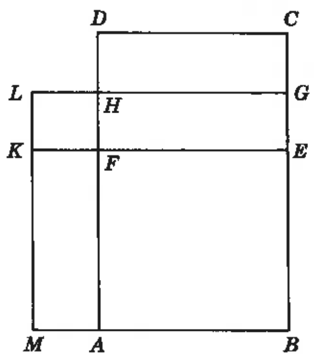


FIG. 12.2

Now construct a rectangle with diagonal equal to LG and with shorter side HF. Then the longer side of this rectangle is the side of the square desired.

So conjectural are the origin and period of the *Sulvasūtras* that we cannot tell whether or not the rules are related to early Egyptian surveying or to the later Greek problem of altar doubling. They are variously dated within an interval of almost a thousand years stretching from the eighth century B.C. to the second century of our era. Chronology in ancient cultures of the Far East is scarcely reliable when orthodox Hindu tradition boasts of important astronomical work more than 2,000,000 years ago[12] and when calculations lead to billions of days from the beginning of the life of Brahman to about A.D. 400.[13] References to arithmetic and geometric series in Vedic literature that purport to go back to 2000 B.C.[14] may be more reliable, but there are no contemporary documents from India to confirm this. It has been claimed also that the first recognition of incommensurables is to be found in India during the *Sulvasūtra* period,[15] but such claims are not well substantiated.

[12] G. R. Kaye, "Indian Mathematics," *Isis*, **2** (1914), 326–356.

[13] *Alberuni's India*, ed. by E. C. Sachan (London, 1960, 2 vols.), II, 32 f.

[14] A. N. Singh, "On the use of Series in Hindu Mathematics," *Osiris*, **1** (1936), 606–628.

[15] A. N. Singh, "A Review of Hindu Mathematics up to the XIIth Century," *Archeion* **18** (1936), 43–62; Saradakanta Ganguli, "On the Indian Discovery of the Irrational at the Time of the Sulvasutras," *Scripta Mathematica*, **1** (1932), 135–141.

The case for early Hindu awareness of incommensurable magnitudes is rendered most unlikely by the failure of Indian mathematicians to come to grips with fundamental concepts.

The period of the *Sulvasūtras*, which closed in about the second century, 12
was followed by the age of the *Siddhāntas*, or systems (of astronomy). The establishment of the dynasty of King Gupta (290) marked the beginning of a renaissance in Sanskrit culture, and the *Siddhāntas* seem to have been an outcome of this revival. Five different versions of the *Siddhāntas* are known by name, *Pauliśha Siddhānta*, *Sūrya Siddhānta*, *Vasisishta Siddhānta*, *Paitamaha Siddhānta*, and *Romanka Siddhānta*. Of these, the *Sūrya Siddhānta* ("System of the Sun"), written about 400, is the only one that seems to be completely extant. According to the text, written in epic stanzas, it is the work of Sūrya, the Sun God.[16] The main astronomical doctrines evidently are Greek, but with the retention of considerable old Hindu folklore. The *Pauliśha Siddhānta*, which dates from about 380, was summarized by the Hindu mathematician Varahamihira (fl. 505) and was referred to frequently by the Arabic scholar Al-Biruni, who suggested a Greek origin or influence. Later writers report that the *Siddhāntas* were in substantial agreement on substance, only the phraseology varying; hence we can assume that the others, like the *Sūrya Siddhānta*, were compendia of astronomy comprising cryptic rules in Sanskrit verse with little explanation and without proof.

It is generally agreed that the *Siddhāntas* stem from the late fourth or the early fifth century, but there is sharp disagreement about the origin of the knowledge that they contain. Hindu scholars insist on the originality and independence of the authors, whereas Western writers are inclined to see definite signs of Greek influence. It is not unlikely, for example, that the *Pauliśha Siddhānta* was derived in considerable measure from the work of the astrologer Paul who lived at Alexandria shortly before the presumed date of composition of the *Siddhāntas*. (Al-Biruni, in fact, explicitly attributes this *Siddhānta* to Paul of Alexandria.) This would account in a simple manner for the obvious similarities between portions of the *Siddhāntas* and the trigonometry and astronomy of Ptolemy. The *Pauliśha Siddhānta*, for example, uses the value 3 177/1250 for π, which is in essential agreement with the Ptolemaic sexagesimal value 3;8,30.

Even if the Hindus did acquire their knowledge of trigonometry from the cosmopolitan Hellenism at Alexandria, the material in their hands took on a significantly new form. Whereas the trigonometry of Ptolemy had been based on the functional relationship between the chords of a circle and the

[16] An English translation by Burgess and Whitney, together with extensive notes, was published in *Journal of the American Oriental Society*, **6** (1860), 141–498. See also George Sarton, *An Introduction to the History of Science* (1927), pp. 386–388.

central angles they subtend, the writers of the *Siddhāntas* converted this to a study of the correspondence between *half* of a chord of a circle and *half* of the angle subtended at the center by the whole chord. Thus was born, apparently in India, the predecessor of the modern trigonometric function known as the sine of an angle; and the introduction of the sine function represents the chief contribution of the *Siddhāntas* to the history of mathematics. Although it is generally assumed that the change from the whole chord to the half chord took place in India, it has been suggested by Paul Tannery, the leading historian of science at the turn of this century, that this transformation of trigonometry may have occurred at Alexandria during the post-Ptolemaic period. Whether or not this suggestion has merit, there is no doubt that it was through the Hindus, and not the Greeks, that our use of the half chord has been derived; and our word "sine", through misadventure in translation (see below), has descended from the Hindu name, *jiva*.

13 During the sixth century, shortly after the composition of the *Siddhāntas*, there lived two Hindu mathematicians who are known to have written books on the same type of material. The older, and more important, of the two was Aryabhata, whose best known work, written in 499 and entitled *Aryabhatiya*, is a slim volume, written in verse, covering astronomy and mathematics. The names of several Hindu mathematicians before this time are known, but nothing of their work has been preserved beyond a few fragments. In this respect, then, the position of the *Aryabhatiya* of Aryabhata in India is somewhat akin to that of the *Elements* of Euclid in Greece some eight centuries before. Both are summaries of earlier developments, compiled by a single author. There are, however, more striking differences than similarities between the two works. The *Elements* is a well-ordered synthesis of pure mathematics with a high degree of abstraction, a clear logical structure, and an obvious pedagogical inclination; the *Aryabhatiya* is a brief descriptive work, in 123 metrical stanzas, intended to supplement rules of calculation used in astronomy and mensurational mathematics, with no feeling for logic or deductive methodology. About a third of the work is on *ganitapada* or mathematics. This section opens with the names of the powers of ten up to the tenth place and then proceeds to give instructions for square and cube roots of integers. Rules of mensuration follow, about half of which are erroneous. The area of a triangle is correctly given as half the product of the base and altitude, but the volume of a pyramid also is taken to be half the product of the base and altitude.[17] The area of a circle is found correctly as the product of the circumference and half the diameter, but the volume of a sphere is incorrectly stated to be the product of the area of a great circle

[17] *The Aryabhatiya of Aryabhata*, trans. by W. E. Clark (1930), p. 26.

and the square root of this area. Again, in the calculation of areas of quadrilaterals, correct and incorrect rules appear side by side. The area of a trapezoid is expressed as half the sum of the parallel sides multiplied by the perpendicular between them; but then follows the incomprehensible assertion that the area of any plane figure is found by determining two sides and multiplying them. One statement in the *Aryabhatiya* to which Hindu scholars have pointed with pride is as follows:[18]

> Add 4 to 100, multiply by 8, and add 62,000. The result is approximately the circumference of a circle of which the diameter is 20,000.

Here we see the equivalent of 3.1416 for π, but it should be recalled that this is essentially the value Ptolemy had used. The likelihood that Aryabhata here was influenced by Greek predecessors is strengthened by his adoption of the myriad, 10,000, as the number of units in the radius.

A typical portion of the *Aryabhatiya* is that involving arithmetic progressions, which contains arbitrary rules for finding the sum of the terms in a progression and for determining the number of terms in a progression when given the first term, the common difference, and the sum of the terms. The first rule had long been known by earlier writers. The second is a curiously complicated bit of exposition:

> Multiply the sum of the progression by eight times the common difference, add the square of the difference between twice the first term, and the common difference, take the square root of this, subtract twice the first term, divide by the common difference, add one, divide by two. The result will be the number of terms.

Here, as elsewhere in the *Aryabhatiya*, no motivation or justification is given for the rule. It was probably arrived at through a solution of a quadratic equation, knowledge of which might have come from Mesopotamia or Greece. Following some complicated problems on compound interest (that is, geometrical progressions), the author turns, in flowery language, to the very elementary problem of finding the fourth term in a simple proportion:

> In the rule of three multiply the fruit by the desire and divide by the measure. The result will be the fruit of the desire.

This, of course, is the familiar rule that if $a/b = c/x$, then $x = bc/a$, where a is the "measure," b the "fruit," c the "desire," and x the "fruit of the desire." The work of Aryabhata is indeed a potpourri of the simple and the complex, the correct and the incorrect. The Arabic scholar al-Biruni, half a millennium later, characterized Hindu mathematics as a mixture of common pebbles and costly crystals, a description quite appropriate to *Aryabhatiya*.

[18] *Aryabhatiya*, p. 28. Translations, here and below, are from the Clark edition cited in footnote 17.

14 The second half of the *Aryabhatiya* is on the reckoning of time and on spherical trigonometry; here we note an element that was to leave a permanent impress on the mathematics of later generations—the decimal place-value numeration. It is not known just how Aryabhata carried out his calculations, but his phrase "from place to place each is ten times the preceding" is an indication that the application of the principle of position was in his mind. "Local value" had been an essential part of Babylonian numeration, and perhaps the Hindus were becoming aware of its applicability to the decimal notation for integers in use in India. The development of numerical notations in India seems to have followed about the same pattern found in Greece. Inscriptions from the earliest period at Mohenjo Daro show at first simple vertical strokes, arranged into groups, but by the time of Asoka (third century B.C.) a system resembling the Herodianic was in use. In the newer scheme the repetitive principle was continued, but new symbols of higher order were adopted for four, ten, twenty, and one hundred. This so-called Karosthi script then gradually gave way to another notation, known as the Brahmi characters, which resembled the alphabetic cipherization in the Greek Ionian system; one wonders if it was only a coincidence that the change in India took place shortly after the period when in Greece the Herodianic numerals were displaced by the Ionian.

From the Brahmi ciphered numerals to our present-day notation for integers two short steps are needed. The first is a recognition that, through the use of the positional principle, the ciphers for the first nine units can serve also as the ciphers for the corresponding multiples of ten, or equally well as ciphers for the corresponding multiples of any power of ten. This recognition would make superfluous all of the Brahmi ciphers beyond the first nine. It is not known when the reduction to nine ciphers occurred, and it is likely that the transition to the more economical notation was made only gradually. It appears from extant evidence that the change took place in India, but the source of the inspiration for the change is uncertain. Possibly the so-called Hindu numerals were the result of internal development alone; perhaps they developed first along the western interface between India and Persia, where remembrance of the Babylonian positional notation may have led to modification of the Brahmi system. It is possible that the newer system arose along the eastern interface with China where the pseudopositional rod numerals may have suggested the reduction to nine ciphers. There is also a theory that this reduction may first have been made at Alexandria within the Greek alphabetic system and that subsequently the idea spread to India.[19] During the later Alexandrian period the earlier Greek habit of writing common fractions with the numerator beneath the denominator was

[19] See Harriet P. Lattin, "The Origin of Our Present System of Notation According to the Theories of Nicholas Bubnov," *Isis*, **19** (1933), 181–194.

reversed, and it is this form that was adopted by the Hindus, without the bar between the two. Unfortunately, the Hindus did not apply the new numeration for integers to the realm of decimal fractions; hence the chief potential advantage of the change from Ionian notation was lost.

The earliest specific reference to the Hindu numerals is found in 662 in the writings of Severus Sebokt, a Syrian bishop. After Justinian closed the Athenian philosophical schools some of the scholars moved to Syria, where they established centers of Greek learning. Sebokt evidently felt piqued by the disdain for non-Greek learning expressed by some associates; hence he found it expedient to remind those who spoke Greek that "there are also others who know something." To illustrate his point he called attention to the Hindus and their "subtle discoveries in astronomy," especially "their valuable methods of calculation, and their computing that surpasses description. I wish only to say that this computation is done by means of nine signs."[20] That the numerals had been in use for some time is indicated by the fact that the first Indian occurrence is on a plate of the year 595, where the date 346 is written in decimal place-value notation.[21]

It should be remarked that the reference to *nine* symbols, rather than *ten*, 15
implies that the Hindus evidently had not yet taken the second step in the transition to the modern system of numeration—the introduction of a notation for a missing position, that is, a zero symbol. The history of mathematics holds many anomalies, and not the least of these is the fact that "the earliest undoubted occurrence of a zero in India is in an inscription of 876"[22] —that is, more than two centuries after the first reference to the other nine numerals. It is not even established that the number zero (as distinct from a symbol for an empty position) arose in conjunction with the other nine Hindu numerals. It is quite possible that zero originated in the Greek world, perhaps at Alexandria, and that it was transmitted to India after the decimal positional system had been established there.[23]

The history of the zero placeholder in positional notation is further complicated by the fact that the concept appeared independently, well before the days of Columbus, in the western, as well as the eastern hemisphere. The Mayas of Yucatan, in their representation of time intervals between dates in their calendar, used a place-value numeration, generally with twenty as the primary base and with five as an auxiliary (corresponding to the Babylonian use of sixty and ten respectively). Units were represented by dots and fives by horizontal bars, so that the number seventeen, for example,

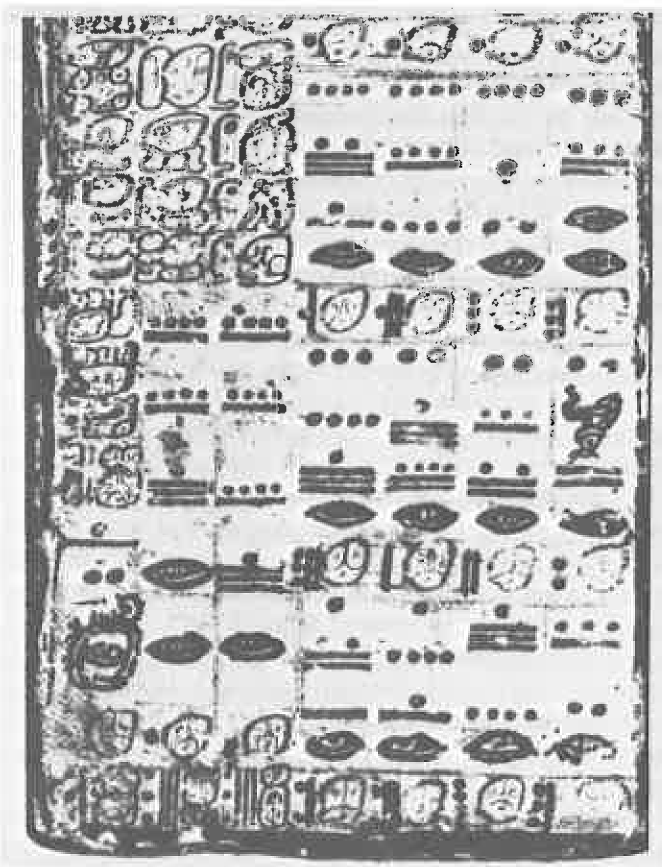
[20] Quoted from D. E. Smith, *History of Mathematics*, I, 167.

[21] See D. J. Struik, *A Concise History of Mathematics*, 3rd ed. (New York: Dover, 1967), p. 71.

[22] Smith, *History of Mathematics*, II, 69.

[23] See, for example, B. L. van der Waerden, *Science Awakening* (1961), pp. 56–58.

would appear as  [that is, as 3(5) + 2]. A vertical positional arrangement was used, with the larger units of time above; hence the notation  denoted 352 [that is 17(20) + 12]. Because the system was primarily for counting days within a calendar having 360 days in a year, the third position usually did not represent multiples of (20)(20), as in a pure vigesimal system, but (18)(20). However, beyond this point the base twenty again prevailed. Within this positional notation the Mayas indicated missing positions



From the Dresden Codex, of the Maya, displaying numbers. The second column on the left, from above down, displays the numbers 9, 9, 16, 0, 0, which stand for 9 × 144,000 + 9 × 7200 + 16 × 360 + 0 + 0 = 1,366,560. In the third column are the numerals 9, 9, 9, 16, 0, representing 1,364,360. The original appears in black and red colors. (Taken from Morley, *An Introduction to the Study of the Maya Hieroglyphs*, p. 266.)

through the use of a symbol, appearing in variant forms, somewhat resembling a half-open eye. In their scheme, then, the notation  denoted 17(20·18·20) + 0(18·20) + 13(20) + 0.

With the introduction, in the Hindu notation, of the tenth numeral, a round goose egg for zero, the modern system of numeration for integers was completed. Although the Medieval Hindu forms of the ten numerals differ considerably from those in use today, the principles of the system were established. The new numeration, which we generally call the Hindu system, is merely a new combination of three basic principles, all of ancient origin: (1) a decimal base; (2) a positional notation; and (3) a ciphered form for each of the ten numerals. Not one of these three was due originally to the Hindus, but it presumably is due to them that the three were first linked to form the modern system of numeration.

It may be well to say a word about the form of the Hindu symbol for zero—which is also ours. It once was assumed that the round form stemmed originally from the Greek letter omicron, initial letter in the word "ouden" or empty, but recent investigations seem to belie such an origin. Although the symbol for an empty position in some of the extant versions of Ptolemy's tables of chords does seem to resemble an omicron, the early zero symbols in Greek sexagesimal fractions are round forms variously embellished and differing markedly from a simple goose egg. Moreover, when in the fifteenth century in the Byzantine Empire a decimal positional system was fashioned out of the old alphabetic numerals by dropping the last eighteen letters and adding a zero symbol to the first nine letters, the zero sign took forms quite unlike an omicron.[24] Sometimes it resembled an inverted form of our small letter h, sometimes it appeared as a dot.

The development of our system of notation for integers was one of the 16
two most influential contributions of India to the history of mathematics. The other was the introduction of an equivalent of the sine function in trigonometry to replace the Greek tables of chords. The earliest tables of the sine relationship that have survived are those in the *Siddhāntas* and the *Aryabhatiya*. Here the sines of angles up to 90° are given for twenty-four equal intervals of $3\frac{3}{4}°$ each. In order to express arc length and sine length in terms of the same unit, the radius was taken as 3438 and the circumference as $360 \cdot 60 = 21{,}600$. This implies a value of π agreeing to four significant figures with that of Ptolemy. In another connection Aryabhata used the value $\sqrt{10}$ for π, which appeared so frequently in India that it sometimes is known as the Hindu value.

[24] See O. Neugebauer, *The Exact Sciences in Antiquity*, 2nd ed. (Providence, R.I.: Brown University Press, 1957), p. 14.

For the sine of $3\frac{3}{4}^\circ$ the *Siddhāntas* and the *Aryabhatiya* took the number of units in the arc—that is, $60 \times 3\frac{3}{4}$ or 225. In modern language, the sine of a small angle is very nearly equal to the radian measure of the angle (which is virtually what the Hindus were using). For further items in the sine table the Hindus used a recursion formula which may be expressed as follows. If the nth sine in the sequence from $n = 1$ to $n = 24$ is designated as s_n, and if the sum of the first n sines is S_n, then $s_{n+1} = s_n + s_1 - S_n/s_1$. From this rule one easily deduces that $\sin 7\frac{1}{2}^\circ = 449$, $\sin 11\frac{1}{4}^\circ = 671$, $\sin 15^\circ = 890$, and so on up to $\sin 90^\circ = 3438$—the values listed in the table in the *Siddhāntas* and the *Aryabhatiya*. Moreover, the table also includes values for what we call the versed sine of the angle—that is, $1 - \cos\theta$ in modern trigonometry or $3438\ (1 - \cos\theta)$ in Hindu trigonometry—from vers $3\frac{3}{4}^\circ = 7$ to vers $90^\circ = 3438$. If we divide the items in the table by 3438, the results are found to be in close agreement with the corresponding values in modern trigonometric tables.[25]

17 Hindu trigonometry evidently was a useful and accurate tool in astronomy. How the Hindus arrived at results such as the recursion formula is uncertain, but it has been suggested[26] that an intuitive approach to difference equations and interpolation may have prompted such rules. Indian mathematics frequently is described as "intuitive," in contrast to the stern rationalism of Greek geometry. Although in Hindu trigonometry there is evidence of Greek influence, the Indians seem to have had no occasion to borrow Greek geometry, concerned as they were with simple mensurational rules. Of the classical geometrical problems, or the study of curves other than the circle, there is little evidence in India, and even the conic sections seem to have been overlooked by the Hindus, as by the Chinese. Hindu mathematicians were fascinated instead by work with numbers, whether it involved the ordinary arithmetic operations or the solution of determinate or indeterminate equations. Addition and multiplication were carried out in India much as they are by us today, except that they seem at first to have preferred to write numbers with the smaller units on the left, hence to work from left to right, using small blackboards with white removable paint or a board covered with sand or flour. Among the devices used for multiplications was one that is known under various names: lattice multiplication, *gelosia* multiplication, or cell or grating or quadrilateral multiplication. The scheme behind this is readily recognized in two examples. In the first example (Fig. 12.3) the number 456 is multiplied by 34. The multiplicand has been written above the lattice and the multiplier appears to the left, with the partial products

[25] The table from the *Sūrya Siddhānta* is reproduced in Smith, *History of Mathematics*, II.
[26] E. S. Kennedy in the article "Trigonometry," to appear in the *Yearbook on History of Mathematics* of the National Council of Teachers of Mathematics.

occupying the square cells. Digits in the diagonal rows are added, and the product 15,504 is read off at the bottom and the right. To indicate that other arrangements are possible, a second example is given in Fig. 12.4, in which

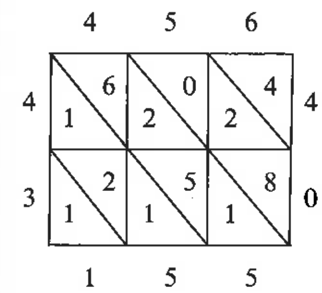


FIG. 12.3

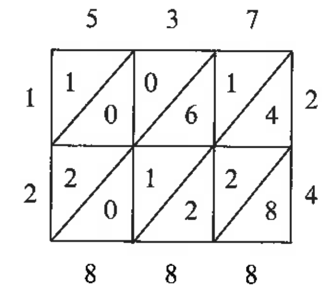
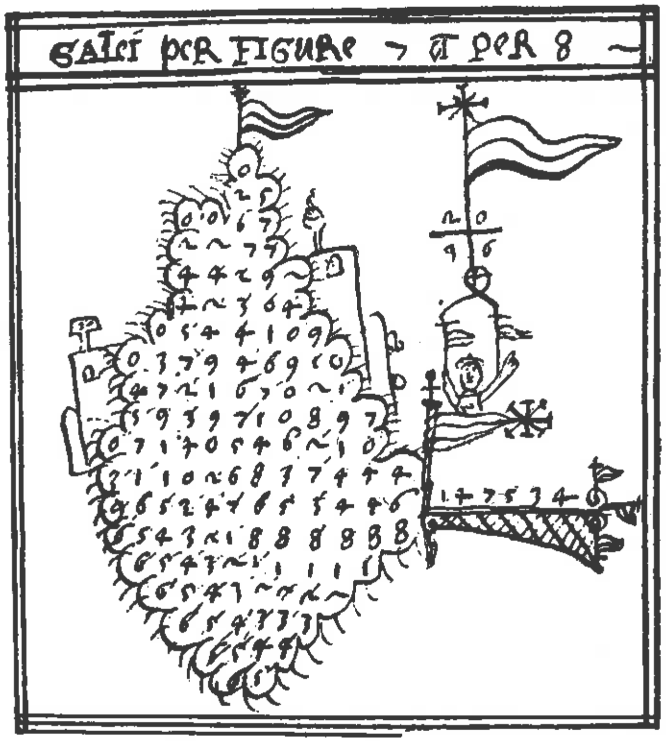


FIG. 12.4

the multiplicand 537 is placed at the top and the multiplier 24 is on the right, the product 12,888 appearing to the left and along the bottom. Still other modifications are easily devised. In fundamental principle *gelosia* multiplication is of course the same as our own, the cell arrangement being merely a convenient device for relieving the mental concentration called for in "carrying over" from place to place the tens arising in the partial products. The only "carrying" required in lattice multiplication is in the final additions along the diagonals.

It is not known when or where *gelosia* multiplication arose, but India 18
seems to be the most likely source. It was used there at least by the twelfth century, and from India it seems to have been carried to China and Arabia. From the Arabs it passed over to Italy in the fourteenth and fifteenth centuries, where the name *gelosia* was attached to it because of the resemblance to gratings placed before windows in Venice and elsewhere. (The current word jalousie seems to stem from the Italian *gelosia* and is used for Venetian blinds in France, Germany, Holland, and Russia.) The Arabs (and through them the later Europeans) appear to have adopted most of their arithmetic devices from the Hindus, and so it is likely that the pattern of long division known as the "scratch method" or the "galley method" (from its resemblance to a boat) came also from India. To illustrate the method, let it be required to divide 44,977 by 382. In Fig. 12.5 we give the modern method, in Fig. 12.6 the galley method.[27] The latter parallels the former closely except that the

[27] For further description of the innumerable computational devices that have been used, see F. A. Yeldham, *The Story of Reckoning in the Middle Ages* (1926).



Galley division, sixteenth century. From an unpublished manuscript of a Venetian monk. The title of the work is "Opus Arithmeticá D. Honorati veneti monachj coenobij S. Lauretig." From Mr. Plimpton's library.

dividend appears in the middle, for subtractions are performed by canceling digits and placing differences *above* rather than *below* the minuends. Hence the remainder, 283, appears above and to the right, rather than below.

```
     117
382)44977
    382
    677
    382
    2957
    2674
     283
```

FIG. 12.5

```
          2
         2 3
         3 9 8
      1 6 7 5 3
382 | 4 4 9 7 7 | 117
      3 8 2 2 4
         3 8 7
         2 6
```

FIG. 12.6

The process in Fig. 12.6 is easy to follow if we note that the digits in a given subtrahend, such as 2674, or in a given difference, such as 2957, are not necessarily all in the same row and that subtrahends are written below the middle and differences above the middle. Position in a column is significant, but not position in a row. The determination of roots of numbers probably followed a somewhat similar "galley" pattern, coupled in the later years with the binomial theorem in "Pascal triangle" form; but Hindu writers did not provide explanations for their calculations or proofs for their statements. It is possible that Babylonian and Chinese influences played a role in the problem of evolution or root extraction. It is often said that the "proof by nines," or the "casting out of nines," is a Hindu invention, but it appears that the Greeks knew earlier of this property, without using it extensively, and that the method came into common use only with the Arabs of the eleventh century.

The last few paragraphs may leave the unwarranted impression that there 19
was a uniformity in Hindu mathematics, for frequently we have localized developments as merely "of Indian origin," without specifying the period. The trouble is that there is a high degree of uncertainty in Hindu chronology. Material in the important Bakshali manuscript, containing an anonymous arithmetic, is supposed by some to date from the third or fourth century, by others from the sixth century, by others from the eighth or ninth century or later; and there is a suggestion that it may not even be of Hindu origin.[28] We have placed the work of Aryabhata around the year 500, but the date is doubtful since there were two mathematicians named Aryabhata and we cannot with certainty ascribe results to our Aryabhata, the elder. Hindu mathematics presents more historical problems than does Greek mathematics, for Indian authors referred to predecessors infrequently, and they exhibited surprising independence in mathematical approach. Thus it is that Brahmagupta (fl. 628), who lived in Central India somewhat more than a century after Aryabhata, has little in common with his predecessor, who had lived in eastern India. Brahmagupta mentions two values of π—the "practical value" 3 and the "neat value" $\sqrt{10}$—but not the more accurate value of Aryabhata; in the trigonometry of his best-known work, the *Brahmasphuta Siddhānta*, he adopted a radius of 3270 instead of Aryabhata's 3438. In one respect he does resemble his predecessor—in the juxtaposition of good and bad results. He found the "gross" area of an isosceles triangle by multiplying half the base by one of the equal sides; for the scalene triangle with base fourteen and sides thirteen and fifteen he found the "gross area" by multiplying half the base by the arithmetic mean of the other sides. In finding the

[28] See Florian Cajori, *A History of Mathematics* (1919), pp. 84–85; Smith, *History of Mathematics*, I, 164; Hofmann, *Geschichte der Mathematik*, I, 59.

"exact" area he utilized the Archimedean-Heronian formula. For the radius of the circle circumscribed about a triangle he gave the equivalent of the correct trigonometric result $2R = a/\sin A = b/\sin B = c/\sin C$, but this of course is only a reformulation of a result known to Ptolemy in the language of chords. Perhaps the most beautiful result in Brahmagupta's work is the generalization of "Heron's" formula in finding the area of a quadrilateral. This formula—$K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$, where a, b, c, d are the sides and s is the semiperimeter—still bears his name; but the glory of his achievement is dimmed by failure to remark that the formula is correct only in the case of a *cyclic* quadrilateral.[29] (The correct formula for an arbitrary quadrilateral is $\sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\alpha}$, where α is half the sum of two opposite angles.) As a rule for the "gross" area of a quadrilateral Brahmagupta gave the pre-Hellenic formula, the product of the arithmetic means of the opposite sides. For the quadrilateral with sides $a = 25$, $b = 25$, $c = 25$, $d = 39$, for example, he found a "gross" area of 800.

20 Brahmagupta's contributions to algebra are of a higher order than are his rules of mensuration, for here we find general solutions of quadratic equations, including two roots even in cases in which one of them is negative. The systematized arithmetic of negative numbers and zero is, in fact, first found in his work. The equivalents of rules on negative magnitudes were known through the Greek geometrical theorems on subtraction, such as $(a - b)(c - d) = ac + bd - ad - bc$, but the Hindus converted these into numerical rules on positive and negative numbers. Moreover, although the Greeks had a concept of nothingness, they never interpreted this as a number, as did the Hindus. However, here again Brahmagupta spoiled matters somewhat by asserting that $0 \div 0 = 0$, and on the touchy matter of $a \div 0$, for $a \neq 0$, he did not commit himself:

> Positive divided by positive, or negative by negative, is affirmative. Cipher divided by cipher is naught. Positive divided by negative is negative. Negative divided by affirmative is negative. Positive or negative divided by cipher is a fraction with that for denominator.[30]

It should be mentioned also that the Hindus, unlike the Greeks, regarded irrational roots of numbers as numbers. This was of enormous help in algebra, and Indian mathematicians have been much praised for taking this step; but one must remember that the Hindu contribution in this case was the result of logical innocence rather than of mathematical insight. We have

[29] A proof of the formula can be found in R. A. Johnson, *Modern Geometry* (New York: Houghton Mifflin, 1929), pp. 81–82.

[30] See H. T. Colebrooke, *Algebra, with Arithmetic and Mensuration, from the Sanscrit of Brahmagupta and Bhaskara* (1817).

seen the lack of nice distinction on the part of Hindu mathematicians between exact and inexact results, and it was only natural that they should not have taken seriously the difference between commensurable and incommensurable magnitudes. For them there was no impediment to the acceptance of irrational numbers, and later generations followed their lead uncritically until in the nineteenth century mathematicians established the real number system on a sound basis.

Indian mathematics was, as we have said, a mixture of good and bad. But some of the good was superlatively good, and here Brahmagupta deserves high praise. Hindu algebra is especially noteworthy in its development of indeterminate analysis, to which Brahmagupta made several contributions. For one thing, in his work we find a rule for the formation of Pythagorean triads expressed in the form $m, \frac{1}{2}(m^2/n - n), \frac{1}{2}(m^2/n + n)$; but this is only a modified form of the old Babylonian rule, with which he may have become familiar. Brahmagupta's area formula for a quadrilateral, mentioned above, was used by him in conjunction with the formulas

$$\sqrt{(ab+cd)(ac+bd)/(ad+bc)} \quad \text{and} \quad \sqrt{(ac+bd)(ad+bc)/(ab+cd)}$$

for the diagonals,[31] to find quadrilaterals whose sides, diagonals, and areas are all rational. Among them was the quadrilateral with sides $a = 52$, $b = 25$, $c = 39$, $d = 60$, and diagonals 63 and 56. Brahmagupta gave the "gross" area as $1933\frac{3}{4}$, despite the fact that his formula provides the exact area, 1764, in this case.

Like many of his countrymen, Brahmagupta evidently loved mathematics 21
for its own sake, for no practical-minded engineer would raise questions such as those Brahmagupta asked about quadrilaterals. One admires his mathematical attitude even more when one finds that apparently he was the first one to give a *general* solution of the linear Diophantine equation $ax + by = c$, where a, b, and c are integers. For this equation to have integral solutions, the greatest common divisor of a and b must divide c; and Brahmagupta knew that if a and b are relatively prime, all solutions of the equation are given by $x = p + mb$, $y = q - ma$, where m is an arbitrary integer. He suggested also the Diophantine quadratic equation $x^2 = 1 + py^2$, named mistakenly for John Pell (1611–1685), but first appearing in the Archimedean cattle problem. The Pell equation was solved for some cases by Brahmagupta's countryman, Bhaskara (1114–ca. 1185).

It is greatly to the credit of Brahmagupta that he gave *all* integral solutions of the linear Diophantine equation, whereas Diophantus himself had been

[31] For indications of a proof of these formulas see Howard Eves, *An Introduction to the History of Mathematics* (1964), pp. 202–203.

satisfied to give one particular solution of an indeterminate equation. Inasmuch as Brahmagupta used some of the same examples as Diophantus, we see again the likelihood of Greek influence in India—or the possibility that they both made use of a common source, possibly from Babylonia. It is interesting to note also that the algebra of Brahmagupta, like that of Diophantus, was syncopated. Addition was indicated by juxtaposition, subtraction by placing a dot over the subtrahend, and division by placing the divisor below the dividend, as in our fractional notation but without the bar. The operations of multiplication and evolution (the taking of roots), as well as unknown quantities, were represented by abbreviations of appropriate words.

:2 India produced a number of later Medieval mathematicians, but we shall describe the work of only one of these—Bhaskara (1114–ca. 1185), the leading mathematician of the twelfth century. It was he who filled some of the gaps in Brahmagupta's work, as by giving a general solution of the Pell equation and by considering the problem of division by zero. Aristotle once had remarked that there is no ratio by which a number such as four exceeds the number zero;[32] but the arithmetic of zero had not been part of Greek mathematics, and Brahmagupta had been noncommittal on the division of a number other than zero by the number zero. It is therefore in Bhaskara's *Vija-Ganita* that we find the first statement that such a quotient is infinite.

> Statement: Dividend 3. Divisor 0. Quotient the fraction 3/0. This fraction of which the denominator is cipher, is termed an infinite quantity. In this quantity consisting of that which has cipher for a divisor, there is no alteration, though many be inserted or extracted; as no change takes place in the infinite and immutable God.

This statement sounds promising, but lack of clear understanding of the situation is suggested by Bhaskara's further assertion that $a/0 \cdot 0 = a$.

Bhaskara was the last significant Medieval mathematician from India, and his work represents the culmination of earlier Hindu contributions. In his best known treatise, the *Lilavati*, he compiled problems from Brahmagupta and others, adding new observations of his own. The very title of this book may be taken to indicate the uneven quality of Hindu thought, for the name in the title is that of Bhaskara's daughter who, according to legend, lost the opportunity to marry because of her father's confidence in his astrological predictions. Bhaskara had calculated that his daughter might propitiously marry only at one particular hour on a given day. On what was to have been her wedding day the eager girl was bending over the water clock, as the hour

[32] See C. B. Boyer, "An Early Reference to Division by Zero," *American Mathematical Monthly*, **50** (1943), 487–491.

for the marriage approached, when a pearl from her headdress fell, quite unnoticed, and stopped the outflow of water. Before the mishap was noted, the propitious hour had passed. To console the unhappy girl, the father gave her name to the book we are describing.

The *Lilavati*, like the *Vija-Ganita*, contains numerous problems dealing 23
with favorite Hindu topics: linear and quadratic equations, both determinate and indeterminate, simple mensuration, arithmetic and geometric progressions, surds, Pythagorean triads, and others. The "broken bamboo" problem, popular in China (and included also by Brahmagupta), appears in the following form: If a bamboo 32 cubits high is broken by the wind so that the tip meets the ground 16 cubits from the base, at what height above the ground was it broken? Also making use of the Pythagorean theorem is the following problem: A peacock is perched atop a pillar at the base of which is a snake's hole. Seeing the snake at a distance from the pillar which is three times the height of the pillar, the peacock pounced upon the snake in a straight line before it could reach its hole. If the peacock and the snake had gone equal distances, how many cubits from the hole did they meet?

These two problems illustrate well the heterogeneous nature of the *Lilavati*, for despite their apparent similarity and the fact that only a single answer is required, one of the problems is determinate and the other is indeterminate. In treating of the circle and the sphere the *Lilavati* fails also to distinguish between exact and approximate statements. The area of the circle is correctly given as one-quarter the circumference multiplied by the diameter and the volume of the sphere as one-sixth the product of the surface area and the diameter, but for the ratio of circumference to diameter in a circle Bhaskara suggests either 3927 to 1250 or the "gross" value 22/7. The former is equivalent to the ratio mentioned, but not used, by Aryabhata. There is no hint in Bhaskara or other Hindu writers that they were aware that all ratios that had been proposed were approximations only. However, Bhaskara severely condemns his predecessors for using the formulas of Brahmagupta for the area and diagonals of a general quadrilateral, because he saw that a quadrilateral is not uniquely determined by its sides. Evidently he did not realize that the formulas are indeed exact for all cyclic quadrilaterals.

Many of Bhaskara's problems in the *Lilavati* and the *Vija-Ganita* evidently were derived from earlier Hindu sources; hence it is no surprise to note that the author is at his best in dealing with indeterminate analysis. In connection with the Pell equation, $x^2 = 1 + py^2$, proposed earlier by Brahmagupta, Bhaskara gave particular solutions for the five cases $p = 8, 11, 32, 61$, and 67. For $x^2 = 1 + 61y^2$, for example, he gave the solution $x = 1{,}776{,}319{,}049$ and $y = 22{,}615{,}390$. This is an impressive feat in calculation, and its verification alone will tax the efforts of the reader.

Bhaskara's books are replete with other instances of Diophantine problems.[33]

24 Bhaskara died toward the end of the twelfth century, and for several hundred years there were few mathematicians in India of comparable stature. It is of interest to note, nevertheless, that Srinivasa Ramanujan (1887–1920), the twentieth-century Hindu genius, had the same uncanny manipulative ability in arithmetic and algebra that is found in Bhaskara. The British mathematician G. H. Hardy once visited Ramanujan in a hospital at Putney and mentioned to his ill friend that he had arrived in a taxi with the dull number 1729, whereupon Ramanujan without hesitation pointed out that this number is indeed interesting, for it is the least integer that can be represented in two different ways as the sum of two cubes—$1^3 + 12^3 = 1729 = 9^3 + 10^3$. In Ramanujan's work we note also the disorganized character, the strength of intuitive reasoning, and the disregard for geometry that stood out so clearly in his predecessors. Although in Ramanujan these characteristics had perhaps developed largely because he was self-taught, we cannot help but see how strikingly different the development of mathematics in India has been from that in Greece. Even when the Hindus borrowed from their neighbors, they fashioned the material in their own peculiar manner. Although in attitude and interests they had more in common with the Chinese, they did not share the latter's fascination with accurate approximations, such as led to Horner's method. And although they shared with the Mesopotamians a preponderately algebraic view, they tended to avoid sexagesimal numeration. In short, the eclectic Hindu mathematicians adopted and developed only such aspects as appealed to them. In one respect it was unfortunate that their first love should have been theory of numbers in general, and indeterminate analysis in particular, for it was not from these aspects that later developments in mathematics grew. Analytic geometry and calculus had Greek rather than Indian roots, and European algebra came from the Islamic countries rather than India. Nevertheless, in modern mathematics there are at least two reminders that mathematics owes its development to India, as well as to many other lands. The trigonometry of the sine function came presumably from India; our own system of numeration for integers is appropriately called the Hindu-Arabic system to indicate its probable origin in India and its transmission through Arabia.

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[33] An exceptionally full account of Bhaskara's work is found in J. F. Scott, *A History of Mathematics* (1958). See also Colebrooke, *op. cit.*

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EXERCISES

1. Compare Hindu and Chinese mathematics with respect to favorite topics and level of achievement.
2. Which had the greater influence on modern thought, Chinese or Hindu mathematics? Explain clearly.