EUCLID'S ELEMENTS.

BOOK I.

DEFINITIONS.

1. A POINT is that which has no parts, or which has no magnitude.

2. A line is length without breadth.

3. The extremities of a line are points.

4. A straight line is that which lies evenly between its extreme points.

5. A superficies is that which has only length and breadth.

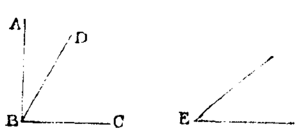
6. The extremities of a superficies are lines.

7. A plane superficies is that in which any two points being taken, the straight line between them lies wholly in that superficies.

8. A plane angle is the inclination of two lines to one another in a plane, which meet together, but are not in the same direction.

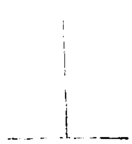
9. A plane rectilineal angle is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.

Note. When several angles are at one point B, any one of them is expressed by three letters, of which the letter which is at the vertex of the angle, that is, at the point at which the straight lines that contain the angle meet one another, is put between the other two letters, and one of these two letters is somewhere on one of those straight lines, and the other letter on the other straight line. Thus, the angle which is contained by the

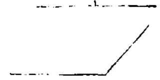


straight lines AB, CB is named the angle ABC, or CBA; the angle which is contained by the straight lines AB, DB is named the angle ABD, or DBA; and the angle which is contained by the straight lines DB, CB is named the angle DBC, or CBD; but if there be only one angle at a point, it may be expressed by a letter placed at that point; as the angle at E.

10. When a straight line standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle; and the straight line which stands on the other is called a perpendicular to it.



11. An obtuse angle is that which is greater than a right angle.

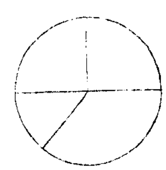


12. An acute angle is that which is less than a right angle.

13. A term or boundary is the extremity of any thing.

14. A figure is that which is enclosed by one or more boundaries.

15. A circle is a plane figure contained by one line, which is called the circumference, and is such, that all straight lines drawn from a certain point within the figure to the circumference are equal to one another:



16. And this point is called the centre of the circle.

17. A diameter of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

[A radius of a circle is a straight line drawn from the centre to the circumference.]

18. A semicircle is the figure contained by a diameter and the part of the circumference cut off by the diameter.

19. A segment of a circle is the figure contained by a straight line and the circumference which it cuts off.

20. Rectilineal figures are those which are contained by straight lines:

21. Trilateral figures, or triangles, by three straight lines:

22. Quadrilateral figures by four straight lines:

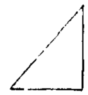
23. Multilateral figures, or polygons, by more than four straight lines.

24. Of three-sided figures,

An equilateral triangle is that which has three equal sides:

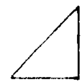
25. An isosceles triangle is that which has two sides equal:

26. A scalene triangle is that which has three unequal sides:

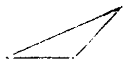


27. A right-angled triangle is that which has a right angle:

[The side opposite to the right angle in a right-angled triangle is frequently called the hypotenuse.]



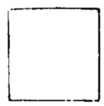
28. An obtuse-angled triangle is that which has an obtuse angle:



29. An acute-angled triangle is that which has three acute angles.

Of four-sided figures,

30. A square is that which has all its sides equal, and all its angles right angles:



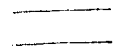
31. An oblong is that which has all its angles right angles, but not all its sides equal:

32. A rhombus is that which has all its sides equal, but its angles are not right angles:

33. A rhomboid is that which has its opposite sides equal to one another, but all its sides are not equal, nor its angles right angles:

34. All other four-sided figures besides these are called trapeziums.

35. Parallel straight lines are such as are in the same plane, and which being produced ever so far both ways do not meet.



[*Note.* The terms *oblong* and *rhomboid* are not often used. Practically the following definitions are used. Any four-sided figure is called a *quadrilateral.* A line joining two opposite angles of a quadrilateral is called a *diagonal.* A quadrilateral which has its opposite sides parallel is called a *parallelogram.* The words *square* and *rhombus* are used in the sense defined by Euclid; and the word *rectangle* is used instead of the word *oblong.*

Some writers propose to restrict the word *trapezium* to a quadrilateral which has two of its sides parallel; and it would certainly be convenient if this restriction were universally adopted.]

POSTULATES.

Let it be granted,

1. That a straight line may be drawn from any one point to any other point:

2. That a terminated straight line may be produced to any length in a straight line:

3. And that a circle may be described from any centre, at any distance from that centre.

AXIOMS.

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals the wholes are equal.

3. If equals be taken from equals the remainders are equal.

4. If equals be added to unequals the wholes are unequal.

5. If equals be taken from unequals the remainders are unequal.

6. Things which are double of the same thing are equal to one another.

7. Things which are halves of the same thing are equal to one another.

8. Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

9. The whole is greater than its part.

10. Two straight lines cannot enclose a space.

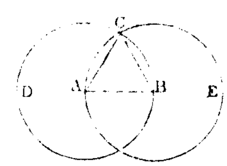
11. All right angles are equal to one another.

12. If a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles.

PROPOSITION 1. *PROBLEM.*

To describe an equilateral triangle on a given finite straight line.

Let AB be the given straight line: it is required to describe an equilateral triangle on AB.



From the centre A, at the distance AB, describe the circle BCD. [*Postulate* 3.

From the centre B, at the distance BA, describe the circle ACE. [*Postulate* 3.

From the point C, at which the circles cut one another, draw the straight lines CA and CB to the points A and B. [*Post.* 1.

ABC shall be an equilateral triangle.

Because the point A is the centre of the circle BCD, AC is equal to AB. [*Definition* 15.

And because the point B is the centre of the circle ACE, BC is equal to BA. [*Definition* 15.

But it has been shewn that CA is equal to AB;

therefore CA and CB are each of them equal to AB.

But things which are equal to the same thing are equal to one another. [*Axiom* 1.

Therefore CA is equal to CB.

Therefore CA, AB, BC are equal to one another.

Wherefore *the triangle ABC is equilateral,* [*Def.* 24.
and it is described on the given straight line AB. Q.E.F.

BOOK XII.

LEMMA.

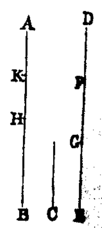
If from the greater of two unequal magnitudes there be taken more than its half, and from the remainder more than its half, and so on, there shall at length remain a magnitude less than the smaller of the proposed magnitudes.

Let AB and C be two unequal magnitudes, of which AB is the greater: if from AB there be taken more than its half, and from the remainder more than its half, and so on, there shall at length remain a magnitude less than C.

For C may be multiplied so as at length to become greater than AB.

Let it be so multiplied, and let DE its multiple be greater than AB, and let DE be divided into DF, FG, GE, each equal to C.

From AB take BH, greater than its half, and from the remainder AH take HK greater than its half, and so on, until there be as many divisions in AB as in DE; and let the divisions in AB be AK, KH, HB, and the divisions in DE be DF, FG, GE.



Then, because DE is greater than AB;
and that EG taken from DE is not greater than its half;
but BH taken from AB is greater than its half;
therefore the remainder DG is greater than the remainder AH.

Again, because DG is greater than AH;
and that GF is not greater than the half of DG, but HK is greater than the half of AH;
therefore the remainder DF is greater than the remainder AK.

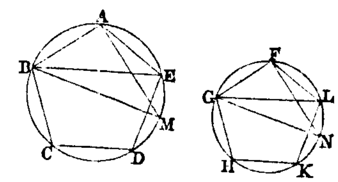
But DF is equal to C;
therefore C is greater than AK;
that is, AK is less than C. Q.E.D.

And if only the halves be taken away, the same thing may in the same way be demonstrated.

PROPOSITION 1. *THEOREM.*

Similar polygons inscribed in circles are to one another as the squares on their diameters.

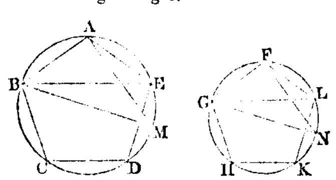
Let $ABCDE$, $FGHKL$ be two circles, and in them the similar polygons $ABCDE$, $FGHKL$; and let BM, GN be the diameters of the circles: the polygon $ABCDE$ shall be to the polygon $FGHKL$ as the square on BM is to the square on GM.



Join AM, BE, FN, GL.

Then, because the polygons are similar,
therefore the angle BAE is equal to the angle GFL,
and BA is to AE as GF is to FL. [VI. *Definition* 1.
Therefore the triangle BAE is equiangular to the triangle GFL; [VI. 6.
therefore the angle AEB is equal to the angle FLG.
But the angle AEB is equal to the angle AMB, and the angle FLG is equal to the angle FNG; [III. 21.
therefore the angle AMB is equal to the angle FNG.

And the angle BAM is equal to the angle GFN, for each of them is a right angle. [III. 31.



Therefore the remaining angles in the triangles AMB, FNG are equal, and the triangles are equiangular to one another;
therefore BA is to BM as GF is to GN, [VI. 4.
and, alternately, BA is to GF as BM is to GN; [V. 16.
therefore the duplicate ratio of BA to GF is the same as the duplicate ratio of BM to GN. [V. *Definition* 10, V. 22.

But the polygon $ABCDE$ is to the polygon $FGHKL$ in the duplicate ratio of BA to GF; [VI. 20.
and the square on BM is to the square on GN in the duplicate ratio of BM to GN; [VI. 20.
therefore the polygon $ABCDE$ is to the polygon $FGHKL$ as the square on BM is to the square on GN. [V. 11.

Wherefore, *similar polygons* &c. Q.E.D.

PROPOSITION 2. *THEOREM.*

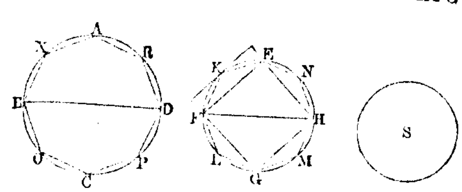
Circles are to one another as the squares on their diameters.

Let $ABCD$, $EFGH$ be two circles, and BD, FH their diameters: the circle $ABCD$ shall be to the circle $EFGH$ as the square on BD is to the square on FH.

For, if not, the square on BD must be to the square on FH as the circle $ABCD$ is to some space either less than the circle $EFGH$, or greater than it.

First, if possible, let it be as the circle $ABCD$ is to a space S less than the circle $EFGH$.

In the circle $EFGH$ inscribe the square $EFGH$. [IV. 6.
This square shall be greater than half of the circle $EFGH$.



For the square $EFGH$ is half of the square which can be formed by drawing straight lines to touch the circle at the points E, F, G, H;
and the square thus formed is greater than the circle;
therefore the square $EFGH$ is greater than half of the circle.

Bisect the arcs EF, FG, GH, HE at the points K, L, M, N;
and join EK, KF, FL, LG, GM, MH, HN, NE. Then each of the triangles EKF, FLG, GMH, HNE shall be greater than half of the segment of the circle in which it stands.

For the triangle EKF is half of the parallelogram which can be formed by drawing a straight line to touch the circle at K, and parallel straight lines through E and F, and the parallelogram thus formed is greater than the segment FEK; therefore the triangle EKF is greater than half of the segment.

And similarly for the other triangles.

Therefore the sum of all these triangles is together greater than half of the sum of the segments of the circle in which they stand.

Again, bisect EK, KF, &c. and form triangles as before; then the sum of these triangles is greater than half of the sum of the segments of the circle in which they stand.

If this process be continued, and the triangles be supposed to be taken away, there will at length remain segments of circles which are together less than the excess of the circle *EFGH* above the space *S*, by the preceding Lemma.

Let then the segments *EK*, *KF*, *FL*, *LG*, *GM*, *MH*, *HN*, *NE* be those which remain, and which are together less than the excess of the circle above *S*;

therefore the rest of the circle, namely the polygon *EKFLGMHN*, is greater than the space *S*.

In the circle *ABCD* describe the polygon *AXBOCPDR* similar to the polygon *EKFLGMHN*.

Then the polygon *AXBOCPDR* is to the polygon *EKFLGMHN* as the square on *BD* is to the square on *FH*, [XII. 1.

that is, as the circle *ABCD* is to the space *S*. [*Hyp.*, V. 11.

But the polygon *AXBOCPDR* is less than the circle *ABCD* in which it is inscribed,

therefore the polygon *EKFLGMHN* is less than the space *S*; [V. 14.

but it is also greater, as has been shewn;

which is impossible.

Therefore the square on *BD* is not to the square on *FH* as the circle *ABCD* is to any space less than the circle *EFGH*.

In the same way it may be shewn that the square on *FH* is not to the square on *BD* as the circle *EFGH* is to any space less than the circle *ABCD*.

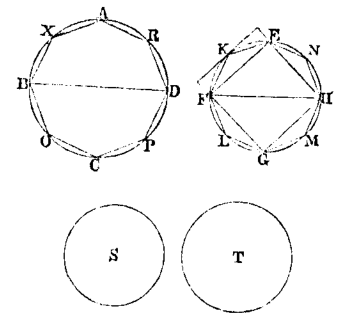
Nor is the square on *BD* to the square on *FH* as the circle *ABCD* is to any space greater than the circle *EFGH*.

For, if possible, let it be as the circle *ABCD* is to a space *T* greater than the circle *EFGH*.

Then, inversely, the square on *FH* is to the square on *BD* as the space *T* is to the circle *ABCD*.

But as the space *T* is to the circle *ABCD* so is the circle *EFGH* to some space, which must be less than the circle

ABCD, because, by hypothesis, the space *T* is greater than the circle *EFGH*. [V. 14.



Therefore the square on *FH* is to the square on *BD* as the circle *EFGH* is to some space less than the circle *ABCD*;

which has been shewn to be impossible.

Therefore the square on *BD* is not to the square on *FH* as the circle *ABCD* is to any space greater than the circle *EFGH*.

And it has been shewn that the square on *BD* is not to the square on *FH* as the circle *ABCD* is to any space less than the circle *EFGH*.

Therefore the square on *BD* is to the square on *FH* as the circle *ABCD* is to the circle *EFGH*.

Wherefore, *circles* &c. Q.E.D.