(A1) Medieval Indian mathematicians knew that

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots,$$

and used $\arctan 1 = \pi/4$ to get a pretty formula for π. Euler used the arctan series to get a faster-converging expression via

$$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3},$$

obtained by using the trig identity $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$. First, explain why Euler's converges faster.

Later, Machin derived the improved formula $\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}$. We'll derive a related formula by solving for α and β satisfying $\tan(2\alpha+\beta) = 1$.

Let $a = \tan\alpha$ and $b = \tan\beta$. Assuming $\tan(2\alpha+\beta) = 1$, show that $2a + b - a^2b = 1 - a^2 - 2ab$. Solve for a simple a and b and use this to derive a formula for $\pi/4$. Check your formula with a calculator!

(A2) We will derive Viète's formula $2/\pi = (\cos\frac{\pi}{4})(\cos\frac{\pi}{8})(\cos\frac{\pi}{16})\ldots$

Consider the sequence of polygons given by inscribing a square, octagon, 16-gon, and so on into a circle of radius 1.

(a) Show that $2/\pi$ is equal to the ratio of the area of the square to the area of the circle.

(b) Suppose $R(n)$ is the area of the 2^n-gon divided by the area of the 2^{n+1}-gon. Explain why the formula $2/\pi = R(2)\cdot R(3)\cdot R(4)\cdots = \prod_{n=2}^{\infty} R(n)$ is reasonable.

(c) Show that $R(2) = \cos\frac{\pi}{4}$.

(d) Extra Credit: Compute $R(n)$ and so derive Viète's formula.

(A3) Here's a gem of Ramanujan's.

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!(1103+26390k)}{(k!)^4\cdot 396^{4k}}.$$

Remember we already saw an infinite sum formula for π via $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ but it was super-useless because it converged to the truth so extremely slowly.

On the contrary, Ramanujan's formula is amazingly fast-converging and to this day gives one of the fastest known algorithms for computing π.

Find the first and second partial sum in the formula and use a calculator to see what approximations of π they give you. How does the error compare to the continued fraction approximation 355/113? Explain why the continued fraction approximations are good for some purposes and the Ramanujan formula is good for others.

The *partition function* $p(n)$ is defined as the number of ways to express n as an unordered sum of natural numbers, so for instance $3 = 2+1 = 1+1+1$ gives us $p(3) = 3$. We use the convention that $p(0) = 1$ (zero is like the "empty sum" and that's the only way to write it) and $p(n) = 0$ when n is negative.

Recall the formulas of Ramanujan giving congruences for partitions:

$$p(5m+4) \equiv 0 \pmod{5}, \qquad p(7m+5) \equiv 0 \pmod{7}, \qquad p(11m+6) \equiv 0 \pmod{11}.$$

You can find a table of the first $10,000$ values of $p(n)$ online at the URL

http://oeis.org/A000041/b000041.txt

(B1) Use the methods of the Chinese Remainder Theorem to find a number N that satisfies all three congruence relations in Ramanujan's formulas. (It should have remainder 4 when divided by 5, remainder 5 when divided by 7, and remainder 6 when divided by 11.) Now use the website to find $p(N)$ and check that Ramanujan's formulas are correct in this case. (In your write-up, explain how you checked.)

(B2) Before Ramanujan came along, Euler worked on $p(n)$. He found some remarkable formulas, such as this recursive one:

$$p(n) = \sum_{k=1}^{n} (-1)^{k+1} \left[p\big(n - \frac{1}{2}k(3k-1)\big) + p\big(n - \frac{1}{2}k(3k+1)\big) \right].$$

(a) Make a table showing $\frac{1}{2}k(3k-1)$, and $\frac{1}{2}k(3k+1)$ for $k = 1, \ldots, 8$. Now write out the expressions that Euler presents for $p(n)$ for $n = 4, 5, 6, 7, 8$, and 20. Check against the table that Euler's formula is correct for these values.

(b) Roughly how many nonzero terms are in Euler's expression for $p(n)$ when n is large?

(B3) Another formula of Euler's was what is called a *generating function* for $p(n)$: it is a power series whose **coefficients** are the values we're looking for. Here's how it works. First he defined a function

$$f(x) = \prod_{m=1}^{\infty} (1 - x^m).$$

Then the generating function for the partitions is given by its reciprocal: $F(x) = 1/f(x)$.

(a) If you expanded the product expression for $f(x)$, you would get

$$f(x) = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + \cdots$$

Verify that this is correct up to x^7, showing all work.

(b) Now set $F(x) = 1/f(x)$. To derive its expansion, write $F(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$. Since $F(x) \cdot f(x) = 1$, or in other words

$$(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) \cdot (1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + \cdots) = 1,$$

we can solve for the coefficients a_n by considering the coefficient of x^n on the left and the coefficient of x^n on the right. For instance, the constant term on the left is a_0, and on the right it is 1, so $a_0 = 1$. Find the values of a_1 through a_6. How do the a_n compare to the values $p(n)$?

(c) Extra Credit: explain *why* $1/f(x)$ gives coefficients which are the partition numbers.