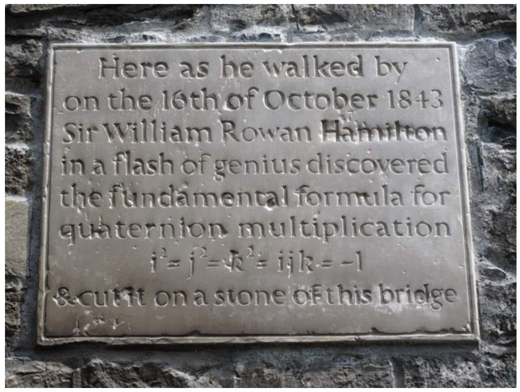

(A1) If $q = a + bi + cj + dk$ is a quaternion, let $\bar{q} = a - bi - cj - dk$ denote its conjugate. If the *norm* of a quaternion is defined to be $|q| = \sqrt{a^2 + b^2 + c^2 + d^2}$, check that $q \cdot \bar{q} = |q|^2$. Using this or by any other method, find inverses of $3i$ and of $3i + 2j$. (In other words, what do you have to multiply those numbers by to get 1? Put your answer in $a + bi + cj + dk$ form.)

(A2) Recall that a set is called *countable* (or countably infinite) if there is a bijection (a one-to-one correspondence) with that set and the natural numbers $\mathbb{N} = \{1, 2, 3, ...\}$. Prove that the integers $\mathbb{Z}$ are a countable set, by giving a bijection $f : \mathbb{N} \to \mathbb{Z}$. For your map f, what are $f(1)$, $f(5)$, and $f(1002)$? What value m satisfies $f(m) = 1002$?

(A3) Let's define some new words: *homodescriptual* and *heterodescriptual.* A word is called homodescriptual if it describes itself, like "word," or "ostentatious," and it's called heterodescriptual if it doesn't. First list several examples of homodescriptual words and examples of heterodescriptual words (be creative!). Then find a Russell-type paradox using these concepts, and explain the connection.

(B1) The "Liar's Paradox" is the famous sentence *This sentence is false.* If true, it's false. If false, it's true. That seems bad: it's a sentence that has no consistent truth value.

Let's consider a different kind of self-referential sentence.

If this sentence is true, then I owe \$50 to Moon Duchin.

If the sentence is true, what does that tell you about our financial relationship? Then doesn't that make the sentence true? Discuss.

Note: this is NOT the same as a liar's paradox, which can't be true and it can't be false. This is also NOT the same as just saying that the sentence if true must be true. (ANY sentence has that boring property.) This is a sentence that (a) causes you to owe me money, and actually (b) *proves itself to be true.* Try to understand why.

(P.S. I take cash or check.)

(B2) In class, we discussed algebraic structures with axiomatic definitions, such as groups, rings, and fields. The integers $\mathbb{Z}$ are an example of a ring. In any ring R, let's define an *ideal* to be any subset $I \subseteq R$ such that:

- $a + b \in I$ for all $a, b \in I$; and
- $ma \in I$ for all $m \in R$ and $a \in I$.

That is, it's closed under addition and under ring-multiplication. In $\mathbb{Z}$, let's write (n) or $n\mathbb{Z}$ for the ideal containing all multiples of n. For instance, (2) represents the even numbers.

We say that one ideal is I_1 is "smaller" than another ideal I_2 if it's contained properly inside it: $I_1 \subsetneq I_2$.

(a) Suppose an ideal I contains the numbers 6 and 8. Using the axioms, explain why it must also contain the number 4. What is the smallest ideal in $\mathbb{Z}$ containing 6 and 8?

(b) What is the smallest ideal in $\mathbb{Z}$ containing 2 and 3?

(c) What are all ideals in $\mathbb{Z}$? (Classify/list them.)

(d) Extra credit: figure out how multiplication of ideals should work, and show that $\sqrt{(4)} = (2)$.

(B3) Let's say that shape A and shape B are *equidecomposable* if you can cut A into finitely many pieces, rearrange them with rigid motions, and reassemble them to form B. So the Banach-Tarski paradox says that a solid ball in space is equidecomposable with two copies of itself.

An even stronger condition is *scissors congruence*—say two shapes are scissors congruent if they are equidecomposable via pieces with straight cuts. Here's an example:

Using the idea from the picture below, prove that every rectangle is scissors congruent to a square.