numbers developed to the point where the same digit 1 represented 60 as well. We do not know why the Babylonians decided to have one large unit represent 60 small units and then adapt this method for their numeration system. One conjecture is that 60 is evenly divisible by many small integers. Therefore, fractional values of the "large" unit could easily be expressed as integral values of the "small" unit. The Babylonian base 60 place-value system is still in use in our units for angle and time measurement, units preserved over the centuries in astronomical contexts and today an irreplaceable part of world culture.

There is no record of the written number system of ancient India, but there is literary evidence that numerical symbols did exist. It is only from about the third century B.C.E. that examples of written numbers are available. Originally, the system was mixed. There was a ciphered system similar to the hieratic with separate symbols for the numbers 1 through 9 and 10 through 90. For larger numbers, the system was a multiplicative one similar to the Chinese. For example, the symbol for 200 was a combination of the symbol for 2 and that for 100, and the symbol for 70,000 combined the symbols for 70 and 1000. As will be discussed in Chapter 6, it was in or near India that the modern base 10 place-value system developed, but not until about the seventh century C.E.

1.3 ARITHMETIC COMPUTATIONS

Once their system of writing numbers came into existence, all of the civilizations under discussion devised rules for the basic arithmetic operations—addition, subtraction, multiplication, and division—and as a consequence of the last operation, rules for writing and operating with fractions. These rules may be considered as some of the earliest algorithms.

An **algorithm** is an ordered list of instructions designed to produce an answer to a given type of problem. Ancient peoples produced algorithms of all sorts to handle many different problems. In fact, ancient mathematics can be characterized as algorithmic in nature, as opposed to the Greek mathematics, which emphasized theory. In most of the available documents of ancient mathematics, the author describes a problem to be solved and then proceeds to use an algorithm, either explicit or implicit, to obtain the solution. There is little concern in the documents as to how the algorithm was discovered, why it works, or what its limitations are. Instead, we simply are shown many examples of the use of the algorithm, often in increasingly complex situations. Nevertheless, in our discussion of these algorithms, we will describe the possible origins and justifications of each one and will present the possible answers that the Babylonian, Chinese, or Egyptian scribes gave to their students who asked the eternal question "why?"

In the Egyptian hieroglyphic grouping system, addition is simple enough: Combine the units, then the tens, then the hundreds, and so on. Whenever a group of ten of one type of symbol appears, replace it by one of the next. Hence, to add 783 and 275, put 𓏤𓏤𓏤𓎆𓎆𓎆𓎆𓎆𓎆𓎆𓎆𓍢𓍢𓍢𓍢𓍢𓍢𓍢 and 𓏤𓏤𓏤𓏤𓏤𓎆𓎆𓎆𓎆𓎆𓎆𓎆𓍢𓍢 together to get 𓏤𓏤𓏤𓏤𓏤𓏤𓏤𓏤𓎆𓎆𓎆𓎆𓎆𓎆𓎆𓎆𓎆𓎆𓎆𓎆𓎆𓎆𓎆𓍢𓍢𓍢𓍢𓍢𓍢𓍢𓍢𓍢. Since there are fifteen 𓎆s, replace ten of them by one 𓍢. This then gives ten of the latter. Replace these by one 𓆼. The final answer is 𓏤𓏤𓏤𓏤𓏤𓏤𓏤𓏤𓎆𓎆𓎆𓎆𓎆𓆼, or 1058. Subtraction is done similarly. In this case, of course, whenever "borrowing" is needed, one of the symbols would be converted to ten of the next lower symbol.

Such a simple algorithm for addition and subtraction is not possible in the hieratic system. For these operations, the mathematical papyri do not provide much evidence; the answers to addition and subtraction problems are merely written down. Most probably, the scribes had addition tables. At some point these would have existed in written form, but a competent scribe would, of course, have memorized them. The scribes presumably used the addition tables in reverse for subtraction problems.

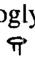
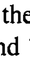
The Egyptian algorithm for multiplication was based on a continual doubling process. To multiply two numbers a and b, the scribe would first write down the pair 1, b. He would then double each number in the pair repeatedly, until the next doubling would cause the first element of the pair to exceed a. Then, having determined the powers of 2 that add to a, the scribe would add the corresponding multiples of b to get the answer. For example, to multiply 12 by 13 the scribe would set down the following lines:

1	12
2	24
4	48
8	96

At this point, he would notice that the next doubling would produce 16 in the first column, which is larger than 13. He would then check off those multipliers that added to 13, namely 1, 4, and 8, and add the corresponding numbers in the other column. The result would be written as: Totals 13 156.

As before, there is no record of how the scribe did the doubling. The answers are simply written down. Perhaps the scribe had memorized an extensive 2 times table. In fact, there is some evidence that doubling was a standard method of computation in areas of Africa to the south of Egypt, so it is likely that the Egyptian scribes learned from their southern colleagues.[6] In addition, the scribes were somehow aware that every positive integer could be uniquely expressed as the sum of powers of 2. That fact provides the justification for the procedure. How was it discovered? Our best guess is that it was discovered by experimentation and then passed down as tradition.

Because division is the inverse of multiplication, a problem such as $156 \div 12$ would be stated as "multiply 12 so as to get 156." The scribe would then write down the same lines listed before. This time, however, he would check off the lines having the numbers in the right-hand column that sum to 156; in this case, 12, 48, and 96. Then the sum of the corresponding numbers on the left, namely 1, 4, and 8, would give the answer 13. Of course, division does not always "come out even." When it did not, the Egyptians used fractions.

The kind of fractions that the Egyptians used were unit fractions, or "parts" (fractions with numerator 1), with the single exception of 2/3, perhaps because these fractions are the most "natural." The fraction $1/n$ (the nth part) is represented in hieroglyphics by the symbol for the integer n with the symbol  above. In the hieratic a dot is used instead. Thus 1/7 is denoted in hieroglyphics by and in the hieratic by . The single exception, 2/3, had a special symbol:  in hieroglyphics and  in hieratic. (The former symbol is indicative of the reciprocal of 1 1/2.) In the remainder of this text, however, the notation $\overline{n}$ will be used to represent $1/n$ and $\overline{\overline{3}}$ to represent 2/3.

Because fractions show up as the result of divisions that do not come out evenly, we need to be able to deal with fractions other than unit fractions. It was in this connection that the most intricate of the Egyptian mathematical techniques developed, the representation of any fraction in terms of unit fractions. The Egyptians did not put the question this way, however. Where we would use a nonunit fraction, they would simply write a sum of unit fractions. For example, problem 3 of the *Rhind Mathematical Papyrus* asks how to divide 6 loaves among 10 men. The answer is given that each man gets $\overline{2}\ \overline{10}$ loaves (that is, $1/2 + 1/10$). The scribe checks this by multiplying this value by 10. We may regard the scribe's answer as more cumbersome than our answer of $3/5$, but in some sense the actual division is easier to accomplish this way. If we divide five of the loaves in half and the sixth one in tenths, and then give each man one-half plus one-tenth, it is then clear to all concerned that everyone has the same portion of bread. Cumbersome or not, this Egyptian unit-fraction method was used throughout the Mediterranean basin for over 2000 years.

In multiplying whole numbers, the important step is the doubling step. Likewise in multiplying fractions, the scribe had to be able to express the double of any unit fraction. For example, in the preceding problem, the check of the solution is written as follows:

1	$\overline{2}\ \overline{10}$
'2	$1\ \overline{5}$
4	$2\ \overline{3}\ \overline{15}$
'8	$4\ \overline{\overline{3}}\ \overline{10}\ \overline{30}$
10	6

How are these doubles formed? To double $\overline{2}\ \overline{10}$ is easy: Because each denominator is even, it is merely halved. In the next line, however, $\overline{5}$ must be doubled. To perform calculations like this, the scribe had to use a table to get the answer $\overline{3}\ \overline{15}$ (that is, $2 \cdot 1/5 = 1/3 + 1/15$). In fact, the first section of the *Rhind Papyrus* is a table of the division of 2 by every odd integer from 3 to 101 (Fig. 1.7), and the Egyptian scribes realized that the result of multiplying $\overline{n}$ by 2 is the same as that of dividing 2 by n. Although it is not known how the division table was constructed, several scholarly accounts present hypotheses for the scribes' methods.[7] In any case, the solution of problem 3 depends on using that table twice, first as already indicated and second, in the next step, where the double of $\overline{15}$ is given as $\overline{10}\ \overline{30}$ (or $2 \cdot 1/15 = 1/10 + 1/30$). The final step in this problem involves the addition of $1\ \overline{5}$ to $4\ \overline{\overline{3}}\ \overline{10}\ \overline{30}$, and here the scribe just gave the answer. Again, the conjecture is that for such addition problems an extensive table existed. The *Egyptian Mathematical Leather Roll*, which dates from about 1600 B.C.E., contains a short version of such an addition table. There are also extant several other tables for dealing with unit fractions and a multiplication table for the special fraction $2/3$. It thus appears that the arithmetic algorithms used by the Egyptian scribes involved extensive knowledge of basic tables for addition, subtraction, and doubling and then a definite procedure for reducing multiplication and division problems into steps, each of which could be performed using the tables.

Often in dealing with division, the scribes replaced the doubling procedure by halving. For example, in calculating $2 \div 7$, the first steps are:

1	7
$\overline{2}$	$3\ \overline{2}$
$\overline{4}$	$1\ \overline{2}\ \overline{4}$

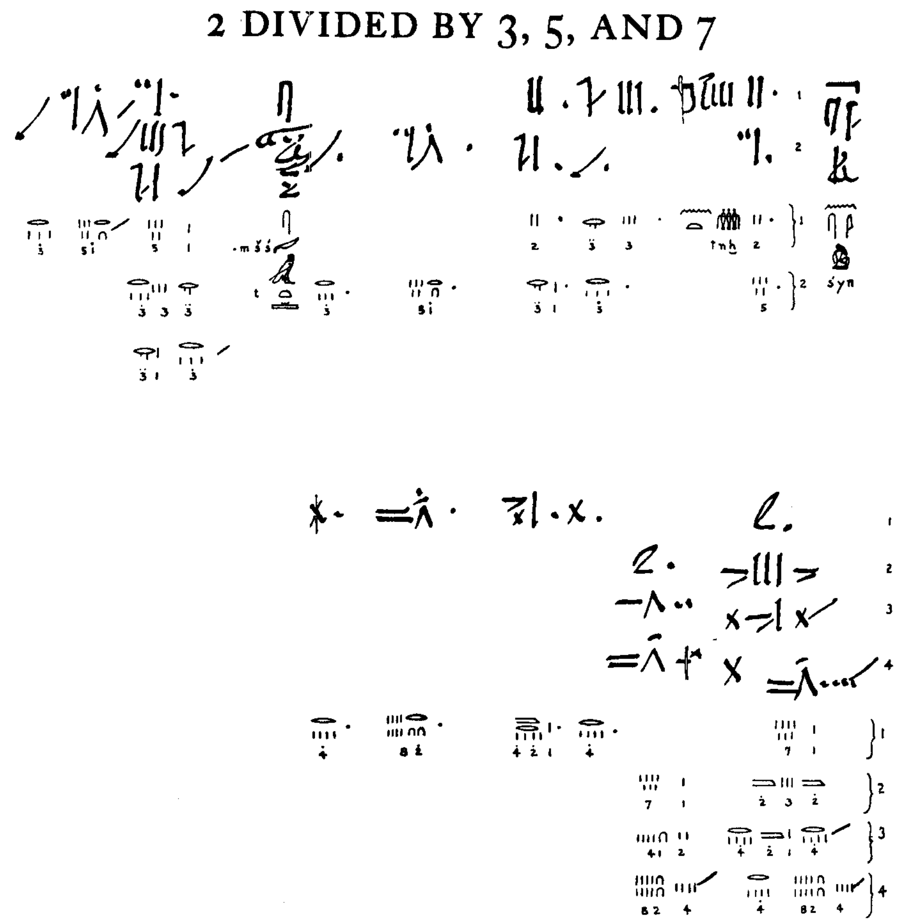


FIGURE 1.7

Transcription and hieroglyphic translation of $2 \div 3$, $2 \div 5$, and $2 \div 7$ from the *Rhind Mathematical Papyrus*. (*Source*: *The Rhind Mathematical Papyrus*, N.C.T.M.)

To get 2 as a total in the right-hand column requires the addition of $\overline{4}$ to $1\ \overline{2}\ \overline{4}$ in the third line. Thus the scribe needed to determine by what 7 should be multiplied to get $\overline{4}$. To do this, he inverted the known result that $4 \times 7 = 28$ to get $1/28$ of 7 is $1/4$. Then he added the line $\overline{28}\quad \overline{4}$ to his calculation and added the last two lines together to get Total: $\overline{4}\ \overline{28}\quad 2$, or $2 \div 7 = \overline{4}\ \overline{28}$.

Problem 21 of the *Rhind Papyrus* presents a different type of calculation: Complete $\overline{\overline{3}}\ \overline{15}$ to 1. In other words, we need to determine what must be added to $2/3 + 1/15$ to get 1. The scribe notes that $2/3$ of 15 is 10 and $1/15$ of 15 is 1, for a total of 11. Thus he needs to "multiply 15 to get 4." The steps are set down as follows:

1	15
$\overline{10}$	$1\ \overline{2}$
$\overline{5}$	$3'$
$\overline{15}$	$1'$
$\overline{5}\ \overline{15}$	4

Here the scribe doubles from the second line to the third, but, since he realized that 1 is $1/3$ of 3, he took thirds to get from the third line to the fourth. The answer to the original problem is then $\overline{5}$ $\overline{15}$.

As an example of other modifications that the scribes made to their basic procedure, consider problem 69 of the *Rhind Papyrus*, which includes the division of 80 by 3 $\overline{2}$ and its subsequent check:

1	3 $\overline{2}$	\1	22 $\overline{\overline{3}}$ $\overline{7}$ $\overline{21}$
10	35	\2	45 $\overline{3}$ $\overline{4}$ $\overline{14}$ $\overline{28}$ $\overline{42}$
\20	70	\\$\overline{2}$	11 $\overline{3}$ $\overline{14}$ $\overline{42}$
\2	7	3 $\overline{2}$	80
\\$\overline{\overline{3}}$	2 $\overline{3}$		
\\$\overline{21}$	$\overline{6}$		
\\$\overline{7}$	$\overline{2}$		
22 $\overline{\overline{3}}$ $\overline{7}$ $\overline{21}$	80		

In the second line the scribe has taken advantage of the decimal nature of his notation to get the product of 3 $\overline{2}$ by 10. In the fifth line he has used the $2/3$ multiplication table mentioned earlier. The scribe then realized that since the numbers in the second column of the third through the fifth lines added to 79 $\overline{3}$, he needed to add $\overline{2}$ and $\overline{6}$ in that column to get 80. Thus, because $6 \times 3\ \overline{2} = 21$ and $2 \times 3\ \overline{2} = 7$, it follows that $\overline{21} \times 3\ \overline{2} = \overline{6}$ and $\overline{7} \times 3\ \overline{2} = \overline{2}$, as indicated in the sixth and seventh lines. The check shows several uses of the table of division by 2 as well as great facility in addition.

That the Babylonians used tables in the process of performing arithmetic computations is proved by extensive direct evidence. Many of the preserved tablets are in fact multiplication tables. No addition tables have ever been found, however. Because over 200 Babylonian table texts have been analyzed, we may assume that they did not exist and that the scribes knew their addition procedures by heart and simply wrote down the answers when needed. On the other hand, there do exist many examples of "scratch tablets" on which a scribe has performed various calculations in the process of solving a problem. In any case, since the Babylonian number system was a place-value system, the actual algorithms for addition and subtraction, including carrying and borrowing, may well have been similar to modern ones. For example, to add 23,37 (= 1417) to 41,32 (= 2492), one first adds 37 and 32 to get 1,09 (= 69). One writes down 09 and carries 1 to the next column. Then $23 + 41 + 1 = 1{,}05$ (= 65), and the final result is 1,05,09 (= 3909).

Because the place-value system was based on 60, the multiplication tables were extensive. Any given one listed the multiples of a particular number, say 9, from 1×9 to 20×9 and then gave 30×9, 40×9, and 50×9 (Fig. 1.8). To obtain the product 34×9, the scribe simply added the two results $30 \times 9 = 4{,}30$ (= 270) and $4 \times 9 = 36$ to get 5,06 (= 306). For multiplication of two- or three-digit sexagesimal numbers, several such tables were needed. The exact algorithm the Babylonians used for such multiplications—where the partial products are written and how the final result is obtained—is not known, but it may well have been similar to our own.

One might think that for a complete system of tables, the Babylonians would have one for each integer from 2 to 59. Such was not the case, however. In fact, although there are no tables for 11, 13, or 17, for example, there are tables for 1,15, 3,45, and 44,26,40.

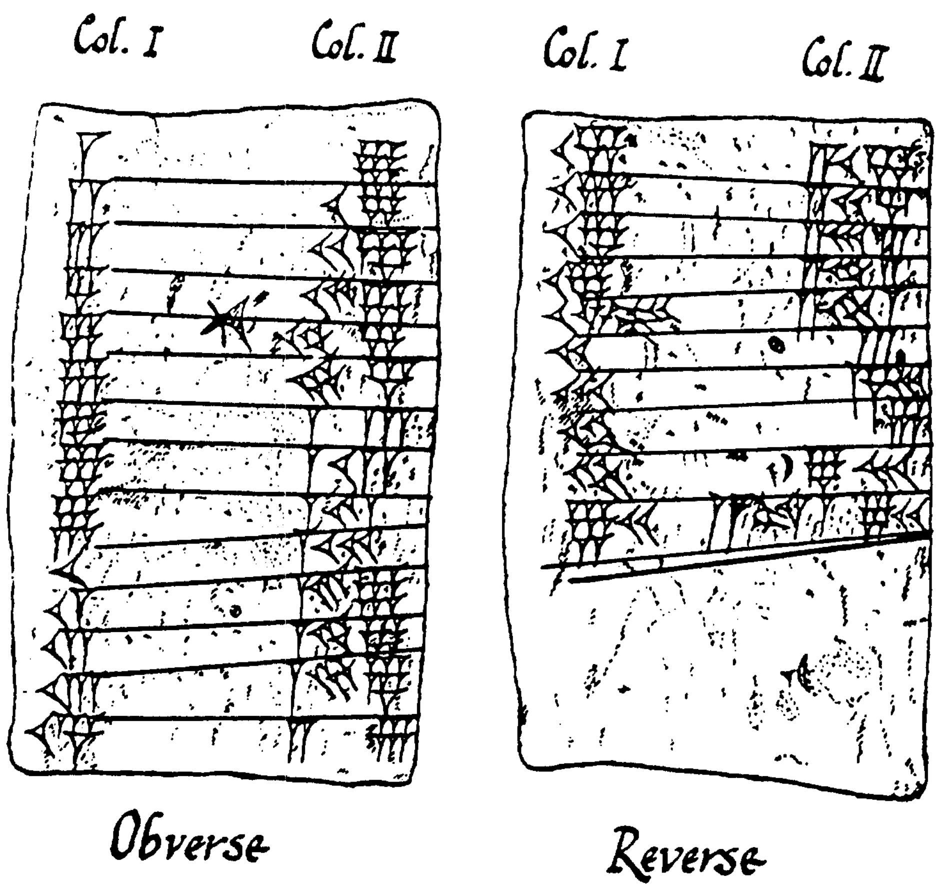


FIGURE 1.8
A Babylonian multiplication table for 9 (Department of Archaeology, University of Pennsylvania).

Although we do not know precisely why the Babylonians made these choices, we do know that, with the single exception of 7, all the multiplication tables found so far are for **regular** sexagesimal numbers—that is, numbers whose reciprocal is a terminating sexagesimal fraction. The Babylonians treated all fractions as sexagesimal fractions, analogous to our use of decimal fractions. Namely, the first place after the "sexagesimal point," denoted by ";", represents 60ths, the next place 3600ths, and so on. Thus, the reciprocal of 48 is the sexagesimal fraction 0;1,15, which represents $1/60 + 15/60^2$, while the reciprocal of 1,21 (= 81) is 0;0,44,26,40, or $44/60^2 + 26/60^3 + 40/60^4$. Because the Babylonians did not indicate an initial 0 or the sexagesimal point, this last number would just be written as 44,26,40. As noted, there exist multiplication tables for this regular number. Such tables provide no indication of the absolute size of the number, nor is one necessary. When the Babylonians used the table they, of course, realized that the eventual placement of the sexagesimal point depended on the absolute size of the numbers involved, so the placement was finally determined by context.

Besides multiplication tables, the Babylonians also used extensive tables of reciprocals; part of one is reproduced here. A **table of reciprocals** is a list of pairs of numbers whose product is 1 (where the 1 can represent any power of 60). Like the multiplication tables, these tables only contained regular sexagesimal numbers.

2	30	16	3,45	48	1,15
3	20	25	2,24	1,04	56,15
10	6	40	1,30	1,21	44,26,40

The reciprocal tables were used in conjunction with the multiplication tables to do division. Thus the multiplication table for 1,30 (= 90) served not only to give multiples of that number, but also, since 40 is the reciprocal of 1,30, to do divisions by 40. In other words, the Babylonians considered the problem $50 \div 40$ to be equivalent to $50 \times 1/40$, or, in sexagesimal notation, to $50 \times 0;1,30$. The multiplication table for 1,30, part of which appears here, then gives 1,15 (or 1,15,00) as the product. The appropriate placement of the sexagesimal point gives 1;15(= 1 1/4) as the correct answer to the division problem.

1	1,30	10	15	30	45
2	3	11	16,30	40	1
3	4,30	12	18	50	1,15

In ancient China, arithmetic calculations were made on the counting board. In general, whenever fractions were needed, they were expressed as common fractions. The Chinese actually used our modern rules of calculation with fractions, including our device of common denominators. There is some evidence, however, of the early use of decimal fractions simply as additional columns on the counting board, particularly in dealing with measures of length and weight. A fully developed decimal fraction system was not in place until much later.

1.4 LINEAR EQUATIONS

Most of the mathematical sources from ancient times are concerned with the solution of problems, to which various mathematical techniques are applied. Our study of these problems begins with several methods for solving what are today known as linear equations. Of course one must always remember that none of the ancient peoples had any of the symbolism for operations or unknowns that we use today. Nevertheless, the scribes were able to solve problems using purely verbal techniques.

The Egyptian papyri present several different procedures for dealing with linear equations. For example, the *Moscow Papyrus* uses the current technique to find the number such that if it is taken 1 1/2 times and then 4 is added, the sum is 10. In modern notation, the equation is simply $(1\ 1/2)x + 4 = 10$. The scribe proceeds the same way we would today: He first subtracts 4 from 10 to get 6, then multiplies 6 by 2/3 (the reciprocal of 1 1/2) to get 4 as the solution. Similarly, problem 31 of the *Rhind Papyrus* asks to find a quantity such that the sum of itself, its 2/3, its 1/2, and its 1/7 becomes 33—that is, to find x such that $x + (2/3)x + (1/2)x + (1/7)x = 33$. The problem is not conceptually difficult, but it is arithmetically challenging. It and the three following problems were probably put in to demonstrate methods of division, for the scribe solved the problem by dividing 33 by $1 + 2/3 + 1/2 + 1/7$. His answer—and this should be checked—is written as $14\ \overline{4}\ \overline{56}\ \overline{97}\ \overline{194}\ \overline{388}\ \overline{679}\ \overline{776}$ (or, in modern notation, 14 28/97). These two problems are

[redacted]