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$$\sum_{k=1}^{\infty} \frac{1}{k} = \prod_{p} \frac{1}{1-1/p}$$

Euler
The Master of Us All

William Dunham

Finally, and most sincerely, I wish to recognize my wife and colleague Penny Dunham, who made numerous improvements to the final manuscript and who applied her computer wizardry to generate the figures contained herein. For such assistance—and for so much more—I dedicate this book to her with love and thanks.

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"Read Euler, read Euler. He is the master of us all."
–*Laplace*

Preface

In the crypt beneath St. Paul's Cathedral lies the tomb of Christopher Wren, architect of that great and beautiful building. The accompanying inscription ranks among the most famous of epitaphs:

Lector, si monumentum requiris, circumspice.

This translates as, "Visitor, if you seek his monument, look around." Indeed, an architect could have no finer memorial than the huge church soaring overhead. From nave to dome, from transepts to choir, St. Paul's is Wren's masterpiece.

Mathematics lacks the tactile solidity of architecture. It is intangible, existing not in stone and mortar but in the human imagination. Yet, like architecture, it is real. And, like architecture, it has its masters.

This book is about one of the undisputed geniuses of mathematics, Leonhard Euler. His insight was breathtaking, his vision profound, his influence as significant as that of anyone in history. Euler contributed to long-established branches of mathematics like number theory, analysis, algebra, and geometry. He also ventured into the largely unexplored territory of analytic number theory, graph theory, and differential geometry. In addition, he was his century's foremost *applied* mathematician, as his work in mechanics, optics, and acoustics amply demonstrates. There was hardly an aspect of the subject that escaped Euler's penetrating gaze. As the twentieth-century mathematician André Weil put it, "All his life... he seems to have carried in his head the whole of the mathematics of his day, both pure and applied."[1]

If the quality of his achievement was extraordinary, so too was its sheer quantity. At present, 73 volumes of the *Opera Omnia* (his collected works) are in print—a publishing project that began in 1911—and many volumes of scientific correspondence and other manuscripts are yet to appear. Euler was

[1] André Weil, *Number Theory: An Approach through History*, Birkhäuser, Boston, 1984, p. 284.

a veritable Niagara, one who wrote mathematics faster than most people can absorb it.

As an expositor, Euler has no peers. He produced classic texts in algebra, differential and integral calculus, and the calculus of variations—works that continue to shape the nature of their subjects down to the present day. Further, his writing was fresh and enthusiastic, in contrast to the modern tendency of obscuring a scholar's passion behind the façade of detached, technical prose. Euler was clearly having fun, pursuing the game for its own enjoyment, and exhibiting a pervasive confidence that his quest would be successful.

In beholding such productivity, one is apt to be humble. In all honesty, one is apt to be *overwhelmed*. No author can do justice to the tens of thousands of pages Euler penned over six decades of his career, and it is hard not to feel both inadequate and foolhardy even to consider an undertaking such as this.

Yet his achievements deserve a look. For all the mathematicians who revere Euler's name, relatively few have picked up a volume of the *Opera Omnia* and plunged in. On the contrary, it is the custom of modern mathematicians to learn the subject from textbooks rather than from original sources. Because of changes in notation and emphasis that occur over time, not to mention real advances that can render a prior discussion obsolete, this is not an inherently bad idea.

But something is lost if we deal only in substitutes, only in proxies. Original mathematics, even if centuries old, can be as stirring as the theorems proved last week. This is especially true of Euler's work, as Raymond Ayoub so cogently observed when he wrote:

> Reading his papers is an exhilarating experience; one is struck by the great imagination and originality. Sometimes a result familiar to the reader will take on an original and illuminating aspect, and one wishes that later writers had not tampered with it.[2]

No student of literature would be satisfied with a mere synopsis of *Hamlet*. In like fashion, no mathematician should go through a career without meeting Euler face to face. To do otherwise suggests not only an indifference about the past but also, in some fundamental way, a genuine selfishness.

My ground rules for this book are simple: I focus each chapter upon a subject to which Euler made a significant contribution. Chapters begin with a discussion of what was known prior to Euler; this provides an opportunity to

[2]Raymond Ayoub, "Euler and the Zeta Function," *The American Mathematical Monthly*, Vol. 81, No. 10, 1974, p. 1069.

introduce such predecessors as Euclid, Heron, Briggs, and Bernoulli—giants upon whose shoulders Euler would stand. I next examine an Eulerian "great theorem" that pushed the frontiers as only he could. In so doing, I pledge to be as faithful as possible in explaining his original line of attack. Each chapter concludes with an epilogue, either discussing Euler's subsequent work on the topic or describing how later mathematicians further developed his ideas.

As a consequence, this book meanders through number theory, analysis, complex variables, algebra, geometry, and combinatorics—these being but a few of the areas in which Euler made an impact. Selections of theorems—indeed, selections of the areas themselves—are my own. Moreover, because Euler was a master at devising multiple proofs of the same result, one must choose among equally intriguing routes to the same end. Fifty different authors operating under the same ground rules would come up with fifty different books (and I'd be interested to read the other forty-nine). But this one is mine.

What mathematical prerequisites are necessary for the chapters ahead? On the one hand, this volume is not aimed at the rank beginner. Readers should be familiar with such concepts as "integration by parts" or "prime numbers" or "geometric series." I imagine that a few college math courses would provide more than sufficient background for everything I cover.

On the other hand, the book certainly does not assume a graduate school mastery of any branch of mathematics. In a very real sense, that would defeat my purpose. I hope I have made the material accessible to the widest possible audience of "mathematically literate" readers so that it is, in the best sense of the term, *expository*.

As I begin, I make both an observation and a request.

The observation is that Euler was far from infallible. He operated in an era whose standards of mathematical rigor were far more primitive than those of today. As we shall see, some of his arguments were questionable, and others were simply wrong. After all, it was Euler who, without hesitation, introduced expressions like

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots = 0.66215 + \frac{1}{2}\ln(\infty)\ ^{3}$$

or

$$\frac{1 - x^0}{0} = -\ln x.\ ^{4}$$

[3]Leonhard Euler, *Elements of Algebra*, trans. John Hewlett, Springer-Verlag, New York (1840 Reprint), p. 296.
[4]Euler, *Opera Omnia*, Ser. 1, Vol. 14, p. 12.

The modern reader may dismiss these with a knowing smirk, but one dare not laugh too quickly. Because both the left and right sides of the first equation are infinite, it is not really incorrect (even if the 0.66215 on the right seems absurdly superfluous). And the second equation, if slightly modified to read "$\lim_{t \to 0^+} (1 - x^t)/t = -\ln x$ for $x > 0$," makes perfect sense. Here, as often happens with Euler's "mistakes," we come to realize that, though this be mathematical madness, yet there is method in it.

I also make my request, namely that irate readers not complain about the omission of their favorite Eulerian theorem. At the outset, I plead guilty to such charges, for I have omitted virtually *all* of Euler's work. This book represents just the tip of the mathematical iceberg or, perhaps more appropriately, of the mathematical glacier.

At best, I hope to share my personal enthusiasm for a tiny fragment of Euler's remarkable vision. In spite of the passage of centuries, his contributions remain of the highest order, and his impact upon mathematics is everywhere evident. No matter their speciality, mathematicians of today may truly say of Euler what was once said of Wren:

"If you seek his monument, look around."

Biographical Sketch

Euler's life fits snugly within the eighteenth century. Born in the spring of 1707, he lived 76 years until the autumn of 1783. This makes him the close contemporary of another quintessential citizen of that century, Benjamin Franklin (1706–1790). Although of different temperaments, different interests, and even different hemispheres, both Franklin and Euler were widely esteemed in their own day, and both had a profound impact upon the course of Western civilization.

Although focusing primarily on Euler's mathematics, this book should also provide at least a quick survey of his life. In one sense, that life was not especially exciting. Euler was a fairly conventional person, by all accounts kind and generous, but one who lacked the flair of some of his century's better known figures. Unlike Washington (1732–1799), he did not command armies to victory; unlike Robespierre (1758–1794), he did not lead—or succumb to—a political revolution; unlike Captain Cook (1728–1779), he did not sail the seas to explore unknown continents.

Yet in another sense, Euler *was* a great adventurer. His adventures, of course, were of the intellectual sort, carrying him not across the physical world but through a wonderful mathematical landscape. Exploration, after all, can take many forms.

Leonhard Euler was born near Basel, Switzerland. His father was a Protestant clergyman of modest means who entertained the hope that Leonhard would follow him into the pulpit. His mother also came from a pastoral family, so the deck seemed stacked: young Euler appeared destined for the ministry.

He was a precocious youth, blessed with a gift for languages and an extraordinary memory. Euler eventually carried in his head an assortment of curious information, including orations, poems, and lists of prime powers. He also was a fabulous mental calculator, able to perform intricate arithmetical computations without benefit of pencil and paper. These uncommon talents would serve him well later in life.

After entering the University of Basel at age 14, Leonhard encountered its most famous professor, Johann Bernoulli (1667–1748). Two facts about Bernoulli should be noted. First, he was a proud and arrogant man, as quick to demean the work of others as to praise that of himself. Second, any such praise was probably deserved. In 1721, Johann Bernoulli could claim to be the world's greatest active mathematician (Leibniz had died a few years before, and the aged Newton had long since abandoned mathematics). It was only by chance that he found himself in Basel, a small city that was hardly the intellectual capital of the world. Yet there he was when Euler needed a mentor.

Not Euler's teacher in a modern sense of the term, Bernoulli instead became a guide for the young scholar, suggesting mathematical readings and making himself available to discuss those points that seemed especially difficult. As Euler recalled years later,

> I was given permission to visit [Johann Bernoulli] freely every Saturday afternoon and he kindly explained to me everything I could not understand.[1]

Euler asserted that this loose tutorial arrangement was "undoubtedly... the best method to succeed in mathematical subjects," and crusty Johann Bernoulli came to realize that his young tutee was something special. As the years passed and their relationship matured, it was Bernoulli who more and more seemed to become the pupil. Johann, a man not easily given to compliments, once wrote to Euler these generous lines:

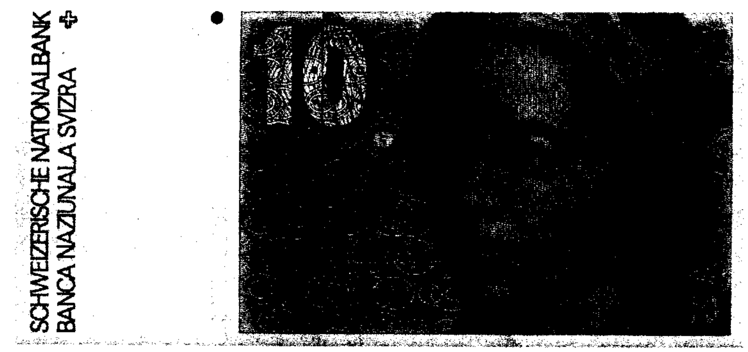
> I present higher analysis as it was in its childhood, but you are bringing it to man's estate.[2]

At university, Euler's education was not limited to mathematics. He spoke on the subject of temperance, wrote on the history of law, and eventually completed a master's degree in philosophy. Then, fulfilling his apparent destiny, Euler entered divinity school to study for the ministry.

But the call of mathematics was too strong. He later remembered:

> I had to register in the faculty of theology, and I was to apply myself ... to the Greek and Hebrew languages, but not much progress was

[1]Charles C. Gillispie, ed., *Dictionary of Scientific Biography*, Leonhard Euler, p. 468.
[2]Morris Kline, *Mathematical Thought from Ancient to Modern Times*, Oxford U. Press, New York, 1972, p. 592.



Euler on Swiss currency

> made, for I turned most of my time to mathematical studies, and by my happy fortune the Saturday visits to Johann Bernoulli continued.[3]

He left the ministry to others—Euler would become a mathematician.

His progress was rapid. At age 20, he earned recognition in an international scientific competition for his analysis of the placement of masts on a sailing ship. This was remarkable for one so young and so landlocked (after all, Euler had spent his entire life in Switzerland). It was the harbinger of successes to come.

Then, as now, it did not hurt to have friends in high places. In 1725, Johann's son Daniel Bernoulli (1700–1782) arrived in Russia to assume a position in mathematics at the new St. Petersburg Academy, and the next year Euler was invited to join him. The only opening at the time was in physiology/medicine, but jobs were scarce, so Euler accepted the offer. Knowing nothing of the medical arts, he set about learning the subject in characteristically industrious fashion—albeit from a somewhat *geometrical* point of view.

Upon his 1727 arrival in St. Petersburg, Euler learned that he had been reassigned to physics rather than physiology—surely a fortuitous development not only for him but also for those patients whom he might have operated upon with compass and straightedge. During the early years in Russia, Euler resided at the home of Daniel Bernoulli, and the two engaged in extended discussions of physics and mathematics that to some extent previewed the course of European science over the coming decades.

[3]Clifford Truesdell, "Leonhard Euler, Supreme Geometer," in Euler's *Elements of Algebra*, p. xii.

In 1733, Daniel Bernoulli left for an academic post in Switzerland. On the one hand, the departure of his good friend left a hole in Euler's life. On the other, it opened up the chair in mathematics, which Euler soon occupied.

With such professional advancement, Euler found himself comfortably situated, and soon thereafter he married. His wife was Katharina Gsell (?–1773), daughter of a Swiss painter living in Russia. Over four decades of their happy and productive marriage, the Eulers had 13 children. Unfortunately, as was common at the time, only five survived to adolescence, and only three outlived their parents.

Intellectual life at the St. Petersburg Academy suited Euler perfectly. He devoted a vast amount of effort to research but was constantly at the disposal of the state—which, after all, paid his salary. Time and again he found himself as a scientific consultant to the government, in which capacity he prepared maps, advised the Russian navy, and even tested designs for fire engines. However, he drew the line when asked to cast a horoscope for the young Czar, a job he quickly passed to another.

Meanwhile his fame was growing. One of his earliest triumphs was a solution of the so-called "Basel Problem" that had perplexed mathematicians for the better part of the previous century. The issue was to determine the *exact* value of the infinite series

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\cdots+\frac{1}{k^2}+\cdots.$$

Numerical approximations had revealed that the series sums to a number somewhere in the vicinity of $8/5$, but the exact answer eluded a string of mathematicians ranging from Pietro Mengoli (1625–1686), who posed the problem in 1644, through Jakob Bernoulli (1654–1705)—Johann's brother and Daniel's uncle—who brought it to the attention of the broader mathematical community in 1689. Well into the next century the problem remained unsolved, and anyone capable of summing the series was certain to make a major splash.

When it happened in 1735, the splash was Euler's.[4] The answer was not only a mathematical *tour de force* but a genuine surprise, for the series sums to $\pi^2/6$. This highly non-intuitive result made the solution all the more spectacular and its solver all the more famous. (Euler's reasoning is described in Chapter 3 of this book).

[4] Ron Calinger, "Leonhard Euler: The First St. Petersburg Years (1727–1741)," *Historia Mathematica*, Vol. 23, 1996, pp. 121–166 contains an account of the Basel problem and an excellent survey of Euler's first stay in Russia.

With the Basel problem behind him and the promise of good things ahead, Euler pursued his research at a breathtaking pace. Paper after paper flowed from his pen into the journal of the St. Petersburg Academy, so that for some issues half the articles in the publication were his. He seemed to be living in a mathematician's paradise.

But three problems darkened this period. First was the political turmoil that swirled across Russia in the aftermath of the unexpected death of Catherine I. Her absence left a leadership vacuum that, in conjunction with the suspicions and intrigues of the day, had dangerous consequences. Among these were an intolerance of dissent and a growing suspicion of foreigners. The fact that the Academy was staffed almost exclusively by non-Russians led Euler to describe his situation as "rather awkward."[5]

Second, the Academy was run by a pompous bureaucrat named Johann Schumacher. In the words of Clifford Truesdell, Schumacher's primary occupation lay "in the suppression of talent wherever it might rear its inconvenient head."[6] Although Euler was diplomatic in dealing with his boss, he surely could not have been comfortable working under a martinet with such undeserved self-importance.

The final problem was physical: the deterioration of Euler's eyesight. As early as 1738 he experienced a loss of vision in his right eye. Euler attributed this to overwork, particularly to his intense efforts at cartography, but modern medical opinion suggests it more likely was the result of a severe infection he had recently suffered.

The impact of his visual decline was—in terms of Euler's mathematics—nil. Visual impairment or no, Euler continued his program of research. He wrote about ship-building, acoustics, and the theory of musical harmony. With the encouragement of his friend Christian Goldbach (1690–1764), Euler made seminal discoveries in classical number theory (see Chapter 1) and pushed into the uncharted waters of analytic number theory (see Chapter 4). In response to a letter from Philippe Naudé (1684–1745), he lay the groundwork for the theory of partitions (Chapter 8). And it was during this period that he wrote his text, *Mechanica*, which presented the Newtonian laws of motion within a framework of calculus. For this, the *Mechanica* has been called "a landmark in the history of physics."[7]

[5] Truesdell, p. xx.
[6] Ibid., p. xv.
[7] Calinger, p. 143.

With such an output came a matching reputation, which in turn generated an offer from Prussia's Frederick the Great (1712–1786) to become a member of the newly revitalized Berlin Academy. Because of the uneasy political situation in Russia, which Euler described as "a country where every person who speaks is hanged," the offer looked appealing.[8] Thus in 1741 Leonhard, Katharina, and family made the move to Germany.

Berlin was home for a quarter of a century, the middle phase of Euler's mathematical career. During this time he published two of his greatest works—a 1748 text on functions, the *Introductio in analysin infinitorum* (discussed in Chapter 2), and a 1755 volume on differential calculus, the *Institutiones calculi differentialis*. This period also saw him investigate complex numbers and discover "Euler's identity"—$e^{i\theta} = \cos\theta + i\sin\theta$ (see Chapter 5)—as well as offer a proof of the fundamental theorem of algebra (which we treat in Chapter 6).

While in Berlin, Euler was asked to provide instruction in elementary science to the Princess of Anhalt Dessau. The result was a multi-volume masterpiece of exposition, subsequently published as *Letters of Euler on Different Subjects in Natural Philosophy Addressed to a German Princess*.[9] This compilation of over 200 "letters" introduced subjects as diverse as light, sound, gravity, logic, language, magnetism, and astronomy. In the course of the work, Euler explained why it is cold atop a high mountain in the tropics, why the moon looks larger when it rises, and why the sky is blue. He ranged further afield when he discussed the origin of evil, the conversion of sinners, and the intriguing topic of "Electrization of Men and Animals."

Writing about vision in a "letter" dated August 1760, Euler began with these words: "I am now enabled to explain the phenomena of vision, which is undoubtedly one of the greatest operations of nature that the human mind can contemplate."[10] The poignancy of this remark, coming as it did from a partially—and soon to be totally—blind author, is striking. But Euler was not one to let personal misfortune interfere with his attitude toward the wonders of Nature.

Letters to a German Princess became an international hit. The work was translated into a host of languages across Europe and eventually published (in

[8] Euler, *Letters of Euler on Different Subjects in Natural Philosophy*, Arno Press, New York, 1975, p. 19.
[9] Item #8 is an English translation of Euler's *Letters to a German Princess*.
[10] Ibid., p. 155.

1833) in the United States. In the preface to the American edition, the publisher gushed over Euler's expository skill in guaranteeing that

> the delight of the reader is, at every step, commensurate with her improvement, and each succeeding acquisition of knowledge becomes a source of still increasing gratification.[11]

In the end, this was Euler's most widely read book. It is not always the case that a scholar working at the very frontier of research can step back to write a treatise accessible to the layman, but this Euler surely did. *Letters to a German Princess* remains to this day one of history's finest examples of popular science.[12]

In spite of the fact that Euler had deserted his colleagues in Russia, they bore him no ill will. From Germany he continued to edit the St. Petersburg journal, to publish article after article in its pages, and to receive a regular stipend from his old employer. Such cordiality continued even through the Seven Years' War, which saw Russian troops invade Berlin. A friendly relationship with St. Petersburg would prove significant in the years to come.

Beyond his mathematical research, Euler was deeply involved in administrative duties at the Berlin Academy. Although not officially the Academy's director, he informally played that role. In the process, he assumed a peculiar array of responsibilities, from juggling budgets to overseeing greenhouses.

But all was not well in Berlin, for Frederick the Great had developed an inexplicable contempt for his most famous scholar-in-residence. The animosity seems to have stemmed as much from a personality conflict as anything. Frederick regarded himself as an erudite, witty savant. He loved philosophy, poetry, and anything French. In fact, affairs at the Academy were conducted in French, not German. To the King, Euler was something of a bumpkin—a brilliant bumpkin to be sure, but a bumpkin all the same. Conventional in his tastes, Euler was a hard-working family man and a devout Protestant. "As long as he preserved his sight," we are told,

> he assembled the whole of his family every evening, and read a chapter of the Bible, which he accompanied with an exhortation. Theology was one of his favourite studies, and the doctrines which he held were the most rigid doctrines of Calvinism.[13]

[11] Ibid., p. ii.
[12] See also Ron Calinger, "Euler's letters to a princess of Germany as an expression of his mature scientific outlook," *Archives of the History of the Exact Sciences*, Vol. 15, No. 3, 1975/76, pp. 211–233.
[13] Euler, *Letters of Euler on Different Subjects in Natural Philosophy*, p. 26.

Here was someone of a different breed than the glittering sophisticates at the Berlin Academy. Before long, Frederick took to calling him "my cyclops," a cruel reference to Euler's limited vision.

Making matters worse was the frosty relationship that developed between Euler and the Academy's other superstar, Voltaire (1694–1778). At least for a time, Voltaire enjoyed several advantages in the circle of Frederick the Great—he was celebrated as an author and satirist; he was as sophisticated as the King; and he was thoroughly French. Euler was not spared Voltaire's caustic wit. The latter characterized him as one who "never learnt philosophy" and thus had to satisfy himself "with the fame of being the mathematician who in a given time has filled more sheets of paper with calculations than any other."[14]

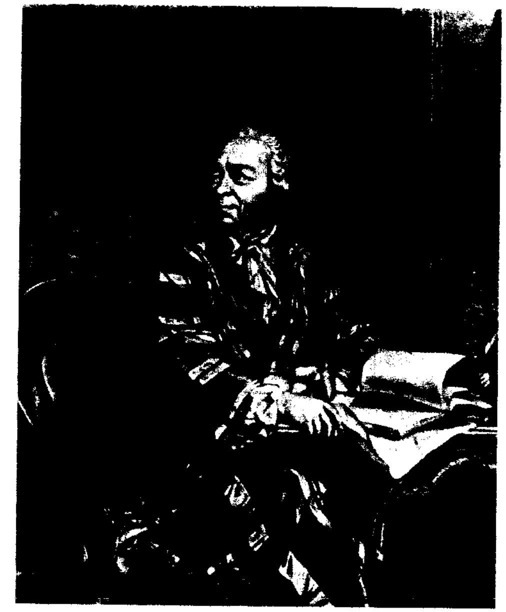
Thus, despite bringing to the Berlin Academy a mathematical glory it would never again achieve, Euler was forced out. Matters in Russia had improved during his absence, particularly with the installation of Catherine the Great (1729–1796), so Euler was only too happy to return. The St. Petersburg Academy must have barely believed its good fortune when, in 1766, it welcomed back the greatest mathematician in the world. This time, Euler would stay for good.

Although his scientific life proceeded apace, the next few years brought two personal tragedies. First, he suffered the failure of his remaining good eye. By 1771 Euler was virtually blind. This left him without the ability to write or read anything other than very large characters. Then, late in 1773, Katharina died. Coupled with his recent blindness, this loss could well have marked the end of Euler's productive years.

Euler, however, was no ordinary man. Although unable to see, he not only maintained but even increased his scientific output. In the year 1775, for instance, he wrote an average of one mathematical paper *per week*. Such productivity came in spite of the fact that he now had to have others read him the contents of scientific papers, and he in turn had to dictate his work to diligent scribes. During this descent into blindness, he wrote an influential textbook on algebra, a 775-page treatise on the motion of the moon, and a massive, three-volume development of integral calculus, the *Institutiones calculi integralis*. Never was his remarkable memory more useful than when he could see mathematics only in his mind's eye.

That this blind and aging man forged ahead with such gusto is a remarkable lesson, a tale for the ages. Euler's courage, determination, and utter unwillingness to be beaten serves, in the truest sense of the word, as an inspiration for mathematician and non-mathematician alike. The long history of mathematics provides no finer example of the triumph of the human spirit.

[14]Truesdell, p. xxix.



Portrait of the mature Euler

Three years after his wife's death Euler married her half sister, thereby finding a companion with whom to share his last years. These stretched until September 18, 1783. On that day, Euler spent time with his grandchildren and then took up mathematical questions associated with the flight of balloons. This was a topic of interest due to the Montgolfier brothers' recent ascent above Paris in a hot-air balloon—an event witnessed, incidentally, by a diplomat of the new American nation, Benjamin Franklin.[15]

After lunch Euler made some calculations on the orbit of the planet Uranus. Undoubtedly he would have found the behavior of Uranus a rich source of new

[15]Roger Burlingame, *Benjamin Franklin: Envoy Extraordinary*, Coward-McCann, New York, 1967, p. 182.

problems. In the decades to come, the planet's peculiar orbit, analyzed in light of equations that Euler had refined, led astronomers to search for—and to discover—the even more distant planet Neptune. Had Euler the time, he would have enjoyed the challenge of seeking a new planet mathematically.

But Euler was not to have such an opportunity. In the late afternoon of that typically busy September day, he was struck down by a massive hemorrhage that caused his immediate death. Mourned by his family, by his colleagues at the Academy, and by the world's scientific community, Leonhard Euler was laid to rest in St. Petersburg. Only then did this great engine of mathematics fall silent.

Euler left behind a legacy of epic proportions. So prolific was he that the journal of the St. Petersburg Academy was still publishing the backlog of his papers a full 48 years after his death. There is hardly a branch of mathematics—or for that matter of physics—in which he did not play a significant role.

In his eulogy, the Marquis de Condorcet observed that whosoever pursues mathematics in the future will be "guided and sustained by the genius of Euler" and asserted, with much justification, that "all mathematicians ... are his disciples."[16]

In the eight chapters that follow, sustained by this genius, we shall examine a tiny fraction of Euler's output. It is only a sampler. But, heeding the advice of Laplace, we shall be reading the work of a master.

[16]Euler, *Opera Omnia*, Ser. 3, Vol. 12, p. 308.

CHAPTER 1

Euler and Number Theory

Of all branches of mathematics, none is so natural—nor so deceptively difficult—as the theory of numbers. Its object is to understand the positive integers, surely the most fundamental of mathematical entities. To the uninitiated, number theory seems far simpler than its more sophisticated cousins like trigonometry or calculus. After all, any eight-year old can count to fifty, but how many know the Law of Cosines or the Chain Rule?

It takes very little number theoretic exposure to disabuse the uninitiated of this notion. In fact, the innocent-looking whole numbers are the source of some of the deepest, most vexing problems in mathematics. Hiding their secrets with an embarrassing ease, the integers provide a worthy challenge for the greatest of mathematicians.

Perfect numbers, the subject of this chapter, were of interest as far back as classical times. Euclid (ca. 300 BCE) included a major theorem about such numbers in his masterpiece, the *Elements*, and twenty centuries later Leonhard Euler revisited the topic to finish what Euclid had begun. Yet even Euler left important questions unanswered. To this day, as with so many issues in number theory, the final chapter remains to be written, and the quest for perfect numbers, in the words of Victor Klee and Stan Wagon, "is perhaps the oldest unfinished project of mathematics."[1]

Prologue

Euclid's *Elements* is recognizable even by non-mathematicians as the foremost geometry text of the ancient Greeks. But many are surprised to learn that Euclid devoted three of the thirteen books (or chapters) of the *Elements* to number theory.

[1]Victor Klee and Stan Wagon, *Old and New Unsolved Problems in Plane Geometry and Number Theory*, Mathematical Association of America, 1991, p. 178.