***51·511.** $\vdash . \breve{\iota}\text{‘}\iota\text{‘}x = x$ $\left[*51{\cdot}51 \dfrac{\iota\text{‘}x}{\alpha} . *20{\cdot}2\right]$

***51·52.** $\vdash : E ! \breve{\iota}\text{‘}\alpha . \equiv . \alpha = \iota\text{‘}\breve{\iota}\text{‘}\alpha$ $\left[*51{\cdot}51 \dfrac{\breve{\iota}\text{‘}\alpha}{x} . *14{\cdot}21{\cdot}18\right]$

***51·53.** $\vdash : E ! \breve{\iota}\text{‘}\alpha . \equiv . \breve{\iota}\text{‘}\alpha \in \alpha$ $[*51{\cdot}52{\cdot}16 . *14{\cdot}21{\cdot}18]$

***51·54.** $\vdash : E ! \breve{\iota}\text{‘}\alpha . \equiv . (\exists x) . \alpha = \iota\text{‘}x$ $[*51{\cdot}51 . *14{\cdot}204]$

***51·55.** $\vdash : E ! \breve{\iota}\text{‘}\alpha . \equiv . E ! (\imath x)(x \in \alpha)$

Dem.

$\vdash . *51{\cdot}54{\cdot}14 . \supset \vdash :. E ! \breve{\iota}\text{‘}\alpha . \equiv : (\exists x) : y \in \alpha . \equiv_y . y = x :$

$[*14{\cdot}11] \quad \equiv : E ! (\imath x)(x \in \alpha) :. \supset \vdash . \text{Prop}$

***51·56.** $\vdash : b = \breve{\iota}\text{‘}\hat{y}(\phi y) . \equiv . \hat{y}(\phi y) = \iota\text{‘}b . \equiv . b = (\imath x)(\phi x)$

Dem.

$\vdash . *51{\cdot}51 . \supset \vdash :. b = \breve{\iota}\text{‘}\hat{y}(\phi y) . \equiv : \hat{y}(\phi y) = \iota\text{‘}b :$ (1)

$[*20{\cdot}15 . *51{\cdot}11] \quad \equiv : \phi y . \equiv_y . y = b :$

$[*14{\cdot}202] \quad \equiv : b = (\imath x)(\phi x)$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***51·57.** $\vdash : E ! \breve{\iota}\text{‘}\hat{y}(\phi y) . \equiv . \breve{\iota}\text{‘}\hat{y}(\phi y) = (\imath x)(\phi x) . \equiv . E ! (\imath x)(\phi x)$

Dem.

$\vdash . *14{\cdot}204 . *51{\cdot}56 . \supset \vdash : E ! \breve{\iota}\text{‘}\hat{y}(\phi y) . \equiv . E ! (\imath x)(\phi x)$ (1)

$\vdash . *14{\cdot}205 . \supset \vdash : (\imath x)(\phi x) = \breve{\iota}\text{‘}\hat{y}(\phi y) . \equiv . (\exists b) . b = (\imath x)(\phi x) . b = \breve{\iota}\text{‘}\hat{y}(\phi y) .$

$[*51{\cdot}56 . *4{\cdot}71] \quad \equiv . (\exists b) . b = (\imath x)(\phi x) .$

$[*14{\cdot}204{\cdot}13] \quad \equiv . E ! (\imath x)(\phi x)$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***51·58.** $\vdash : E ! \breve{\iota}\text{‘}\alpha . \equiv . \breve{\iota}\text{‘}\alpha = (\imath x)(x \in \alpha)$ $[*51{\cdot}57 . *20{\cdot}3 . *14{\cdot}272]$

***51·59.** $\vdash : \psi\{\breve{\iota}\text{‘}\hat{z}(\phi z)\} . \equiv . \psi(\imath x)(\phi x)$ $[*51{\cdot}56 . *14{\cdot}205]$

*52. THE CARDINAL NUMBER 1

*Summary of *52.*

In this number, we introduce the cardinal number 1, defined as the class of all unit classes. The fact that 1 so defined is a cardinal number is not relevant at present, and cannot of course be proved until "cardinal number" has been defined. For the present, therefore, 1 is to be regarded simply as the class of all unit classes, unit classes being such classes as are of the form $\iota\text{‘}x$ for some x.

Like Λ and V, 1 is ambiguous as to type: it means "all unit classes of the type in question." The symbol "$1(\alpha)$," where α is a type, will mean "all unit classes whose sole members belong to the type α" (cf. *65). Thus *e.g.* "$\xi \in 1$ (Indiv)" will mean "ξ is a class consisting of one individual," if "Indiv" stands for the class of individuals.

The properties of 1 to be proved in the present number are what we may call *logical* as opposed to *arithmetical* properties, *i.e.* they are not concerned with the arithmetical operations (addition, etc.) which can be performed with 1, but with the relations of 1 to unit classes. The arithmetical properties of 1 will be considered later, in Part III.

The propositions of the present number which are most used are the following:

***52·16.** $\vdash :. \alpha \in 1 . \equiv : \exists ! \alpha : x, y \in \alpha . \supset_{x,y} . x = y$

I.e. α is a unit class if, and only if, it is not null, and all its members are identical.

***52·22.** $\vdash . \iota\text{‘}x \in 1$

***52·4.** $\vdash :. \alpha \in 1 \cup \iota\text{‘}\Lambda . \equiv : x, y \in \alpha . \supset_{x,y} . x = y$

We shall define 0 as $\iota\text{‘}\Lambda$. Thus the above proposition states that a class has one member or none when, and only when, all its members are identical.

***52·41.** $\vdash : \exists ! \alpha . \alpha \sim\in 1 . \equiv . (\exists x, y) . x, y \in \alpha . x \neq y$

This proposition is obtainable from *52·4 by transposition, *i.e.* by negating each side of the equivalence.

***52·46.** $\vdash :. \alpha, \beta \in 1 . \supset : \alpha \subset \beta . \equiv . \alpha = \beta . \equiv . \exists ! (\alpha \cap \beta)$

I.e. two unit classes are identical when, and only when, one is contained in the other, and when and only when they have a common part.

***52·01.** $1 = \hat{\alpha}\{(\exists x) . \alpha = \iota\text{‘}x\}$ Df

***52·1.** $\vdash : \alpha \in 1 . \equiv . (\exists x) . \alpha = \iota\text{‘}x$ $[*20{\cdot}3 . (*52{\cdot}01)]$

***52·11.** $\vdash :. \alpha \epsilon 1 . \equiv : (\exists x) : y \epsilon \alpha . \equiv_y . y = x$ [*52·1 . *51·14]

***52·12.** $\vdash : \hat{z}(\phi z) \epsilon 1 . \equiv . E!(\iota x)(\phi x)$

Dem.

$\vdash$. *52·11 . $\supset \vdash :. \hat{z}(\phi z) \epsilon 1 . \equiv : (\exists x) : y \epsilon \hat{z}(\phi z) . \equiv_y . y = x :$

[*20·3] $\equiv : (\exists x) : \phi y . \equiv_y . y = x :$

[*14·11] $\equiv : E!(\iota x)(\phi x) :. \supset \vdash . \text{Prop}$

***52·13.** $\vdash . 1 = D\text{‘}\iota$

Dem.

$\vdash$. *51·131 . $\supset \vdash : \alpha = \iota\text{‘}x . \equiv . \alpha \iota x :$

[*10·11·281] $\supset \vdash : (\exists x) . \alpha = \iota\text{‘}x . \equiv . (\exists x) . \alpha \iota x :$

[*52·1] $\supset \vdash : \alpha \epsilon 1 . \equiv . (\exists x) . \alpha \iota x$

[*33·13] $\equiv . \alpha \epsilon D\text{‘}\iota : \supset \vdash . \text{Prop}$

***52·14.** $\vdash . 1 = \iota\text{“}V$ [*52·13 . *37·28]

***52·15.** $\vdash : \alpha \epsilon 1 . \equiv . E ! \breve{\iota}\text{‘}\alpha$ [*51·54 . *52·1]

***52·16.** $\vdash :. \alpha \epsilon 1 . \equiv : \exists ! \alpha : x, y \epsilon \alpha . \supset_{x,y} . x = y$ [*52·15 . *51·55 . *14·203]

***52·17.** $\vdash : \alpha \epsilon 1 . \equiv . \breve{\iota}\text{‘}\alpha = (\iota x)(x \epsilon \alpha)$ [*51·58 . *52·15]

***52·171.** $\vdash : \alpha \epsilon 1 . \equiv . E ! (\iota x)(x \epsilon \alpha)$ [*51·55 . *52·15]

***52·172.** $\vdash : \alpha \epsilon 1 . \equiv . \iota\text{‘}\breve{\iota}\text{‘}\alpha = \alpha$ [*51·52 . *52·15]

***52·173.** $\vdash : \alpha \epsilon 1 . \equiv . \breve{\iota}\text{‘}\alpha \epsilon \alpha$ [*51·53 . *52·15]

***52·18.** $\vdash :. \alpha \epsilon 1 . \equiv : (\exists x) : x \epsilon \alpha : y \epsilon \alpha . \supset_y . y = x$

Dem.

$\vdash$. *51·141 . $\supset \vdash :. (\exists x) . \alpha = \iota\text{‘}x . \equiv : (\exists x) : x \epsilon \alpha : y \epsilon \alpha . \supset_y . y = x$ (1)

$\vdash . (1)$. *52·1 . $\supset \vdash . \text{Prop}$

***52·181.** $\vdash :. \alpha \sim\epsilon 1 . \equiv : x \epsilon \alpha . \supset_x . (\exists y) . y \epsilon \alpha . y \neq x$ [*52·18 . *10·51]

***52·2.** $\vdash . 1 \subset \text{Cls}$

Dem.

$\vdash$. *52·1 . $\supset \vdash : \alpha \epsilon 1 . \supset . (\exists x) . \alpha = \iota\text{‘}x .$

[*51·11] $\supset . (\exists x) . \alpha = \hat{z}(z = x) .$

[*20·54] $\supset . (\exists x, \phi) . \hat{z}(\phi ! z) = \hat{z}(z = x) . \alpha = \hat{z}(\phi ! z) .$

[*10·5] $\supset . (\exists \phi) . \alpha = \hat{z}(\phi ! z) .$

[*20·4] $\supset . \alpha \epsilon \text{Cls} : \supset \vdash . \text{Prop}$

***52·21.** $\vdash . \Lambda \sim\epsilon 1$

Dem.

$\vdash$. *52·16 . $\supset \vdash : \alpha \epsilon 1 . \supset_\alpha . \exists ! \alpha :$

[*24·63] $\supset \vdash : \Lambda \sim\epsilon 1$

***52·22.** $\vdash . \iota\text{‘}x \epsilon 1$ [*51·12 . *14·28 . *10·24 . *52·1]

***52·23.** $\vdash . \exists ! 1 . \exists ! -1$

Dem.

$\vdash$. *52·22 . *10·24 . $\supset \vdash . (\exists x) . \iota\text{‘}x \epsilon 1 .$

[*20·54] $\supset \vdash . (\exists x, \alpha) . \alpha = \iota\text{‘}x . \alpha \epsilon 1 .$

[*10·5] $\supset \vdash . (\exists \alpha) . \alpha \epsilon 1$ (1)

$\vdash$. *52·21 . *22·35 . $\supset \vdash . \Lambda \epsilon -1 .$

[*10·24] $\supset \vdash . (\exists \alpha) . \alpha \epsilon -1$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***52·24.** $\vdash . 1 \neq \Lambda \cap \text{Cls} . 1 \neq V \cap \text{Cls}$ [*52·23 . *24·54 . *24·17 . Transp]

***52·3.** $\vdash . \iota\text{“}\alpha \subset 1$

Dem.

$\vdash$. *52·22 . *2·02 . $\supset \vdash : y \epsilon \alpha . \supset . \iota\text{‘}y \epsilon 1 :$

[*51·12.*10·11.*37·61] $\supset \vdash . \iota\text{“}\alpha \subset 1$

***52·31.** $\vdash : \kappa \subset 1 . \equiv . (\exists \alpha) . \kappa = \iota\text{“}\alpha$

Dem.

$\vdash$. *52·14 . $\supset \vdash : \kappa \subset 1 . \equiv . \kappa \subset \iota\text{“}V .$

[*37·66.*51·12] $\equiv . (\exists \alpha) . \alpha \subset V . \kappa = \iota\text{“}\alpha .$

[*24·11] $\equiv . (\exists \alpha) . \kappa = \iota\text{“}\alpha : \supset \vdash . \text{Prop}$

***52·4.** $\vdash :. \alpha \epsilon 1 \cup \iota\text{‘}\Lambda . \equiv : x, y \epsilon \alpha . \supset_{x,y} . x = y$

Dem.

$\vdash$. *52·16 . *24·54 . $\supset$

$\vdash :. \alpha \epsilon 1 . \equiv : \alpha \neq \Lambda : x, y \epsilon \alpha . \supset_{x,y} . x = y :.$

[*4·37] $\supset \vdash :: \alpha \epsilon 1 . \mathrm{v} . \alpha = \Lambda : \equiv :. \alpha = \Lambda :. \mathrm{v} :. \alpha \neq \Lambda : x, y \epsilon \alpha . \supset_{x,y} . x = y :.$

[*5·63] $\equiv :. \alpha = \Lambda : \mathrm{v} : x, y \epsilon \alpha . \supset_{x,y} . x = y$ (1)

$\vdash$. *24·51 . *10·53 . *11·62 . $\supset \vdash :. \alpha = \Lambda . \supset : x, y \epsilon \alpha . \supset_{x,y} . x = y$ (2)

$\vdash . (1) . (2)$. *4·72 . $\supset \vdash :: \alpha \epsilon 1 . \mathrm{v} . \alpha = \Lambda : \equiv :. x, y \epsilon \alpha . \supset_{x,y} . x = y$ (3)

$\vdash . (3)$. *51·236 . $\supset \vdash . \text{Prop}$

This proposition is frequently useful. We shall define the number 0 as $\iota\text{‘}\Lambda$; thus the above proposition states that a class has one member or none when, and only when, all its members are identical. It will be seen that $x, y \epsilon \alpha . \supset_{x,y} . x = y$ does not imply $\exists ! \alpha$, and therefore allows the possibility of α having no members.

***52·41.** $\vdash : \exists ! \alpha . \alpha \sim\epsilon 1 . \equiv . (\exists x, y) . x, y \epsilon \alpha . x \neq y$

Dem.

$\vdash$. *24·54 . $\supset \vdash :. \exists ! \alpha . \alpha \sim\epsilon 1 . \equiv : \alpha \neq \Lambda . \alpha \sim\epsilon 1 :$

[*4·56] $\equiv : \sim\{\alpha \epsilon 1 . \mathrm{v} . \alpha = \Lambda\} :$

[*51·236] $\equiv : \sim(\alpha \epsilon 1 \cup \iota\text{‘}\Lambda) :$

[*52·4.Transp] $\equiv : \sim\{x, y \epsilon \alpha . \supset_{x,y} . x = y\} :$

[*11·52] $\equiv : (\exists x, y) . x, y \epsilon \alpha . x \neq y :. \supset \vdash . \text{Prop}$

***52·42.** $\vdash :. \alpha \in 1 . \supset : \exists ! \alpha \cap \beta . \equiv . \alpha \cap \beta \in 1$

Dem.

$\vdash . *51\cdot31 . \supset \vdash :. \exists ! \iota' x \cap \beta . \equiv . \iota' x \cap \beta = \iota' x :.$

$[*20\cdot53] \quad \supset \vdash :. \alpha = \iota' x . \supset : \exists ! \alpha \cap \beta . \equiv . \alpha \cap \beta = \iota' x :.$

$[*10\cdot11\cdot28] \quad \supset \vdash :. (\exists x) . \alpha = \iota' x . \supset : (\exists x) : \exists ! \alpha \cap \beta . \equiv . \alpha \cap \beta = \iota' x :$

$[*10\cdot37] \quad \supset : \exists ! \alpha \cap \beta . \supset . (\exists x) . \alpha \cap \beta = \iota' x$ (1)

$\vdash . (1) . *52\cdot1 . \supset \vdash :. \alpha \in 1 . \supset : \exists ! \alpha \cap \beta . \supset . \alpha \cap \beta \in 1$ (2)

$\vdash . *52\cdot16 . \supset \vdash : \alpha \cap \beta \in 1 . \supset . \exists ! \alpha \cap \beta$ (3)

$\vdash . (2) . (3) . \supset \vdash .$ Prop

***52·43.** $\vdash : \alpha \in 1 . \exists ! \alpha \cap \beta . \equiv . \alpha \in 1 . \alpha \cap \beta \in 1$ [*52·42 . *5·32]

***52·44.** $\vdash :. \alpha \in 1 . \supset : \exists ! \alpha \cap \beta . \equiv . \alpha \subset \beta . \equiv . \alpha \cap \beta = \alpha$

Dem.

$\vdash . *51\cdot31 . \supset \vdash : \exists ! \iota' x \cap \beta . \equiv . \iota' x \subset \beta :$

$[*13\cdot13 . \text{Exp}] \quad \supset \vdash :. \alpha = \iota' x . \supset : \exists ! \alpha \cap \beta . \equiv . \alpha \subset \beta :.$

$[*10\cdot11\cdot23] \quad \supset \vdash :. (\exists x) . \alpha = \iota' x . \supset : \exists ! \alpha \cap \beta . \equiv . \alpha \subset \beta :.$

$[*52\cdot1] \quad \supset \vdash :. \alpha \in 1 . \supset : \exists ! \alpha \cap \beta . \equiv . \alpha \subset \beta$ (1)

$\vdash . (1) . *22\cdot621 . \supset \vdash .$ Prop

***52·45.** $\vdash :: \alpha, \beta \in 1 . \supset :. \alpha \subset \beta \cup \gamma . \equiv : \alpha = \beta . \vee . \alpha \subset \gamma$

Dem.

$\vdash . *51\cdot236 \frac{x, y, \gamma}{z, x, \beta} . \supset$

$\vdash :. x \in \iota' y \cup \gamma . \equiv : x = y . \vee . x \in \gamma :.$

$[*51\cdot2\cdot23] \quad \supset \vdash :. \iota' x \subset \iota' y \cup \gamma . \equiv : \iota' x = \iota' y . \vee . \iota' x \subset \gamma :.$

$[*13\cdot21] \quad \supset \vdash :: \alpha = \iota' x . \beta = \iota' y . \supset :. \alpha \subset \beta \cup \gamma . \equiv : \alpha = \beta . \vee . \alpha \subset \gamma ::$

$[*11\cdot11\cdot35] \quad \supset \vdash :: (\exists x, y) . \alpha = \iota' x . \beta = \iota' y . \supset :. \alpha \subset \beta \cup \gamma . \equiv : \alpha = \beta . \vee . \alpha \subset \gamma$ (1)

$\vdash . (1) . *52\cdot1 . \supset \vdash .$ Prop

***52·46.** $\vdash :. \alpha, \beta \in 1 . \supset : \alpha \subset \beta . \equiv . \alpha = \beta . \equiv . \exists ! (\alpha \cap \beta)$

Dem.

$\vdash . *51\cdot2\cdot23 . \supset \vdash : \iota' x \subset \iota' y . \equiv . \iota' x = \iota' y$ (1)

$\vdash . (1) . *13\cdot21 . \supset \vdash :. \alpha = \iota' x . \beta = \iota' y . \supset : \alpha \subset \beta . \equiv . \alpha = \beta$ (2)

$\vdash . (2) . *11\cdot11\cdot35 . *52\cdot1 . \supset \vdash :. \alpha, \beta \in 1 . \supset : \alpha \subset \beta . \equiv . \alpha = \beta$ (3)

$\vdash . (3) . *52\cdot44 . \supset \vdash .$ Prop

***52·6.** $\vdash :. \alpha \in 1 . \supset : x \in \alpha . \equiv . \iota' x = \alpha . \equiv . x = \breve{\iota}' \alpha$

Dem.

$\vdash . *51\cdot23 . \supset \vdash : x \in \iota' y . \equiv . \iota' x = \iota' y :$

$[*13\cdot13 . \text{Exp}] \quad \supset \vdash :. \alpha = \iota' y . \supset : x \in \alpha . \equiv . \iota' x = \alpha :.$

$[*10\cdot11\cdot23 . *52\cdot1] \quad \supset \vdash :. \alpha \in 1 . \supset : x \in \alpha . \equiv . \iota' x = \alpha .$ (1)

$[*51\cdot51] \quad \equiv . x = \breve{\iota}' \alpha$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

***52·601.** $\vdash :: \alpha \in 1 . \supset :. \phi (\breve{\iota}' \alpha) . \equiv : x \in \alpha . \supset_x . \phi x : \equiv : (\exists x) . x \in \alpha . \phi x$

Dem.

$\vdash . *52\cdot15 . \supset \vdash :. \text{Hp} . \supset : E ! \breve{\iota}' \alpha :$ (1)

$[*30\cdot4] \quad \supset : x \breve{\iota} \alpha . \equiv . x = \breve{\iota}' \alpha .$

$[*52\cdot6] \quad \equiv . x \in \alpha$ (2)

$\vdash . (1) . *30\cdot33 . \supset$

$\vdash :: \text{Hp} . \supset :. \phi (\breve{\iota}' \alpha) . \equiv : x \breve{\iota} \alpha . \supset_x . \phi x : \equiv : (\exists x) . x \breve{\iota} \alpha . \phi x$ (3)

$\vdash . (2) . (3) . \supset \vdash .$ Prop

***52·602.** $\vdash :. \hat{z} (\phi z) \in 1 . \supset : \psi (\imath x) (\phi x) . \equiv . \phi x \supset_x \psi x . \equiv . (\exists x) . \phi x . \psi x$

[*52·12 . *14·26]

***52·61.** $\vdash :. \alpha \in 1 . \supset : \breve{\iota}' \alpha \in \beta . \equiv . \alpha \subset \beta . \equiv . \exists ! (\alpha \cap \beta)$ $\left[*52\cdot601 \frac{x \in \beta}{\phi x} \right]$

***52·62.** $\vdash :. \alpha, \beta \in 1 . \supset : \alpha = \beta . \equiv . \breve{\iota}' \alpha = \breve{\iota}' \beta$

Dem.

$\vdash . *52\cdot601 . \supset \vdash :: \text{Hp} . \supset :. \breve{\iota}' \alpha = \breve{\iota}' \beta . \equiv : x \in \alpha . \supset_x . x = \breve{\iota}' \beta :$

$[*52\cdot6] \quad \equiv : x \in \alpha . \supset_x . x \in \beta :$

$[*52\cdot46] \quad \equiv : \alpha = \beta :: \supset \vdash .$ Prop

***52·63.** $\vdash : \alpha, \beta \in 1 . \alpha \neq \beta . \supset . \alpha \cap \beta = \Lambda$ [*52·46 . Transp]

***52·64.** $\vdash : \alpha \in 1 . \supset . \alpha \cap \beta \in 1 \cup \iota' \Lambda$

Dem.

$\vdash . *52\cdot43 . \supset \vdash : \text{Hp} . \exists ! \alpha \cap \beta . \supset . \alpha \cap \beta \in 1 :$

$[*5\cdot6 . *24\cdot54] \quad \supset \vdash :. \text{Hp} . \supset : \alpha \cap \beta = \Lambda . \vee . \alpha \cap \beta \in 1 :$

$[*51\cdot236] \quad \supset : \alpha \cap \beta \in 1 \cup \iota' \Lambda :. \supset \vdash .$ Prop

***52·7.** $\vdash :. \beta - \alpha \in 1 . \alpha \subset \xi . \xi \subset \beta . \supset : \xi = \alpha . \vee . \xi = \beta$

Dem.

$\vdash . *22\cdot41 . \supset \vdash : \text{Hp} . \xi \subset \alpha . \supset . \xi = \alpha$ (1)

$\vdash . *24\cdot55 . \supset \vdash : \sim (\xi \subset \alpha) . \supset . \exists ! \xi - \alpha$ (2)

$\vdash . *22\cdot48 . \supset \vdash : \text{Hp} . \supset . \xi - \alpha \subset \beta - \alpha$ (3)

$\vdash . (2) . (3) . \supset \vdash : \text{Hp} . \sim (\xi \subset \alpha) . \supset . \exists ! \xi - \alpha . \xi - \alpha \subset \beta - \alpha$ (4)

$\vdash . *52\cdot1 . \supset \vdash : \text{Hp} . \supset . (\exists x) . \beta - \alpha = \iota' x$ (5)

$\vdash . (4) . (5) . *51\cdot4 . \supset \vdash : \text{Hp} . \sim (\xi \subset \alpha) . \supset . \xi - \alpha = \beta - \alpha .$

$[*24\cdot411] \quad \supset . \xi = \beta$ (6)

$\vdash . (1) . (6) . \supset \vdash .$ Prop

$*53$. MISCELLANEOUS PROPOSITIONS INVOLVING UNIT CLASSES

Summary of $*53$.

The propositions to be given in this number are mostly such as would have come more naturally at an earlier stage, but could not be given sooner because they involved unit classes. It is to be observed that $\iota‘x \cup \iota‘y$ is the class consisting of the members x and y, while $\iota‘x \uparrow \iota‘y$ is the relation which holds only between x and y. If α and β are classes, $\iota‘\alpha \cup \iota‘\beta$ is a class of classes, its members being α and β. If R and S are relations, $\iota‘R \uparrow \iota‘S$ is a relation of relations; and so on.

The present number begins by connecting products and sums $p‘\kappa$, $s‘\kappa$, $\dot{p}‘\lambda$, $\dot{s}‘\lambda$, in cases where the members of κ or λ are specified, with the products or sums $\alpha \cap \beta$, $\alpha \cup \beta$, $R \mathbin{\dot\cap} S$, $R \mathbin{\dot\cup} S$. We have

$*53{\cdot}01$. $\vdash . p‘\iota‘\alpha = \alpha$

$*53{\cdot}1$. $\vdash . p‘(\iota‘\alpha \cup \iota‘\beta) = \alpha \cap \beta$

$*53{\cdot}14$. $\vdash . p‘(\kappa \cup \iota‘\alpha) = p‘\kappa \cap \alpha$

with similar propositions for s, $\dot{p}$ and $\dot{s}$.

We have next a set of propositions on sums and products of classes of unit classes. The most important of these is

$*53{\cdot}22$. $\vdash . s‘\iota‘‘\alpha = \alpha$

We have next a proposition showing that the sum of κ is null when, and only when, κ is either null or has the null-class for its only member, *i.e.*

$*53{\cdot}24$. $\vdash :. s‘\kappa = \Lambda . \equiv : \kappa = \Lambda \cap \mathrm{Cls} . \vee . \kappa = \iota‘\Lambda$

(Here we write "$\Lambda \cap \mathrm{Cls}$," to show that the "$\Lambda$" in question is of the next type above that of the other two Λ's.)

We have next various propositions on the relations of $\overrightarrow{R}‘x$ and $R‘x$ and $R‘‘\alpha$ in various cases, first for a general relation R, and then for the particular relation s defined in $*40$. Three of these propositions are very frequently used, namely:

$*53{\cdot}3$. $\vdash : \mathrm{E}\,!\, R‘x . \equiv . \overrightarrow{R}‘x \in 1$

$*53{\cdot}301$. $\vdash . R‘‘\iota‘x = \overrightarrow{R}‘x$

$*53{\cdot}31$. $\vdash : \mathrm{E}\,!\, R‘x . \supset . R‘‘\iota‘x = \iota‘R‘x = \overrightarrow{R}‘x$

The remaining propositions of this number are of less importance, and are seldom referred to.

$*53{\cdot}01$. $\vdash . p‘\iota‘\alpha = \alpha$

Dem.

$\vdash . *40{\cdot}1 . \supset \vdash :. x \in p‘\iota‘\alpha . \equiv : \beta \in \iota‘\alpha . \supset_\beta . x \in \beta :$

$[*51{\cdot}15] \qquad \equiv : \beta = \alpha . \supset_\beta . x \in \beta :$

$[*13{\cdot}191] \qquad \equiv : x \in \alpha :. \supset \vdash .$ Prop

$*53{\cdot}02$. $\vdash . s‘\iota‘\alpha = \alpha$

Dem.

$\vdash . *40{\cdot}11 . \supset \vdash : x \in s‘\iota‘\alpha . \equiv . (\exists\beta) . \beta \in \iota‘\alpha . x \in \beta .$

$[*51{\cdot}15] \qquad \equiv . (\exists\beta) . \beta = \alpha . x \in \beta .$

$[*13{\cdot}195] \qquad \equiv . x \in \alpha : \supset \vdash .$ Prop

$*53{\cdot}03$. $\vdash . \dot{p}‘\iota‘R = R$ [Proof as in $*53{\cdot}01$]

$*53{\cdot}04$. $\vdash . \dot{s}‘\iota‘R = R$ [Proof as in $*53{\cdot}02$]

$*53{\cdot}1$. $\vdash . p‘(\iota‘\alpha \cup \iota‘\beta) = \alpha \cap \beta$

Dem.

$\vdash . *40{\cdot}18 . \supset \vdash . p‘(\iota‘\alpha \cup \iota‘\beta) = p‘\iota‘\alpha \cap p‘\iota‘\beta$

$[*53{\cdot}01] \qquad = \alpha \cap \beta . \supset \vdash .$ Prop

This proposition can be extended to $\iota‘\alpha \cup \iota‘\beta \cup \iota‘\gamma$, etc. It shows the connection (for finite classes of classes) between the product $p‘\kappa$ and the product of the members $\alpha \cap \beta \cap \gamma \cap \ldots$.

$*53{\cdot}11$. $\vdash . s‘(\iota‘\alpha \cup \iota‘\beta) = \alpha \cup \beta$

Dem.

$\vdash . *40{\cdot}171 . \supset \vdash . s‘(\iota‘\alpha \cup \iota‘\beta) = s‘\iota‘\alpha \cup s‘\iota‘\beta$

$[*53{\cdot}02] \qquad = \alpha \cup \beta . \supset \vdash .$ Prop

Similar remarks apply to this proposition as to $*53{\cdot}1$.

$*53{\cdot}12$. $\vdash . \dot{p}‘(\iota‘R \cup \iota‘S) = R \mathbin{\dot\cap} S$ [$*41{\cdot}18 . *53{\cdot}03$]

This proposition shows the connection between the product $\dot{p}‘\kappa$ for a class κ consisting of two relations R and S, and the product $R \mathbin{\dot\cap} S$. The proposition can be extended to the product of any given finite class of relations.

$*53{\cdot}13$. $\vdash . \dot{s}‘(\iota‘R \cup \iota‘S) = R \mathbin{\dot\cup} S$ [$*41{\cdot}171 . *53{\cdot}04$]

Similar remarks apply to this proposition as to $*53{\cdot}12$.

$*53{\cdot}14$. $\vdash . p‘(\kappa \cup \iota‘\alpha) = p‘\kappa \cap \alpha$

Dem.

$\vdash . *40{\cdot}18 . \supset \vdash . p‘(\kappa \cup \iota‘\alpha) = p‘\kappa \cap p‘\iota‘\alpha$

$[*53{\cdot}01] \qquad = p‘\kappa \cap \alpha$

$*53{\cdot}15$. $\vdash . s‘(\kappa \cup \iota‘\alpha) = s‘\kappa \cup \alpha$ [Proof as in $*53{\cdot}14$]

$*53{\cdot}16$. $\vdash . \dot{p}‘(\lambda \cup \iota‘R) = \dot{p}‘\lambda \mathbin{\dot\cap} R$ [Proof as in $*53{\cdot}14$]

$*53{\cdot}17$. $\vdash . \dot{s}‘(\lambda \cup \iota‘R) = \dot{s}‘\lambda \mathbin{\dot\cup} R$ [Proof as in $*53{\cdot}14$]

The above proposition and the next are both used in connection with mathematical induction ($*91{\cdot}55$ and $*97{\cdot}46$ respectively).

***53·18.** $\vdash . s\text{‘}(\alpha - \iota\text{‘}\Lambda) = s\text{‘}\alpha$

Dem.

$\vdash . *51\cdot221 . \supset \vdash : \Lambda \in \alpha . \supset . (\alpha - \iota\text{‘}\Lambda) \cup \iota\text{‘}\Lambda = \alpha .$

$[*53\cdot15] \quad \supset . s\text{‘}(\alpha - \iota\text{‘}\Lambda) \cup \Lambda = s\text{‘}\alpha .$

$[*24\cdot24] \quad \supset . s\text{‘}(\alpha - \iota\text{‘}\Lambda) = s\text{‘}\alpha$ (1)

$\vdash . *51\cdot222 . \supset \vdash : \Lambda \sim\in \alpha . \supset . \quad \alpha - \iota\text{‘}\Lambda = \alpha .$

$[*30\cdot37] \quad \supset . s\text{‘}(\alpha - \iota\text{‘}\Lambda) = s\text{‘}\alpha$ (2)

$\vdash . (1) . (2) . \supset \vdash .$ Prop

***53·181.** $\vdash . \dot{s}\text{‘}(\lambda - \iota\text{‘}\dot{\Lambda}) = \dot{s}\text{‘}\lambda$ [Proof as in *53·18]

***53·2.** $\vdash : \kappa \in 1 . \supset . \breve{\iota}\text{‘}\kappa = p\text{‘}\kappa = s\text{‘}\kappa$

This proposition requires, for significance, that κ should be a class of classes. It is used in *88·47, in the number on the existence of selections and the multiplicative axiom.

Dem.

$\vdash . *52\cdot601 . \supset \vdash :: \mathrm{Hp} . \supset :. x \in \breve{\iota}\text{‘}\kappa : \equiv : \alpha \in \kappa . \supset_\alpha . x \in \alpha : \equiv : (\exists\alpha) . \alpha \in \kappa . x \in \alpha$ (1)

$\vdash . (1) . *40\cdot1\cdot11 . \quad \supset \vdash .$ Prop

***53·21.** $\vdash : \lambda \in 1 . \supset . \breve{\iota}\text{‘}\lambda = \dot{p}\text{‘}\lambda = \dot{s}\text{‘}\lambda$ [Similar proof]

This proposition requires, for significance, that λ should be a class of relations.

***53·22.** $\vdash . s\text{‘}\iota\text{“}\alpha = \alpha$

Dem.

$\vdash . *40\cdot11 . \supset \vdash : x \in s\text{‘}\iota\text{“}\alpha . \equiv . (\exists\gamma) . \gamma \in \iota\text{“}\alpha . x \in \gamma .$

$[*37\cdot64 . *51\cdot12] \quad \equiv . (\exists y) . y \in \alpha . x \in \iota\text{‘}y .$

$[*51\cdot15] \quad \equiv . (\exists y) . y \in \alpha . x = y .$

$[*13\cdot195] \quad \equiv . x \in \alpha : \supset \vdash .$ Prop

***53·221.** $\vdash . \iota\text{“}(\iota\text{‘}x \cup \iota\text{‘}y) = \iota\text{‘}\iota\text{‘}x \cup \iota\text{‘}\iota\text{‘}y$

Dem.

$\vdash . *37\cdot1 . \supset \vdash :. \alpha \in \iota\text{“}(\iota\text{‘}x \cup \iota\text{‘}y) . \equiv : (\exists z) . z \in (\iota\text{‘}x \cup \iota\text{‘}y) . \alpha \, \iota \, z :$

$[*51\cdot131] \quad \equiv : (\exists z) . z \in (\iota\text{‘}x \cup \iota\text{‘}y) . \alpha = \iota\text{‘}z :$

$[*51\cdot235] \quad \equiv : \alpha = \iota\text{‘}x . \vee . \alpha = \iota\text{‘}y :$

$[*51\cdot232] \quad \equiv : \alpha \in (\iota\text{‘}\iota\text{‘}x \cup \iota\text{‘}\iota\text{‘}y) :. \supset \vdash .$ Prop

***53·222.** $\vdash : \kappa = \iota\text{“}\alpha . \supset . \alpha = \breve{\iota}\text{“}\kappa$

Dem.

$\vdash . *13\cdot12 . *20\cdot2 . \supset \vdash : \mathrm{Hp} . \supset . \breve{\iota}\text{“}\kappa = \breve{\iota}\text{“}\iota\text{“}\alpha$

$[*51\cdot511 . *14\cdot21 . *37\cdot67] \quad = \hat{x}\{(\exists y) . y \in \alpha . x = \breve{\iota}\text{‘}\iota\text{‘}y\}$

$[*51\cdot511] \quad = \hat{x}\{(\exists y) . y \in \alpha . x = y\}$

$[*13\cdot195] \quad = \alpha : \supset \vdash .$ Prop

***53·23.** $\vdash : \kappa \subset 1 . \supset . s\text{‘}\kappa = \breve{\iota}\text{“}\kappa$

Dem.

$\vdash . *52\cdot31 . \supset \vdash : \mathrm{Hp} . \equiv . (\exists\alpha) . \kappa = \iota\text{“}\alpha$ (1)

$\vdash . *53\cdot22 . \supset \vdash : \kappa = \iota\text{“}\alpha . \supset . s\text{‘}\kappa = \alpha$

$[*53\cdot222] \quad = \breve{\iota}\text{“}\kappa$ (2)

$\vdash . (1) . (2) . *10\cdot11\cdot23 . \supset \vdash .$ Prop

***53·231.** $\vdash :. x \in \alpha . \supset_x . x = y : \equiv : \alpha = \Lambda . \vee . \alpha = \iota\text{‘}y$

Dem.

$\vdash . *51\cdot141 . \supset \vdash :. \exists\,!\,\alpha : x \in \alpha . \supset_x . x = y : \equiv : \alpha = \iota\text{‘}y$ (1)

$\vdash . *10\cdot53 . \quad \supset \vdash :. \sim\exists\,!\,\alpha . \supset : x \in \alpha . \supset_x . x = y :.$

$[*4\cdot71] \quad \supset \vdash :. \sim\exists\,!\,\alpha : x \in \alpha . \supset_x . x = y : \equiv . \sim\exists\,!\,\alpha .$

$[*24\cdot51] \quad \equiv . \alpha = \Lambda$ (2)

$\vdash . (1) . (2) . *4\cdot42\cdot39 . \supset \vdash .$ Prop

***53·24.** $\vdash :. s\text{‘}\kappa = \Lambda . \equiv : \kappa = \Lambda \cap \mathrm{Cls} . \vee . \kappa = \iota\text{‘}\Lambda$

Dem.

$\vdash . *24\cdot15 . *40\cdot11 . \supset$

$\vdash :. s\text{‘}\kappa = \Lambda . \quad \equiv : (x) : \sim\{(\exists\alpha) . \alpha \in \kappa . x \in \alpha\} :$

$[*10\cdot51] \quad \equiv : (x, \alpha) : x \in \alpha . \supset . \alpha \sim\in \kappa :$

$[*11\cdot2 . *10\cdot23] \quad \equiv : (\exists x) . x \in \alpha . \supset_\alpha . \alpha \sim\in \kappa :$

$[*24\cdot54] \quad \equiv : \alpha \neq \Lambda . \supset_\alpha . \alpha \sim\in \kappa :$

$[\mathrm{Transp}] \quad \equiv : \alpha \in \kappa . \supset_\alpha . \alpha = \Lambda :$

$[*53\cdot231] \quad \equiv : \kappa = \Lambda \cap \mathrm{Cls} . \vee . \kappa = \iota\text{‘}\Lambda :. \supset \vdash .$ Prop

In the enunciation and the last line of the proof of the above proposition, we write "$\kappa = \Lambda \cap \mathrm{Cls}$" rather than "$\kappa = \Lambda$," because this Λ must be of the type next above that of the Λ in "$\kappa = \iota\text{‘}\Lambda$."

The following proposition is used in the theory of selections (*83·731).

***53·25.** $\vdash :. s\text{‘}\kappa \cap s\text{‘}\lambda = \Lambda . \supset : \kappa \cap \lambda = \Lambda \cap \mathrm{Cls} . \vee . \kappa \cap \lambda = \iota\text{‘}\Lambda$

Dem.

$\vdash . *40\cdot181 . \supset \vdash :. \mathrm{Hp} . \supset : s\text{‘}(\kappa \cap \lambda) = \Lambda :$

$[*53\cdot24] \quad \supset : \kappa \cap \lambda = \Lambda \cap \mathrm{Cls} . \vee . \kappa \cap \lambda = \iota\text{‘}\Lambda :. \supset \vdash .$ Prop

***53·3.** $\vdash : E\,!\,R\text{‘}x . \equiv . \overrightarrow{R}\text{‘}x \in 1$

Dem.

$\vdash . *30\cdot2 . \supset \vdash :. E\,!\,R\text{‘}x . \equiv : (\exists b) : yRx . \equiv_y . y = b :$

$[*32\cdot18 . *51\cdot15] \quad \equiv : (\exists b) : y \in \overrightarrow{R}\text{‘}x . \equiv_y . y \in \iota\text{‘}b :$

$[*20\cdot31] \quad \equiv : (\exists b) . \overrightarrow{R}\text{‘}x = \iota\text{‘}b :$

$[*52\cdot1] \quad \equiv : \overrightarrow{R}\text{‘}x \in 1 :. \supset \vdash .$ Prop

The above proposition is very frequently used.

***53·301.** $\vdash . R‘‘\iota‘x = \overrightarrow{R}‘x$

Dem.

$\vdash . *37·1 . *51·15 . \supset \vdash : y \in R‘‘\iota‘x . \equiv . (\exists z) . z = x . yRz .$

$[*13·195] \quad \equiv . yRx .$

$[*32·18] \quad \equiv . y \in \overrightarrow{R}‘x : \supset \vdash . \text{Prop}$

***53·302.** $\vdash . R‘‘(\iota‘x \cup \iota‘y) = \overrightarrow{R}‘x \cup \overrightarrow{R}‘y$ $[*37·22 . *53·301]$

The above proposition is used in the cardinal theory of exponentiation (*116·71).

***53·31.** $\vdash : E ! R‘x . \supset . R‘‘\iota‘x = \iota‘R‘x = \overrightarrow{R}‘x$

The above proposition is one of which the subsequent use is frequent.

Dem.

$\vdash . *51·11 . *14·18 . \supset \vdash : \text{Hp} . \supset . \iota‘R‘x = \hat{y}(y = R‘x)$

$[*30·4] \quad = \hat{y}(yRx)$

$[*32·13] \quad = \overrightarrow{R}‘x \quad (1)$

$\vdash . (1) . *53·301 . \supset \vdash . \text{Prop}$

***53·32.** $\vdash : E ! R‘x . E ! R‘y . \supset . R‘‘(\iota‘x \cup \iota‘y) = \iota‘R‘x \cup \iota‘R‘y$

Dem.

$\vdash . *37·22 . \supset \vdash . R‘‘(\iota‘x \cup \iota‘y) = R‘‘\iota‘x \cup R‘‘\iota‘y \quad (1)$

$\vdash . (1) . *53·31 . \supset \vdash . \text{Prop}$

***53·33.** $\vdash . s‘‘\iota‘\kappa = \iota‘s‘\kappa$ $\left[*53·31 \frac{s}{R}\right]$

***53·34.** $\vdash . s‘‘(\iota‘\kappa \cup \iota‘\lambda) = \iota‘s‘\kappa \cup \iota‘s‘\lambda$ $\left[*53·32 \frac{s}{R}\right]$

***53·35.** $\vdash . s‘s‘‘(\iota‘\kappa \cup \iota‘\lambda) = s‘\kappa \cup s‘\lambda = s‘(\kappa \cup \lambda)$

Dem.

$\vdash . *53·34 . \supset \vdash . s‘s‘‘(\iota‘\kappa \cup \iota‘\lambda) = s‘(\iota‘s‘\kappa \cup \iota‘s‘\lambda)$

$[*53·11] \quad = s‘\kappa \cup s‘\lambda$

$[*40·171] \quad = s‘(\kappa \cup \lambda) . \supset \vdash . \text{Prop}$

The above proposition may also be proved as follows:

$\vdash . *42·1 . \supset \vdash . s‘s‘‘(\iota‘\kappa \cup \iota‘\lambda) = s‘s‘(\iota‘\kappa \cup \iota‘\lambda)$

$[*53·11] \quad = s‘(\kappa \cup \lambda)$

$[*40·171] \quad = s‘\kappa \cup s‘\lambda . \supset \vdash . \text{Prop}$

***53·4.** $\vdash : x = R‘y . \equiv . \overrightarrow{R}‘y \in 1 . x \in \overrightarrow{R}‘y . \equiv . \iota‘x = \overrightarrow{R}‘y . \equiv . x = \breve{\iota}‘\overrightarrow{R}‘y$

Dem.

$\vdash . *14·21 . *4·71 . \supset \vdash : x = R‘y . \equiv . E ! R‘y . x = R‘y .$

$[*30·4 . *5·32] \quad \equiv . E ! R‘y . xRy .$

$[*53·3 . *32·18] \quad \equiv . \overrightarrow{R}‘y \in 1 . x \in \overrightarrow{R}‘y . \quad (1)$

$[*52·6 . *5·32] \quad \equiv . \overrightarrow{R}‘y \in 1 . \iota‘x = \overrightarrow{R}‘y .$

$[*52·22] \quad \equiv . \iota‘x = \overrightarrow{R}‘y . \quad (2)$

$[*51·51] \quad \equiv . x = \breve{\iota}‘\overrightarrow{R}‘y \quad (3)$

$\vdash . (1) . (2) . (3) . \supset \vdash . \text{Prop}$

***53·5.** $\vdash : \exists ! \alpha . \equiv . \alpha \in \text{Cls} - \iota‘\Lambda$

Dem.

$\vdash . *20·41 . \supset \vdash : \exists ! \hat{z}(\phi z) . \equiv . \hat{z}(\phi z) \in \text{Cls} . \exists ! \hat{z}(\phi z) .$

$[*24·54] \quad \equiv . \hat{z}(\phi z) \in \text{Cls} . \hat{z}(\phi z) \neq \Lambda .$

$[*51·3] \quad \equiv . \hat{z}(\phi z) \in \text{Cls} - \iota‘\Lambda : \supset \vdash . \text{Prop}$

In the above proof, as usually where "Cls" or other type-symbols occur, it is necessary to abandon the notation by Greek letters and revert to the explicit notation.

***53·51.** $\vdash : \dot{\exists} ! R . \equiv . R \in \text{Rel} - \iota‘\dot{\Lambda}$ [Proof as in *53·5]

***53·52.** $\vdash : \alpha \in \kappa . \exists ! \alpha . \equiv . \alpha \in \kappa - \iota‘\Lambda$

Dem.

$\vdash . *24·54 . \supset \vdash : \alpha \in \kappa . \exists ! \alpha . \equiv . \alpha \in \kappa . \alpha \neq \Lambda .$

$[*51·3] \quad \equiv . \alpha \in \kappa - \iota‘\Lambda : \supset \vdash . \text{Prop}$

***53·53.** $\vdash : R \in \lambda . \dot{\exists} ! R . \equiv . R \in \lambda - \iota‘\dot{\Lambda}$ [Proof as in *53·52]

The following propositions are inserted because of their connection with the definition of $\alpha \rightarrow \beta$ in *70. $\overrightarrow{R}‘‘\text{ᗡ}‘R$ and $\overrightarrow{R}‘‘V$ are both important classes.

***53·6.** $\vdash : R = \dot{\Lambda} . \exists ! \alpha . \supset . \overrightarrow{R}‘‘\alpha = \iota‘\Lambda . \overleftarrow{R}‘‘\alpha = \iota‘\Lambda$

Dem.

$\vdash . *33·15·241 . *24·13 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{R}‘x = \Lambda \quad (1)$

$\vdash . (1) . *37·7 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{R}‘‘\alpha = \hat{\beta}\{(\exists x) . x \in \alpha . \beta = \Lambda\}$

$[*10·35] \quad = \hat{\beta}\{\exists ! \alpha . \beta = \Lambda\}$

$[*4·73] \quad = \hat{\beta}(\beta = \Lambda)$

$[*51·11] \quad = \iota‘\Lambda \quad (2)$

Similarly $\quad \vdash : \text{Hp} . \supset . \overleftarrow{R}‘‘\alpha = \iota‘\Lambda \quad (3)$

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

***53·601.** $\vdash : \exists ! \alpha . \alpha \cap \text{ᗡ}‘R = \Lambda . \supset . \overrightarrow{R}‘‘\alpha = \iota‘\Lambda$

Dem.

$\vdash . *33·41 . \quad \supset \vdash : \text{Hp} . x \in \alpha . \supset . \overrightarrow{R}‘x = \Lambda \quad (1)$

$\vdash . (1) . *37·7 . \supset \vdash : \text{Hp} . \supset . \overrightarrow{R}‘‘\alpha = \hat{\beta}\{(\exists x) . x \in \alpha . \beta = \Lambda\}$

$[*10·35] \quad = \hat{\beta}\{\exists ! \alpha . \beta = \Lambda\}$

$[*4·73 . *51·11] \quad = \iota‘\Lambda : \supset \vdash . \text{Prop}$

***53·602.** $\vdash : \exists ! \alpha . \alpha \cap D‘R = \Lambda . \supset . \overleftarrow{R}‘‘\alpha = \iota‘\Lambda$ [Proof as in *53·601]

***53·603.** $\vdash : \exists ! - \text{ᗡ}‘R . \supset . \overrightarrow{R}‘‘(- \text{ᗡ}‘R) = \iota‘\Lambda$ $[*24·21 . *53·601]$

***53·604.** $\vdash : \exists ! - \mathrm{D}\text{‘}R \,.\, \supset \,.\, \overleftarrow{R}\text{“}(-\mathrm{D}\text{‘}R) = \iota\text{‘}\Lambda$ [*24·21 . *53·602]

***53·61.** $\vdash : \mathrm{C}\text{‘}R \subset \alpha \,.\, \mathrm{C}\text{‘}R \neq \alpha \,.\, \supset \,.\, \overrightarrow{R}\text{“}\alpha = \overrightarrow{R}\text{“}\mathrm{C}\text{‘}R \cup \iota\text{‘}\Lambda$

Dem.

$\vdash . *22{\cdot}92 . \quad \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \alpha = \mathrm{C}\text{‘}R \cup (\alpha - \mathrm{C}\text{‘}R)$ (1)

$\vdash . *24{\cdot}6 . \quad \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \exists ! \,\alpha - \mathrm{C}\text{‘}R \,.$

[*24·21.*53·601] $\supset \,.\, \overrightarrow{R}\text{“}(\alpha - \mathrm{C}\text{‘}R) = \iota\text{‘}\Lambda$ (2)

$\vdash . (1) . *37{\cdot}22 . \supset \vdash : \mathrm{Hp} \,.\, \supset \,.\, \overrightarrow{R}\text{“}\alpha = \overrightarrow{R}\text{“}\mathrm{C}\text{‘}R \cup \overrightarrow{R}\text{“}(\alpha - \mathrm{C}\text{‘}R)$

[(2)] $= \overrightarrow{R}\text{“}\mathrm{C}\text{‘}R \cup \iota\text{‘}\Lambda : \supset \vdash . \mathrm{Prop}$

***53·611.** $\vdash : \mathrm{D}\text{‘}R \subset \alpha \,.\, \mathrm{D}\text{‘}R \neq \alpha \,.\, \supset \,.\, \overleftarrow{R}\text{“}\alpha = \overleftarrow{R}\text{“}\mathrm{D}\text{‘}R \cup \iota\text{‘}\Lambda$ [Proof as in *53·61]

***53·612.** $\vdash : \mathrm{C}\text{‘}R \neq \mathrm{V} \,.\, \supset \,.\, \overrightarrow{R}\text{“}\mathrm{V} = \overrightarrow{R}\text{“}\mathrm{C}\text{‘}R \cup \iota\text{‘}\Lambda$ [*53·61 . *24·11]

***53·613.** $\vdash : \mathrm{D}\text{‘}R \neq \mathrm{V} \,.\, \supset \,.\, \overleftarrow{R}\text{“}\mathrm{V} = \overleftarrow{R}\text{“}\mathrm{D}\text{‘}R \cup \iota\text{‘}\Lambda$ [*53·611 . *24·11]

***53·614.** $\vdash . \overrightarrow{R}\text{“}\mathrm{C}\text{‘}R = \overrightarrow{R}\text{“}\mathrm{V} - \iota\text{‘}\Lambda$

Dem.

$\vdash . *53{\cdot}612 . *22{\cdot}68 . *24{\cdot}21 . \supset$

$\vdash : \mathrm{C}\text{‘}R \neq \mathrm{V} \,.\, \supset \,.\, \overrightarrow{R}\text{“}\mathrm{V} - \iota\text{‘}\Lambda = \overrightarrow{R}\text{“}\mathrm{C}\text{‘}R - \iota\text{‘}\Lambda$ (1)

$\vdash . *22{\cdot}481 . \supset \vdash : \mathrm{C}\text{‘}R = \mathrm{V} \,.\, \supset \,.\, \overrightarrow{R}\text{“}\mathrm{V} - \iota\text{‘}\Lambda = \overrightarrow{R}\text{“}\mathrm{C}\text{‘}R - \iota\text{‘}\Lambda$ (2)

$\vdash . *37{\cdot}772 . *51{\cdot}36 . *22{\cdot}621 . \supset \vdash . \overrightarrow{R}\text{“}\mathrm{C}\text{‘}R - \iota\text{‘}\Lambda = \overrightarrow{R}\text{“}\mathrm{C}\text{‘}R$ (3)

$\vdash . (1) . (2) . (3) . \supset \vdash . \mathrm{Prop}$

***53·615.** $\vdash . \overleftarrow{R}\text{“}\mathrm{D}\text{‘}R = \overleftarrow{R}\text{“}\mathrm{V} - \iota\text{‘}\Lambda$ [Proof as in *53·614]

The two following propositions are used in *70·12.

***53·62.** $\vdash : \overrightarrow{R}\text{“}\mathrm{C}\text{‘}R \subset \gamma \,.\, \equiv \,.\, \overrightarrow{R}\text{“}\mathrm{V} \subset \gamma \cup \iota\text{‘}\Lambda$

Dem.

$\vdash . *53{\cdot}614 . \supset \vdash : \overrightarrow{R}\text{“}\mathrm{C}\text{‘}R \subset \gamma \,.\, \equiv \,.\, \overrightarrow{R}\text{“}\mathrm{V} - \iota\text{‘}\Lambda \subset \gamma \,.$

[*24·43] $\equiv \,.\, \overrightarrow{R}\text{“}\mathrm{V} \subset \gamma \cup \iota\text{‘}\Lambda : \supset \vdash . \mathrm{Prop}$

***53·621.** $\vdash : \overleftarrow{R}\text{“}\mathrm{D}\text{‘}R \subset \gamma \,.\, \equiv \,.\, \overleftarrow{R}\text{“}\mathrm{V} \subset \gamma \cup \iota\text{‘}\Lambda$ [Proof as in *53·62]

***53·63.** $\vdash : \mathrm{C}\text{‘}R \neq \mathrm{V} \,.\, \supset \,.\, \mathrm{D}\text{‘}\overrightarrow{R} = \overrightarrow{R}\text{“}\mathrm{C}\text{‘}R \cup \iota\text{‘}\Lambda$ [*37·78 . *53·612]

***53·631.** $\vdash : \mathrm{D}\text{‘}R \neq \mathrm{V} \,.\, \supset \,.\, \mathrm{D}\text{‘}\overleftarrow{R} = \overleftarrow{R}\text{“}\mathrm{D}\text{‘}R \cup \iota\text{‘}\Lambda$ [*37·781 . *53·613]

***53·64.** $\vdash : \mathrm{C}\text{‘}R = \mathrm{V} \,.\, \supset \,.\, \mathrm{D}\text{‘}\overrightarrow{R} = \overrightarrow{R}\text{“}\mathrm{C}\text{‘}R$ [*37·78]

***53·641.** $\vdash : \mathrm{D}\text{‘}R = \mathrm{V} \,.\, \supset \,.\, \mathrm{D}\text{‘}\overleftarrow{R} = \overleftarrow{R}\text{“}\mathrm{D}\text{‘}R$ [*37·781]

*54. CARDINAL COUPLES

*Summary of *54.*

Couples are of two kinds, namely (1) $\iota\text{‘}x \cup \iota\text{‘}y$, in which there is no order as between x and y, and (2) $\iota\text{‘}x \uparrow \iota\text{‘}y$, in which there is an order. We may distinguish these two kinds of couples as cardinal and ordinal respectively, since (as will be shown hereafter) the class of all couples of the form $\iota\text{‘}x \cup \iota\text{‘}y$ (where $x \neq y$) is the cardinal number 2, while the class of all couples of the form $\iota\text{‘}x \uparrow \iota\text{‘}y$ (where $x \neq y$) is the ordinal number 2, to which, for the sake of distinction, we assign the symbol "2_r," where the suffix "r" stands for "relational," because the ordinal 2 is a class of relations. In the present and the following numbers, we shall define 2 and 2_r as the classes of cardinal and ordinal couples respectively, leaving it to a later stage to show that 2 and 2_r, so defined, are respectively a cardinal and an ordinal number. An ordinal couple will also be called an *ordered* couple or a *couple with sense*. Thus a couple with sense is a couple of which one comes first and the other second.

We introduce here the cardinal number 0, defined as $\iota\text{‘}\Lambda$. That 0 so defined is a cardinal number will be proved at a later stage; for the present, we postpone the proof that 0 so defined has the arithmetical properties of zero.

Cardinal couples are much less important, even in cardinal arithmetic, than ordinal couples, which will be considered in the two following numbers (*55 and *56). It is necessary, however, to prove some of the properties of cardinal couples, and this will be done in the present number. Some properties of cardinal couples which have been already proved are here repeated for convenience of reference. The definitions of 0 and 2 are:

***54·01.** $0 = \iota\text{‘}\Lambda$ Df

***54·02.** $2 = \hat{\alpha}\{(\exists x, y) \,.\, x \neq y \,.\, \alpha = \iota\text{‘}x \cup \iota\text{‘}y\}$ Df

Most of the propositions of the present number, except those that merely embody the definitions (*54·1·101·102), are used very seldom. The following are among the most important.

***54·26.** $\vdash : \iota\text{‘}x \cup \iota\text{‘}y \in 2 \,.\, \equiv \,.\, x \neq y$

***54·3.** $\vdash . 2 = \hat{\alpha}\{(\exists x) \,.\, x \in \alpha \,.\, \alpha - \iota\text{‘}x \in 1\}$

***54·4.** $\vdash :. \beta \subset \iota\text{‘}x \cup \iota\text{‘}y \,.\, \equiv : \beta = \Lambda \,.\, \vee \,.\, \beta = \iota\text{‘}x \,.\, \vee \,.\, \beta = \iota\text{‘}y \,.\, \vee \,.\, \beta = \iota\text{‘}x \cup \iota\text{‘}y$

***54·53.** $\vdash : \alpha \in 2 \,.\, x, y \in \alpha \,.\, x \neq y \,.\, \supset \,.\, \alpha = \iota\text{‘}x \cup \iota\text{‘}y$

***54·56.** $\vdash : \alpha \sim\in 0 \cup 1 \cup 2 \,.\, \equiv \,.\, (\exists x, y, z) \,.\, x, y, z \in \alpha \,.\, x \neq y \,.\, x \neq z \,.\, y \neq z$

***54·01.** $0 = \iota'\Lambda$ Df

***54·02.** $2 = \hat{\alpha}\{(\exists x, y) . x \neq y . \alpha = \iota'x \cup \iota'y\}$ Df

***54·1.** $\vdash . 0 = \iota'\Lambda$ [(*54·01)]

***54·101.** $\vdash : \alpha \epsilon 2 . \equiv . (\exists x, y) . x \neq y . \alpha = \iota'x \cup \iota'y$ [(*54·02)]

***54·102.** $\vdash : \alpha \epsilon 0 . \equiv . \alpha = \Lambda$ [*54·1]

The two following propositions have already occurred in *51, but are here repeated, because they belong to the subject of the present number.

***54·21.** $\vdash : \iota'x \cup \iota'y = \iota'x \cup \iota'z . \equiv . y = z$ [*51·41]

***54·22.** $\vdash :. \iota'x \cup \iota'y = \iota'z \cup \iota'w . \equiv : x = z . y = w . \vee . x = w . y = z$ [*51·43]

***54·25.** $\vdash : \iota'x \cup \iota'y \epsilon 1 . \equiv . x = y$

Dem.

$\vdash . *52·46·1 . *22·58 . \supset \vdash : \iota'x \cup \iota'y \epsilon 1 . \supset . \iota'x \cup \iota'y = \iota'x . \iota'x \cup \iota'y = \iota'y .$

[*20·23] $\supset . \iota'x = \iota'y$ (1)

$\vdash . *22·56 . \supset \vdash : \iota'x = \iota'y . \supset . \iota'x \cup \iota'y = \iota'x .$

[*52·22] $\supset . \iota'x \cup \iota'y \epsilon 1$ (2)

$\vdash . (1) . (2) . \supset \vdash : \iota'x \cup \iota'y \epsilon 1 . \equiv . \iota'x = \iota'y .$

[*51·23] $\equiv . x = y : \supset \vdash . \text{Prop}$

***54·26.** $\vdash : \iota'x \cup \iota'y \epsilon 2 . \equiv . x \neq y$

Dem.

$\vdash . *54·101 . \supset \vdash :: \iota'x \cup \iota'y \epsilon 2 .$

$\equiv :. (\exists z, w) . z \neq w . \iota'x \cup \iota'y = \iota'z \cup \iota'w :.$

[*54·22] $\equiv :. (\exists z, w) : z \neq w : x = z . y = w . \vee . x = w . y = z :.$

[*4·4.*11·41] $\equiv :. (\exists z, w) . z \neq w . x = z . y = w . \vee . (\exists z, w) . z \neq w . x = w . y = z :.$

[*13·22] $\equiv :. x \neq y . \vee . y \neq x :.$

[*13·16] $\equiv :. x \neq y :: \supset \vdash . \text{Prop}$

***54·27.** $\vdash . \iota'x \cup \iota'y \epsilon 1 \cup 2$ [*54·25·26]

***54·271.** $\vdash . 1 \cup 2 = \hat{\alpha}\{(\exists x, y) . \alpha = \iota'x \cup \iota'y\}$

Dem.

$\vdash . *4·42 . \supset$

$\vdash :. \alpha = \iota'x \cup \iota'y . \equiv : x = y . \alpha = \iota'x \cup \iota'y . \vee . x \neq y . \alpha = \iota'x \cup \iota'y$ (1)

$\vdash . (1) . *11·11·341·41 . \supset \vdash :. (\exists x, y) . \alpha = \iota'x \cup \iota'y .$

$\equiv : (\exists x, y) . x = y . \alpha = \iota'x \cup \iota'y . \vee . (\exists x, y) . x \neq y . \alpha = \iota'x \cup \iota'y :$

[*13·195] $\equiv : (\exists x) . \alpha = \iota'x \cup \iota'x . \vee . (\exists x, y) . x \neq y . \alpha = \iota'x \cup \iota'y :$

[*22·56] $\equiv : (\exists x) . \alpha = \iota'x . \vee . (\exists x, y) . x \neq y . \alpha = \iota'x \cup \iota'y :$

[*52·1.*54·101] $\equiv : \alpha \epsilon 1 . \vee . \alpha \epsilon 2 :$

[*22·34] $\equiv : \alpha \epsilon 1 \cup 2 :. \supset \vdash . \text{Prop}$

***54·3.** $\vdash . 2 = \hat{\alpha}\{(\exists x) . x \epsilon \alpha . \alpha - \iota'x \epsilon 1\}$

Dem.

$\vdash . *52·1 . *10·35 . \supset$

$\vdash : (\exists x) . x \epsilon \alpha . \alpha - \iota'x \epsilon 1 . \equiv . (\exists x, y) . x \epsilon \alpha . \alpha - \iota'x = \iota'y .$

$\left[*51·22 \dfrac{\iota'y, \alpha}{\alpha, \beta}\right]$ $\equiv . (\exists x, y) . \iota'x \cap \iota'y = \Lambda . \iota'x \cup \iota'y = \alpha .$

[*51·231.*54·101] $\equiv . \alpha \epsilon 2 : \supset \vdash . \text{Prop}$

***54·4.** $\vdash :. \beta \subset \iota'x \cup \iota'y . \equiv : \beta = \Lambda . \vee . \beta = \iota'x . \vee . \beta = \iota'y . \vee . \beta = \iota'x \cup \iota'y$

Dem.

$\vdash . *51·2 . \supset \vdash : x, y \epsilon \beta . \supset . \iota'x \cup \iota'y \subset \beta :$

[Fact] $\supset \vdash : \beta \subset \iota'x \cup \iota'y . x, y \epsilon \beta . \supset . \beta \subset \iota'x \cup \iota'y . \iota'x \cup \iota'y \subset \beta .$

[*22·41] $\supset . \beta = \iota'x \cup \iota'y$ (1)

$\vdash . *51·25 . \supset \vdash :. \beta \subset \iota'x \cup \iota'y . y \sim\epsilon \beta . \supset : \beta \subset \iota'x :$

[*51·401] $\supset : \beta = \Lambda . \vee . \beta = \iota'x$ (2)

Similarly $\vdash :. \beta \subset \iota'x \cup \iota'y . x \sim\epsilon \beta . \supset : \beta = \Lambda . \vee . \beta = \iota'y$ (3)

$\vdash . (2) . (3) . *3·48 . \supset$

$\vdash :. \beta \subset \iota'x \cup \iota'y . \sim(x, y \epsilon \beta) . \supset : \beta = \Lambda . \vee . \beta = \iota'x . \vee . \beta = \iota'y$ (4)

$\vdash . (1) . (4) . *34·8 . \supset$

$\vdash :. \beta \subset \iota'x \cup \iota'y . \supset : \beta = \Lambda . \vee . \beta = \iota'x . \vee . \beta = \iota'y . \vee . \beta = \iota'x \cup \iota'y$ (5)

$\vdash . *24·12 . *22·58·42 . \supset$

$\vdash :. \beta = \Lambda . \vee . \beta = \iota'x . \vee . \beta = \iota'y . \vee . \beta = \iota'x \cup \iota'y : \supset . \beta \subset \iota'x \cup \iota'y$ (6)

$\vdash . (5) . (6) . \supset \vdash . \text{Prop}$

This proposition shows that a class contained in a couple is either the null-class or a unit class or the couple itself, whence it will follow that 0 and 1 are the only numbers which are less than 2.

***54·41.** $\vdash :: \alpha \epsilon 2 . \supset :. \beta \subset \alpha . \supset : \beta = \Lambda . \vee . \beta \epsilon 1 . \vee . \beta \epsilon 2$

Dem.

$\vdash . *52·1 . \supset \vdash :. \beta = \iota'x . \vee . \beta = \iota'y : \supset . \beta \epsilon 1$ (1)

$\vdash . *54·26 . \supset \vdash :. x \neq y . \supset : \beta = \iota'x \cup \iota'y . \supset . \beta \epsilon 2$ (2)

$\vdash . (1) . (2) . *54·4 . \supset$

$\vdash :: x \neq y . \supset :. \beta \subset \iota'x \cup \iota'y . \supset : \beta = \Lambda . \vee . \beta \epsilon 1 . \vee . \beta \epsilon 2 ::$

[*13·12] $\supset \vdash :: \alpha = \iota'x \cup \iota'y . x \neq y . \supset :. \beta \subset \alpha . \supset : \beta = \Lambda . \vee . \beta \epsilon 1 . \vee . \beta \epsilon 2 ::$

[11·11·35] $\supset$

$\vdash :. (\exists x, y) . \alpha = \iota'x \cup \iota'y . x \neq y . \supset :. \beta \subset \alpha : \beta = \Lambda . \vee . \beta \epsilon 1 . \vee . \beta \epsilon 2$ (3)

$\vdash . (3) . *54·101 . \supset \vdash . \text{Prop}$

***54·411.** $\vdash :. \alpha \epsilon 2 . \supset : \beta \subset \alpha . \supset . \beta \epsilon 0 \cup 1 \cup 2$ [*54·41·102]

***54·42.** $\vdash :: \alpha \in 2 . \supset :. \beta \subset \alpha . \exists ! \beta . \beta \neq \alpha . \equiv . \beta \in \iota''\alpha$

Dem.

$\vdash . *54\cdot4 . \supset \vdash :: \alpha = \iota'x \cup \iota'y . \supset :.$

$\beta \subset \alpha . \exists ! \beta . \equiv : \beta = \Lambda . \vee . \beta = \iota'x . \vee . \beta = \iota'y . \vee . \beta = \alpha : \exists ! \beta :$

[*24·53·56 . *51·161] $\equiv : \beta = \iota'x . \vee . \beta = \iota'y . \vee . \beta = \alpha$ (1)

$\vdash . *54\cdot25 . \text{Transp} . *52\cdot22 . \supset \vdash : x \neq y . \supset . \iota'x \cup \iota'y \neq \iota'x . \iota'x \cup \iota'y \neq \iota'y :$

[*13·12] $\supset \vdash : \alpha = \iota'x \cup \iota'y . x \neq y . \supset . \alpha \neq \iota'x . \alpha \neq \iota'y$ (2)

$\vdash . (1) . (2) . \supset \vdash :: \alpha = \iota'x \cup \iota'y . x \neq y . \supset :.$

$\beta \subset \alpha . \exists ! \beta . \beta \neq \alpha . \equiv : \beta = \iota'x . \vee . \beta = \iota'y :$

[*51·235] $\equiv : (\exists z) . z \in \alpha . \beta = \iota'z :$

[*37·6] $\equiv : \beta \in \iota''\alpha$ (3)

$\vdash . (3) . *11\cdot11\cdot35 . *54\cdot101 . \supset \vdash . \text{Prop}$

***54·43.** $\vdash :. \alpha, \beta \in 1 . \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \in 2$

Dem.

$\vdash . *54\cdot26 . \supset \vdash :. \alpha = \iota'x . \beta = \iota'y . \supset : \alpha \cup \beta \in 2 . \equiv . x \neq y .$

[*51·231] $\equiv . \iota'x \cap \iota'y = \Lambda .$

[*13·12] $\equiv . \alpha \cap \beta = \Lambda$ (1)

$\vdash . (1) . *11\cdot11\cdot35 . \supset$

$\vdash :. (\exists x, y) . \alpha = \iota'x . \beta = \iota'y . \supset : \alpha \cup \beta \in 2 . \equiv . \alpha \cap \beta = \Lambda$ (2)

$\vdash . (2) . *11\cdot54 . *52\cdot1 . \supset \vdash . \text{Prop}$

From this proposition it will follow, when arithmetical addition has been defined, that $1 + 1 = 2$.

***54·44.** $\vdash :. z, w \in \iota'x \cup \iota'y . \supset_{z,w} . \phi(z,w) : \equiv . \phi(x,x) . \phi(x,y) . \phi(y,x) . \phi(y,y)$

Dem.

$\vdash . *51\cdot234 . *11\cdot62 . \supset \vdash :. z, w \in \iota'x \cup \iota'y . \supset_{z,w} . \phi(z,w) : \equiv :$

$z \in \iota'x \cup \iota'y . \supset_z . \phi(z,x) . \phi(z,y) :$

[*51·234 . *10·29] $\equiv : \phi(x,x) . \phi(x,y) . \phi(y,x) . \phi(y,y) :. \supset \vdash . \text{Prop}$

***54·441.** $\vdash :: z, w \in \iota'x \cup \iota'y . z \neq w . \supset_{z,w} . \phi(z,w) : \equiv :. x = y : \vee : \phi(x,y) . \phi(y,x)$

Dem.

$\vdash . *5\cdot6 . \supset \vdash :: z, w \in \iota'x \cup \iota'y . z \neq w . \supset_{z,w} . \phi(z,w) : \equiv :.$

$z, w \in \iota'x \cup \iota'y . \supset_{z,w} : z = w . \vee . \phi(z,w) :.$

[*54·44] $\equiv : x = x . \vee . \phi(x,x) : x = y . \vee . \phi(x,y) :$

$y = x . \vee . \phi(y,x) : y = y . \vee . \phi(y,y) :$

[*13·15] $\equiv : x = y . \vee . \phi(x,y) : y = x . \vee . \phi(y,x) :$

[*13·16 . *4·41] $\equiv : x = y . \vee . \phi(x,y) . \phi(y,x)$

This proposition is used in *163·42, in the theory of relations of mutually exclusive relations.

***54·442.** $\vdash :: x \neq y . \supset :. z, w \in \iota'x \cup \iota'y . z \neq w . \supset_{z,w} . \phi(z,w) : \equiv . \phi(x,y) . \phi(y,x)$ [*54·441]

***54·443.** $\vdash :: x \neq y : \phi(x,y) . \equiv . \phi(y,x) : \supset :.$

$z, w \in \iota'x \cup \iota'y . z \neq w . \supset_{z,w} . \phi(z,w) : \equiv . \phi(x,y)$ [*54·442]

***54·45.** $\vdash :. (\exists z, w) . z, w \in \iota'x \cup \iota'y . \phi(z,w) .$

$\equiv : \phi(x,x) . \vee . \phi(x,y) . \vee . \phi(y,x) . \vee . \phi(y,y)$ [*51·235]

***54·451.** $\vdash :: \sim\phi(x,x) . \sim\phi(y,y) . \supset :. (\exists z, w) . z, w \in \iota'x \cup \iota'y . \phi(z,w) .$

$\equiv : \phi(x,y) . \vee . \phi(y,x)$ [*54·45]

***54·452.** $\vdash :: \sim\phi(x,x) . \sim\phi(y,y) : \phi(x,y) . \equiv . \phi(y,x) : \supset :$

$(\exists z, w) . z, w \in \iota'x \cup \iota'y . \phi(z,w) . \equiv . \phi(x,y)$ [*54·451]

***54·46.** $\vdash : (\exists z, w) . z, w \in \iota'x \cup \iota'y . z \neq w . \equiv . x \neq y$ [*54·452 . *13·15·16]

***54·5.** $\vdash :. \alpha \in 2 . \supset : \alpha \subset \iota'z \cup \iota'w . \equiv . \alpha = \iota'z \cup \iota'w$

Dem.

$\vdash . *54\cdot4 . \supset$

$\vdash :. \alpha \subset \iota'z \cup \iota'w . \supset : \alpha = \Lambda . \vee . \alpha = \iota'z . \vee . \alpha = \iota'w . \vee . \alpha = \iota'z \cup \iota'w$ (1)

$\vdash . *54\cdot3 . *24\cdot54 . \quad \supset \vdash : \text{Hp} . \supset . \alpha \neq \Lambda$ (2)

$\vdash . *54\cdot26 \frac{z, z}{x, y} . *13\cdot15 . \quad \supset \vdash : \text{Hp} . \supset . \alpha \neq \iota'z$ (3)

$\vdash . (3) \frac{w}{z} . \quad \supset \vdash : \text{Hp} . \supset . \alpha \neq \iota'w$ (4)

$\vdash . (1) . (2) . (3) . (4) . *2\cdot53 . \supset \vdash :. \text{Hp} . \supset : \alpha \subset \iota'z \cup \iota'w . \supset . \alpha = \iota'z \cup \iota'w$ (5)

$\vdash . *22\cdot42 . \quad \supset \vdash : \alpha = \iota'z \cup \iota'w . \supset . \alpha \subset \iota'z \cup \iota'w$ (6)

$\vdash . (5) . (6) . \supset \vdash . \text{Prop}$

***54·51.** $\vdash :. \alpha \in 2 . \beta \in 1 \cup 2 . \supset : \alpha \subset \beta . \equiv . \alpha = \beta$

Dem.

$\vdash . *54\cdot5 . \supset \vdash :. \alpha \in 2 . \beta = \iota'z \cup \iota'w . \supset : \alpha \subset \beta . \equiv . \alpha = \beta$ (1)

$\vdash . (1) . *11\cdot11\cdot35\cdot45 . \supset$

$\vdash :. \alpha \in 2 : (\exists z, w) . \beta = \iota'z \cup \iota'w : \supset : \alpha \subset \beta . \equiv . \alpha = \beta$ (2)

$\vdash . (2) . *54\cdot271 . \supset \vdash . \text{Prop}$

***54·52.** $\vdash :. \alpha, \beta \in 2 . \supset : \alpha \subset \beta . \equiv . \alpha = \beta . \equiv . \beta \subset \alpha$ [*54·51]

***54·53.** $\vdash : \alpha \in 2 . x, y \in \alpha . x \neq y . \supset . \alpha = \iota'x \cup \iota'y$

Dem.

$\vdash . *51\cdot2 . \quad \supset \vdash : \text{Hp} . \supset . \iota'x \subset \alpha . \iota'y \subset \alpha .$

[*22·59] $\supset . \iota'x \cup \iota'y \subset \alpha$ (1)

$\vdash . *54\cdot26 . \quad \supset \vdash : \text{Hp} . \supset . \iota'x \cup \iota'y \in 2$ (2)

$\vdash . (1) . (2) . *54\cdot52 . \supset \vdash . \text{Prop}$

***54·531.** $\vdash :. \alpha \in 2 . \supset : x, y \in \alpha . x \neq y . \equiv . \alpha = \iota'x \cup \iota'y$

Dem.

$\vdash . *54\cdot53 . \text{Exp} . \supset \vdash :. \alpha \in 2 . \supset : x, y \in \alpha . x \neq y . \supset . \alpha = \iota'x \cup \iota'y$ (1)

$\vdash . *54\cdot26 . \quad \supset \vdash :. \alpha \in 2 . \supset : \alpha = \iota'x \cup \iota'y . \supset . x \neq y$ (2)

$\vdash . *51\cdot16 . \quad \supset \vdash : \alpha = \iota'x \cup \iota'y . \supset . x, y \in \alpha$ (3)

$\vdash . (2) . (3) . \quad \supset \vdash :. \alpha \in 2 . \supset : \alpha = \iota'x \cup \iota'y . \supset . x, y \in \alpha . x \neq y$ (4)

$\vdash . (1) . (4) . \quad \supset \vdash . \text{Prop}$

***54·54.** $\vdash :. \alpha \in 2 . \equiv : x, y \in \alpha . x \neq y . \supset_{x,y} . \alpha = \iota'x \cup \iota'y : (\exists x, y) . x, y \in \alpha . x \neq y$

Dem.

$\vdash . *54\cdot531 . *11\cdot11\cdot3 . \supset \vdash :. \alpha \in 2 . \supset : x, y \in \alpha . x \neq y . \supset_{x,y} . \alpha = \iota'x \cup \iota'y \quad (1)$

$\vdash . *51\cdot16 . *54\cdot101 . \supset \vdash : \alpha \in 2 . \supset . (\exists x, y) . x, y \in \alpha . x \neq y \quad (2)$

$\vdash . *5\cdot3 . *3\cdot27 . \supset \vdash :. x, y \in \alpha . x \neq y . \supset . \alpha = \iota'x \cup \iota'y : \supset :$
$x, y \in \alpha . x \neq y . \supset . x \neq y . \alpha = \iota'x \cup \iota'y :.$

$[*11\cdot11\cdot32\cdot34] \quad \supset \vdash :. x, y \in \alpha . x \neq y . \supset_{x,y} . \alpha = \iota'x \cup \iota'y : \supset :$
$(\exists x, y) . x, y \in \alpha . x \neq y . \supset . (\exists x, y) . x \neq y . \alpha = \iota'x \cup \iota'y \quad (3)$

$\vdash . (3) . \text{Imp} . *54\cdot101 . \supset \vdash :. x, y \in \alpha . x \neq y . \supset_{x,y} . \alpha = \iota'x \cup \iota'y :$
$(\exists x, y) . x, y \in \alpha . x \neq y : \supset . \alpha \in 2 \quad (4)$

$\vdash . (1) . (2) . (4) . \quad \supset \vdash . \text{Prop}$

In the above proposition, "$x, y \in \alpha . x \neq y . \supset_{x,y} . \alpha = \iota'x \cup \iota'y$" secures that α has not *more* than two members, while "$(\exists x, y) . x, y \in \alpha . x \neq y$" secures that α has not *fewer* than two members.

***54·55.** $\vdash . 0 \cup 1 \cup 2 = \hat{\alpha}\{x, y \in \alpha . x \neq y . \supset_{x,y} . \alpha = \iota'x \cup \iota'y\}$

Dem.

$\vdash . *4\cdot42 . \supset \vdash :: x, y \in \alpha . x \neq y . \supset_{x,y} . \alpha = \iota'x \cup \iota'y : \equiv :.$
$x, y \in \alpha . x \neq y . \supset_{x,y} . \alpha = \iota'x \cup \iota'y : \sim(\exists x, y) . x, y \in \alpha . x \neq y :.$
$\vee :. x, y \in \alpha . x \neq y . \supset_{x,y} . \alpha = \iota'x \cup \iota'y : (\exists x, y) . x, y \in \alpha . x \neq y \quad (1)$

$\vdash . *11\cdot63 . \supset \vdash :. \sim(\exists x, y) . x, y \in \alpha . x \neq y . \supset : x, y \in \alpha . x \neq y . \supset_{x,y} . \alpha = \iota'x \cup \iota'y :.$

$[*4\cdot71] \quad \supset \vdash :. x, y \in \alpha . x \neq y . \supset_{x,y} . \alpha = \iota'x \cup \iota'y : \sim(\exists x, y) . x, y \in \alpha . x \neq y : \equiv :$
$\sim(\exists x, y) . x, y \in \alpha . x \neq y :$

$[*11\cdot521] \quad \equiv : x, y \in \alpha . \supset_{x,y} . x = y :$

$[*52\cdot4] \quad \equiv : \alpha \in 0 \cup 1 \quad (2)$

$\vdash . (1) . (2) . *54\cdot54 . \supset$

$\vdash :. x, y \in \alpha . x \neq y . \supset_{x,y} . \alpha = \iota'x \cup \iota'y : \equiv : \alpha \in 0 \cup 1 . \vee . \alpha \in 2 :$

$[*22\cdot34] \quad \equiv : \alpha \in 0 \cup 1 \cup 2 :. \supset \vdash . \text{Prop}$

***54·56.** $\vdash : \alpha \sim\in 0 \cup 1 \cup 2 . \equiv . (\exists x, y, z) . x, y, z \in \alpha . x \neq y . x \neq z . y \neq z$

Dem.

$\vdash . *54\cdot55 . *11\cdot52 . \supset$

$\vdash :. \alpha \sim\in 0 \cup 1 \cup 2 . \equiv : (\exists x, y) . x, y \in \alpha . x \neq y . \alpha \neq \iota'x \cup \iota'y :$

$[*51\cdot2 . *22\cdot59] \quad \equiv : (\exists x, y) . \iota'x \cup \iota'y \subset \alpha . x \neq y . \alpha \neq \iota'x \cup \iota'y :$

$[*24\cdot6] \quad \equiv : (\exists x, y) . \iota'x \cup \iota'y \subset \alpha . x \neq y . \exists ! \alpha - (\iota'x \cup \iota'y) :$

$[*51\cdot232 . \text{Transp}] \quad \equiv : (\exists x, y) : \iota'x \cup \iota'y \subset \alpha . x \neq y : (\exists z) . z \in \alpha . z \neq x . z \neq y :$

$[*51\cdot2 . *22\cdot59] \quad \equiv : (\exists x, y, z) . x, y, z \in \alpha . x \neq y . x \neq z . y \neq z :. \supset \vdash . \text{Prop}$

In virtue of this proposition, a class which is neither null nor a unit class nor a couple contains at least three distinct members. Hence it will follow that any cardinal number other than 0 or 1 or 2 is equal to or greater than 3. The above proposition is used in *104·43, which is an existence-theorem of considerable importance in cardinal arithmetic.

***54·6.** $\vdash :. \alpha \cap \beta = \Lambda . x, x' \in \alpha . y, y' \in \beta . \supset :$
$\iota'x \cup \iota'y = \iota'x' \cup \iota'y' . \equiv . x = x' . y = y'$

Dem.

$\vdash . *51\cdot2 . \supset \vdash :. \text{Hp} . \supset : \iota'x \subset \alpha . \iota'x' \subset \alpha . \iota'y \subset \beta . \iota'y' \subset \beta . \alpha \cap \beta = \Lambda :$

$[*24\cdot48] \quad \supset : \iota'x \cup \iota'y = \iota'x' \cup \iota'y' . \equiv . \iota'x = \iota'x' . \iota'y = \iota'y' .$

$[*51\cdot23] \quad \equiv . x = x' . y = y' :. \supset \vdash . \text{Prop}$

The above proposition is useful in dealing with sets of couples formed of one member of a class α and one member of a class β, where α and β have no members in common. It is used in the theory of cardinal multiplication (*113·148).