

Let's use the following (made-up) terminology from class about continued fraction approximations. Consider a rational approximation  $p/q$  to an irrational number  $\alpha$ , and let the error be  $|\alpha - p/q|$ . Let's say the approximation

- *doesn't suck* if the error is less than  $1/q$ ;
- it's *pretty efficient* if less than  $1/10q$ ;
- it's *super efficient* if less than  $1/q^2$ ;
- it's *double super efficient* if less than  $1/2q^2$ ;
- it's *crazy efficient* if less than  $1/100q^2$ ; and
- *insanely efficient* if less than  $1/q^3$ .

Passing these tests is relatively easy when  $q$  is small, so we'll reserve this terminology for approximations with  $q \geq 10$ .

- (A1) Using a calculator, show that the first few steps of the anthypharesis pattern for the ratio  $\sqrt{2} : 1$  match with  $[1; 2, 2, 2, \dots]$ . Now figure out two whole numbers  $a : b$  such that their anthypharesis pattern would be *exactly*  $[1; 2, 2, 2]$ . What is the ratio  $a/b$ ?

The Babylonian tablet YBC 7289 gave the approximation  $\sqrt{2} \approx 1 : 24, 51, 10$ .

Each of those two fractions is a rational approximation to  $\sqrt{2}$ . Which is a closer approximation (meaning smaller error)? Which is more efficient? Explain some conditions under which you might prefer a closer approximation, and other conditions under which a more efficient approximation is preferable.

- (A2) The **Efficiency Theorem** for continued fractions states that if  $\alpha$  is irrational and its continued fraction expansion is  $[a_0; a_1, a_2, \dots]$ , then its  $n$ th convergent  $\frac{p_n}{q_n} = [a_0; a_1, a_2, \dots, a_n]$  satisfies

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{a_{n+1} \cdot q_n^2}.$$

With this, explain why all continued fraction convergents are super-efficient approximations. Recall that

$$\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, \dots] \quad \text{and} \quad e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \dots].$$

Show that  $p_3/q_3$  is a crazy-efficient approximation to  $\pi$ . Find a  $k$  so that  $p_k/q_k$  is a crazy-efficient approximation to  $e$ .

- (A3) In our discussion of calendar-making, the measured length of a year was stated to be  $365 \frac{10463}{43200}$  days. Suppose instead that it had been  $365 \frac{23}{66}$  days. Explain why the simplest leap year system obtained by continued fractions would have every other year be a leap year, and the next more accurate one would have one leap year out of every three. The next one would call for how many leap years out of every twenty?

- (B1) Suppose a real number  $\alpha$  has a completely periodic continued fraction expansion  $[a; a, a, a, \dots]$ . Turn this into a quadratic equation in  $a$  and  $\alpha$  and use this to generate a formula for the values of  $\alpha$ . Using this, compute the values of  $[2; 2, 2, 2, 2, \dots]$  and  $[3; 3, 3, 3, 3, \dots]$ . What about  $\beta = [1; 2, 3, 4, 3, 4, 3, 4, 3, 4, \dots]$ ?
- (B2) Those solutions depend on the fact that the infinite continued fraction expressions are converging to a real number at all—that’s what makes it valid to do the arithmetic/algebra you needed in the last problem. Prove that the  $p_n/q_n$  tend to a limit for  $[a; a, a, a, a, \dots]$  by showing that  $q_n \rightarrow \infty$ , then applying the Efficiency Theorem.  
 Bonus: prove that  $q_n \rightarrow \infty$  for every infinite-depth continued fraction.
- (B3) Find a solution to the following Sun Zi-style problem:  
*If we count by fours, there is a remainder 1.*  
*If we count by fives, there is a remainder 3.*  
*If we count by nines, there is a remainder 6.*  
 You must explain your method; it is not enough to have a numerical solution.
- (B4) Give an example of a problem in this style, where the  $p_i$  are distinct but NOT relatively prime, which has no solution. This shows that the “relatively prime” hypothesis in the theorem is necessary.