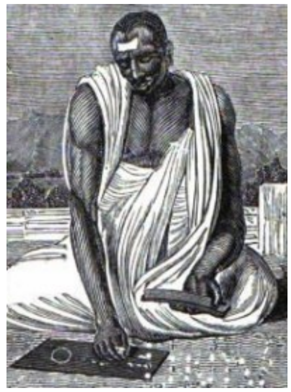

(A1) Run the Euclidean algorithm on $(21, 17)$, and use this to find integers r, s such that $21r + 17s = 1$.

(A2) Fix a value N. Let's say that (x, y, k) is a *Brahmagupta triple* if $x^2 - Ny^2 = k$. Brahmagupta's identity says that if (x_1, y_1, k_1) and (x_2, y_2, k_2) are Brahmagupta triples, then so is

$$(x_1x_2 + Ny_1y_2, \quad x_1y_2 + x_2y_1, \quad k_1k_2).$$

Let $N = 2$ and list the first 15 Brahmagupta triples of natural numbers in lexicographic order. The first few are $(2, 1, 2), (3, 1, 7)$, and $(3, 2, 1)$.

(B1) Prove Brahmagupta's identity using the fact that

$$(x_1 - Ny_1^2)(x_2 - Ny_2^2) = (x_1 - \sqrt{N}y_1)(x_1 + \sqrt{N}y_1)(x_2 - \sqrt{N}y_2)(x_2 + \sqrt{N}y_2)$$

and combining the first factor with the third and the second factor with the fourth.

(A3) For $N = 5$ and $N = 7$, try Brahmagupta's method for solving the equation $x^2 - Ny^2 = 1$. To do this, use trial and error to find a few triples (x, y, k) with a small value of k (but not 1) and then compose them with themselves or each other. This may not produce a solution!

(B2) Explain why a solution to $x^2 - Ny^2 = 1$ satisfies $\frac{x}{y} \approx \sqrt{N}$, and express the error in terms of x and y. What has to be true of x and y to get a close approximation? Going the other way, could you use a good rational approximation to $\sqrt{N}$ to try to solve Pell's equation? Using your knowledge of rational approximations, try this for $N = 5$ and $N = 7$. Now try $N = 77$. (Hint: to help you with the continued fraction calculations, google "continued fraction calculator"...)