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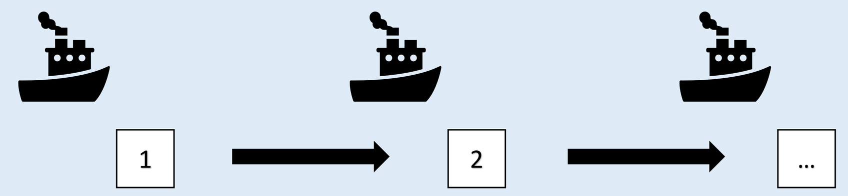
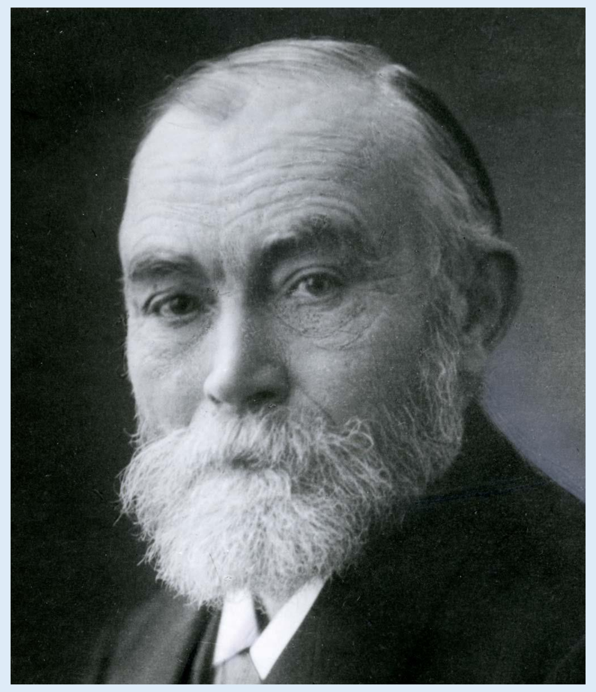
Frege's Development of Logic



How did we go from this, ...to this?

Aristotelian Logic 4th century BCE on	Mathematical Logic 20th century Developed 1879-1931
All humans are mortal, All Greeks are humans, Therefore, all Greeks are mortal	$(\forall x(Gx \rightarrow Hx) \& \forall x(Fx \rightarrow Gx)) \rightarrow \forall x(Fx \rightarrow Hx)$



Gottlob Frege, 1848-1925

Mathematical motives

The late 18th and early 19th centuries saw rapid developments in mathematics (Non-Euclidian geometries, irrational and imaginary numbers) which threw new questions on the objects of math right under the mathematicians' noses. Immanuel Kant went on to claim mathematics as synthetic a priori, meaning they are universal truths that nonetheless depend on our minds because we all share the same faculty: intuition of space and time. Counting natural numbers then occurs because we experience time in sequence. As with the figure above, you count the boats going by in virtue of them going by in time sequentially.

This move bothered Frege because although he could accept geometry as representing our intuition of space, it isn't intuitive at all that arithmetic is a generation of the mind rather than discovery of necessary laws. Arithmetic seems instead to be a matter of purely mechanical manipulation of relations among abstract entities. He thus set out to de-psychologize mathematics by grounding it in a system of logic.

Formal Systems

Frege endeavored to create a formal system strong enough to show that all the truths of mathematics are truths of logic. Aristotle's logic was already considered "formal" in that it recognizes you can capture sentences which are true solely in virtue of their structure. By modeling logic after mathematics, Frege's system presents a logic where such truths of structure can be proven purely through limited manipulation of finite types of symbols. This requires it to be rigorous and expressive in ways Aristotle's logic could never be. He then couldn't get it off the ground without the Concept and Object insight. At the same time, the expressive power Frege's system affords needs the strength of a formal system to prove truths via a mechanical procedure (substitution and modus ponens) exactly because it isn't subject to the limitations that Aristotelian logic is subject to.

Selected Bibliography:
The Cambridge Companion to Frege
Frege, Gottlob: *Begriffsschrift*, *The Foundations of Arithmetic,* "Function and Concept," "On Concept and Object," (Other readings in Beaney's *Frege Reader)*
Kant, Immanuel, *Prolegomena*
Kneale & Kneale, *The Development of Logic*
Russell, Bertrand, *The Principles of Mathematics*

The Picture:

A fundamental change occurred between the system of Aristotelian logic and contemporary formal logic. It wasn't just that they were represented in different ways. The goal of a research project looking into the interim period of development was a historical look into what the major changes were and what motivated them so as to get a sense of what kind of change this was. I looked at the full period across a range of logicians from Gottlob Frege to Kurt Gödel. Frege however played such an important and special role in this transition that he is the one most worth focusing on.

The traditional wisdom, which is generally correct, is that Frege invented his system of formal logic, the *Begriffsschrift* (1879) as a tool to prove the logical basis of arithmetic. He even says as much in his account. However it doesn't explain the particular insights he employed: why those particular insights, and how did he come to them? I have come upon his concept and object distinction as an **extra-logical linguistic insight** which is not only a primitive feature of his philosophy of mathematics, but also as a feature from which the most important features of his system can be derived once brought into a logical system like he was attempting before his insight.

Concept and Object (*Initially "Function" and "Argument"*)

The Aristotelian syllogisms are based on predication (A is B). For the rigor of a formal system it is both too restrictive and too vague. The only syllogisms that are employed must be one out of only four, which may then only be arranged in finite ways. And there are roughly 3 senses of the word "is": predication of a named subject (John is Bald), predication of predicates (Greeks are Humans), and identity (Cicero is Tully). Part of the point of formalization for Frege is to remove these ambiguities of language when dealing with objects of logic, but since it is so embedded in the Aristotelian system he had to abandon that entirely for a new form of predication away from the natural grammar of subject and predicate.

Predication is to be understood as a concept falling under an object. What is a concept? Simply anything with a "blank" in which something is being attributed that property. By eliminating the object, "grass" from "grass is green" we get the concept "___ is green" or Gx. A concept is anything that can be blanked out like so. An object then is anything that could possibly fall under a concept. A variable x is put in place of blanks, where named objects may then be put in its place. <u>For Frege this is the logical grammar which underlies natural language, allowing its expression in logic</u>.

Socrates is Mortal
Is Mortal (____)
Is Mortal (Socrates)
Mx

Breakdown of the Aristotelian Syllogism

A major feature of Aristotle's syllogisms is their rigidity. "All As are Bs" is taken as a unit. When Frege rewrote predication as an object falling under a concept, that in effect shook loose what turned out to be its constitutive parts.

- Quantification

Typically considered the most important contribution to logic, if not philosophy as a whole. A quantifier binds the object falling under the concept to specify how many objects. In Aristotle it was "some" and "all", so here there is the universal quantifier (∀x or "for absolutely every x"), and existential quantifier (∃x or "there is at least one x"). "All As are Bs" is now "for every x, if that x falls under A, then it falls under B" or ∀x(Ax→Bx). Since these are independent quantifiers, they can range over variables across multiple parentheses while other nested quantifiers bind to different variables within that same sentence.

- Multi place relations

With objects quantifying into variables falling under concepts, the notorious problem of depicting relations in logic becomes trivial: it is a concept with two or more variables. Lxy where the concept L stands for "loves" is "x loves y."

- Identity

The Aristotelian system had a hard time distinguishing the "is" of predication (John is bald) from the "is" of identity (Cicero is Tully). Relations already taken care of, identity can then be said to be a privileged kind of relation, x=y.

- Negation

Negation is its own class of syllogism for Aristotle, "No As are Bs" or "Some As are Not Bs." In Frege negation now has its own sign (~) and can be used freely. Not only is this more expressive in general (multiple negation), but you can derive other logical connectives (&, v) through just one connective and negation (same with quantifiers).

Beyond Frege:

Frege was overlooked aside from his theory of numbers. His ideas found popularity in philosophy when Bertrand Russell adopted his use of concept and object and quantification in "On Denoting" and *Principia Mathematica* which continued his logicist program with the incorporation of set theory.