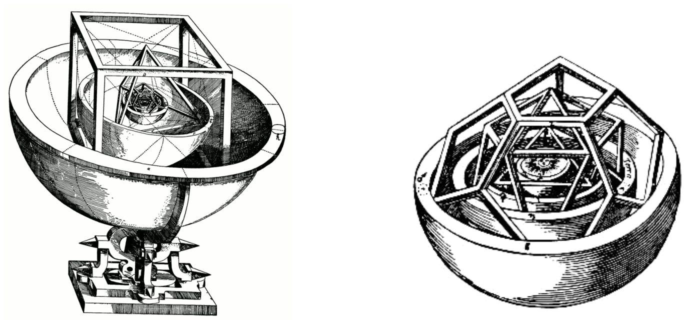
Polyhedra appear in the artifacts and artwork of many cultures.

Stone carvings from neolithic Scotland, circa 2000 BCE.



Johannes Kepler's cosmological diagrams showing nested regular polyhedra, 1596.

A *regular polygon* is one in which all sides have the same length and all internal angles are equal. A *regular polyhedron* is one in which every face is a copy of the same regular polygon, and the same number of faces meet at every vertex. Every edge should have two faces meeting at the same angle, and every vertex should have the same solid angle. In this sequence of problems, we'll *use* $V - E + F = 2$ to classify the regular polyhedra.

Suppose the faces are m-regular, and that n faces meet at each vertex.

(A1) What are (m, n) for a cube? Tetrahedron? Octahedron?

(A2) Prove that there exists a regular m-gon for every $m \geq 3$.

(A3) Explain why there is no regular polyhedron of type $(8, n)$ for any n or $(m, 8)$ for any m with a geometric argument.

(B1) Show that $V = \frac{mF}{n}$ and $E = \frac{mF}{2}$.

(B2) Using this and $V - E + F = 2$, show that $\frac{4n}{2m+2n-mn}$ is a natural number. (That is, show that it's a positive integer.)

(B3) Using the last part, show that $2m + 2n - mn > 0$. Finally, put together all the information that you have to prove that the pair (m, n) must come from this list:

$$\{(3,3),\ (3,4),\ (3,5),\ (4,3),\ (5,3)\}$$

(Hint: once you've established that $m, n \geq 3$, you can get a contradiction from $m \geq 6$.)