(A1) We have recently discussed Viète's formula $2/\pi = (\cos\frac{\pi}{4})(\cos\frac{\pi}{8})(\cos\frac{\pi}{16})\ldots$

First show that $\cos\frac{\pi}{4}$ is a constructible number (that is, given a unit length, this length can be constructed with straightedge and compass).

Then, using the double-angle formula $\cos(2\theta) = 2\cos^2\theta - 1$, explain how to use that length to construct $\cos\frac{\pi}{8}$.

Repeating this logic, conclude that π can be written as an infinite product of constructible numbers.

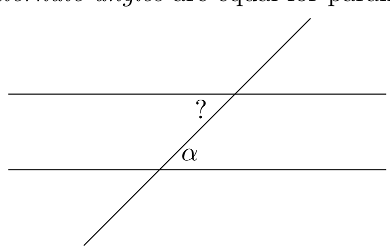
(A2) Give a Euclidean construction to trisect a line segment. Next, prove that trisecting the line at the base of an isosceles triangle does not always trisect the angle opposite that line. Does it ever?

In your solution, include a figure of a case in which it is visually clear that the line trisection does not produce an angle trisection.

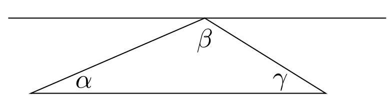
(A3) Euclid's original version of the parallel postulate says that if there are two lines L and L' and they are both crossed by a third line M, then if α and θ are the angles made by L, M and L', M on the same side of M, their sum predicts the location of the intersection: $\alpha + \theta < \pi \implies L, L'$ intersect on the same side of M that the angles appear on; $\alpha + \theta > \pi$ implies an intersection on the other side; and $\alpha + \theta = \pi$ is the case of no intersection (L, L' parallel).

Playfair's version says that for any line L and a point P not on L, there exists exactly one line through P parallel to L.

(a) Draw pictures for the three cases in Euclid's version of the parallel postulate. Using this postulate, prove that *alternate angles* are equal for parallel lines.



(b) With the help of the figure shown below, use Playfair's version to prove that the sum of angles in any triangle equals π. Make sure you explain how Playfair's postulate is used!



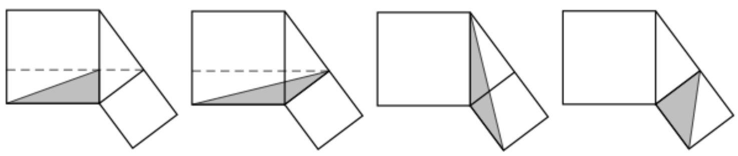
(A4) Use the fact that equiangular triangles (otherwise known as *similar triangles*) have proportional sides to prove *without the Pythagorean theorem* that the diagonal of a unit square has length $\sqrt{2}$.

(B1) This problem concerns the modern concept of cosine (which was not present in Greek mathematics).

Prove that cosines and arccosines are constructible. That is: given two lines that intersect non-perpendicularly, suppose the smaller angle that they make is called θ. Construct a line segment of length $\cos\theta$. In the other direction, given a line segment of length $\ell < 1$, construct the angle $\arccos \ell$.

(B2) One of the famous Greek impossibility problems is that of *squaring the circle*. Prove that you can square the rectangle! We want a geometric proof using no algebra. You may use the fact (which Euclid derives in Prop 4 of Book I) called SAS: if two triangles have a pair of matching sidelengths and a matching angle between those sides, then they are congruent. You may also use the fact (derived in class) that the area of a triangle is one half of base times height.

First explain why all of the shaded triangles in the diagram below have the same area.



Using this, explain how to construct a square of the same area as an arbitrary given rectangle.

(B3) Prove the Pythagorean theorem using the previous construction.
(Suggestion: consider the right triangle pictured in the diagram above; name its short sides a and b and suppose its hypotenuse c is divided into lengths s and t by the dotted line.)

(B4) Get as far as you can in Euclidea (https://www.euclidea.xyz/). For the last one you successfully complete, write it up and hand it in as Problem B4.