An Asymptotically Capacity-Achieving Scheme for the Gaussian Relay Channel with Relay-Destination Cooperation

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***Abstract*—This paper shows that the capacity of the relay channel can be asymptotically achieved by allowing relay-destination cooperation. Consider a scheme with rate splitting and second-order superposition block Markov encoding at the source, partial decode-forward (PDF) relaying and joint decoding over two consecutive blocks at the relay, and quantize-forward relaying and backward decoding at the destination. This scheme includes as special cases the classical PDF scheme and the destination feedback scheme of the relay channel. In Gaussian channels, the proposed scheme outperforms noisy network coding scheme except for weak source-relay link. For all channel gains, it asymptotically achieves the capacity by reaching the cutset bound as the destination power approaches infinity.**

I. Introduction

Recently, cooperative communication has received a heightened research interest as an efficient method to improve the achievable rates of multi-user channels. As the relay channel resembles the basic model for cooperation, it is important to understand the theoretical limits for cooperative scenarios in a relay channel.

The relay channel was first introduced by van der Meulen in [1]. Cover and El Gamal in [2] propose several fundamental coding schemes for this channel including the decode-forward (DF), partial decode-forward (PDF), and compress-forward (CF) relaying schemes. As these schemes assume full-duplex operation at the relay, an extension to the half-duplex mode is studied by Host-Madsen and Zhang in [3].

In the schemes in [2], [3], the relay receives signals from the source only. However, in practical applications such as the uplink in cellular networks, the destination has much more power than the mobiles. It is then of interest to study how the destination can cooperate with either the source or the relay in order to increase the achievable rate. In this paper, we focus on the relay channel with relay-destination cooperation (RDC).

The relay channel with RDC is first defined in [2] as a relay channel with destination feedback. It is shown in [2] that if the relay has causal knowledge of the channel output at the destination, the DF scheme is capacity achieving since the channel becomes a physically degraded channel. Gabbai and Bross in [4] provides coding schemes based on DF and CF for the relay channel with causal, perfect feedback from the destination or the relay to the source. In [5], Jiang et al. consider more realistic scenarios where the feedback information is non-causally available at the relay or the source. They propose different coding schemes for the relay channel with relay-source cooperation and relay-destination cooperation.

In this paper, we consider the relay channel with RDC and show how this cooperation helps achieve the capacity asymptotically. We propose a coding scheme in which the source splits its message into a private and a public part and employs-second order superposition block Markov encoding; the relay employs PDF relaying and simultaneous joint decoding over two consecutive blocks; and the destination employs quantize-forward (QF) relaying and backward decoding. Different from the destination feedback scheme (DFB) in [5] where transmitted signal in block i depends the signal transmitted in block $i-2$ only , the transmission in block i in the proposed RDC scheme depends on the signals transmitted in both blocks $i-1$ and $i-2$. Moreover, the relay in RDC decodes the public message part and the quantization index using joint decoding instead of the sequential decoding as used in the DFB in [5].

This paper proves that the proposed scheme is asymptotically capacity achieving when the destination power approaches infinity. This result agrees with the intuition that when the power approaches infinity, the destination virtually joins the relay in one entity because of the infinite capacity feedback link. This scenario is approximately feasible in practice when the destination power is much higher than the source power as in the uplink of cellular networks. For non-asymptotic regimes, the papers provides numerical results comparing between the proposed RDC, DFB [5], NNC [6] and the cut-set bound. Numerical examples show that the proposed RDC is better than the existing schemes except for weak source-relay link where the NNC scheme is preferred.

II. Channel Model

A. Discrete memoryless channel model

The relay channel with relay-destination (RD) cooperation consists of three input alphabets $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{X}_3$; two output alphabets $\mathcal{Y}_2$ and $\mathcal{Y}_3$; and one conditional transition probability $p(y_2,y_3|x_1,x_2,x_3)$ as shown in Figure 1

A $(\lceil 2^{nR}\rceil, n)$ code for this channel consists of one message set $W_1=\{1,\ldots,\lceil 2^{nR_1}\rceil\}$, three encoding functions $f_{1i}, f_{2i}, f_{3i},\quad i=1,\ldots,n$, and one decoding function g defined as

$$\begin{aligned}
f_{1i} &: W_1 \to \mathcal{X}_1, \quad i=1,\ldots,n\\
f_{2i} &: \mathcal{Y}_2^{i-1} \to \mathcal{X}_2,\ i=1,\ldots,n\\
f_{3i} &: \mathcal{Y}_3^{i-1} \to \mathcal{X}_3,\ i=1,\ldots,n\\
g &: \mathcal{Y}_3^{n} \to W_1.
\end{aligned} \tag{1}$$

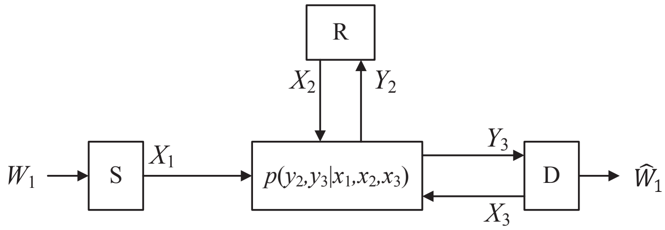

Fig. 1. Model of the relay channel with RD cooperation.

Definitions for the average error probability, achievable rate and capacity region follow the standard ones in [7].

B. Gaussian Channel Model

The discrete-time standard model for the relay channel with RD cooperation over Gaussian noises can be expressed as

$$\begin{aligned} Y_2 &= g_{12}X_1 + g_{32}X_3 + Z_2, \\ Y_3 &= g_{13}X_1 + g_{23}X_2 + Z_3, \end{aligned} \tag{2}$$

where g_{ij} for $i \in \{1,2\}$ and $j \in \{2,3\}$ are the link coefficients from node i to j; $Z_l \sim CN(0,1),\ l \in \{2,3\}$ are independent complex AWGN.

III. Coding Scheme

The coding scheme is based on block Markov superposition encoding at the source, partial decode-forward relaying at the relay, and quantize-forward relaying and backward decoding at the destination. The source splits its message W_1 into two parts (W_{12}, W_{13}) sent at rates R_{12} and R_{13}, respectively $(R_1 = R_{12}+R_{13})$. W_{13} is the private message part decoded at the destination only whereas W_{12} is the public part decoded at both the destination and the relay. In block i, the source performs consecutive superposition encoding for $W_{12,i-2}$, $W_{12,i-1}$, $W_{12,i}$ and $W_{13,i}$ and generates the codewords X_2, U, V and X_1, respectively. The relay utilizes the received signals from the source and the destination to decode $W_{12,i}$ using joint decoding over two consecutive blocks. During the transmission, the destination quantizes its received signal and sends the quantization index in the next block through X_3. Then, it employs backward decoding to decode all transmitted messages.

A. Achievable Rate

Theorem 1. *For the relay channel with RD cooperation, the achievable rate* (R_1) *is given as*

$$R_1 \leq \max_{p_r} \min\{I_1, I_2, I_3\}, \tag{3}$$

where

$$\begin{aligned} I_1 &= I(X_1;Y_3|X_3,X_2,V,U) + I(U;Y_2|X_2,X_3) \\ &\quad + I(V;Y_2,\hat{Y}_3|X_3,X_2,U), \\ I_2 &= I(X_1;Y_3|X_3,X_2,V,U) + I(V,U,X_3;Y_2|X_2) \\ &\quad - I(\hat{Y}_3;Y_3|X_3,U,V,X_2,Y_2), \\ I_3 &= I(X_1,X_2;Y_3|X_3), \end{aligned} \tag{4}$$

and the maximization is over $p_r = p(x_2)p(u|x_2), p(v|u,x_2)$ $p(x_1|u,v,x_2)p(x_3)p(\hat{y}_3|y_3x_3)$ *subject to*

$$I(\hat{Y}_3;Y_3|X_3,U,V,X_2,Y_2) \leq I(X_3;Y_2|X_2,U). \tag{5}$$

Next, we explain the codebook generation, encoding and decoding of coding scheme.

B. Codebook Generation

A sequence of $B-2$ messages is to be sent over the channel in nB transmission. Therefore, no new information is sent in the last two blocks $(B-1$ and $B)$. This reduces the achievable rates in (4) by a factor of $2/B$; however, as $B \to \infty$, this factor becomes negligible.

The codebook generation in block i is given as follows. After fixing p_r as in Theorem 1,

- Generate $2^{nR_{12}}$ i.i.d sequences $x_2^n(w_{12,i-2}) \sim \prod_{k=1}^n p(x_{2k})$.
- For each $x_2^n(w_{12,i-2})$, generate $2^{nR_{12}}$ i.i.d sequences $u^n(w_{12,i-1}, w_{12,i-2}) \sim \prod_{k=1}^n p(u_k|x_{2k})$.
- For each pair $x_2^n(w_{12,i-2})$ and $u^n(w_{12,i-1}, w_{12,i-2})$, generate $2^{nR_{12}}$ i.i.d sequences $v^n(w_{12,i}, w_{12,i-1}, w_{12,i-2}) \sim \prod_{k=1}^n p(v_k|u_k, x_{2k})$.
- For each triplet $x_2^n(w_{12,i-2})$, $u^n(w_{12,i-1}, w_{12,i-2})$ and $v^n(w_{12,i}w_{12,i-1}, w_{12,i-2})$, generate $2^{nR_{13}}$ i.i.d sequences $x_1^n(w_{13,i}, w_{12,i}w_{12,i-1}, w_{12,i-2}) \sim \prod_{k=1}^n p(x_{1k}|v_k, u_k, x_{2k})$.
- Generate 2^{nR_3} i.i.d sequences $x_3^n(l_{i-1}) \sim \prod_{k=1}^n p(x_{3k})$.
- For each $x_3^n(l_{i-1})$, generate 2^{nR_3} i.i.d sequences $\hat{y}_3^n(l_i, l_{i-1}) \sim \prod_{k=1}^n p(\hat{y}_{3k}, x_{3k})$.

1) Encoding: Let (w_{12}, w_{13}) be the new messages to be sent in block i. R has an estimate $(\hat{w}_{12,i-1})$ of $(w_{12,i-1})$ while D has l_{i-1}. Therefore, S sends $x_1^n(w_{13,i}, w_{12,i}w_{12,i-1}, w_{12,i-2})$, R sends $x_2^n(w_{12,i-2})$, and D sends $x_3^n(l_{i-1})$.

2) Decoding:

At the relay: At the end of block i, R already knows $l_{i-1} = L_{i-1}$, $w_{12,i-2} = 1$ and $w_{12,i-1} = 1$ from the decoding in blocks $i-1$ and i. Then, R employs joint decoding over blocks i and $i+1$ to find a unique pair $(\hat{w}_{12,i}, \hat{l}_i)$ such that

$$\begin{aligned} &\big(x_2^n(1), u^n(1,1), v^n(\hat{w}_{12,i-1},1,1), x_3^n(L_{i-1}), \\ &\qquad\qquad \hat{y}_3^n(\hat{l}_i|L_{i-1}), y_2^n(i)\big) \in A_\epsilon^n \\ &\text{and } \big(x_2^n(1), u^n(\hat{w}_{12,i-1},1), x_3^n(\hat{l}_i), y_3^n(i+1)\big) \in A_\epsilon^n \end{aligned} \tag{6}$$

Following standard joint typicality analysis [7], we obtain the following rate constraints:

$$\begin{aligned} R_3 &\leq I(X_3;Y_2|X_2,U) + I(\hat{Y}_3;X_2,V,U,Y_2|X_3), \\ R_{12} &\leq I(U;Y_2|X_2,X_3) + I(V;\hat{Y}_3,Y_2|X_3,X_2,U), \\ R_{12}+R_3 &\leq I(V,U,X_3;Y_2|X_2) + I(\hat{Y}_3;X_2,V,U,Y_2|X_3) \end{aligned} \tag{7}$$

At the destination: D employs two steps of decoding. First, the D quantizes what it receives in each block, i.e. in block i, the destination already knows $l_{i-1} = L_{i-1}$ and it finds an index l_i such that

$$(\hat{y}_3^n(l_i|L_{i-1}), x_3^n(L_{i-1}), y_3^n(i)) \in A_\epsilon^n \tag{8}$$

By the covering lemma, such l_i exists if

$$R_3 > I(\hat{Y}_3;Y_3|X_3) \tag{9}$$

Then, at the end of the last transmission block, the destination employs backward decoding to decode all transmitted messages. In block i, the destination knows $w_{12,i} = 1$ and $w_{12,i-1} = 1$ from the decoding in block $i+2$ and $i+1$, respectively. Hence, it declares that the message vector $(\hat{w}_{13,i}, \hat{w}_{12,i-2})$ is sent if it is the unique vector such that

$$\big(x_2^n(\hat{w}_{12,i-2}), u^n(1, \hat{w}_{12,i-2}), v^n(1,1,\hat{w}_{12,i-2}), \\ x_1^n(\hat{w}_{13,i}, 1, 1, \hat{w}_{12,i-2}), x_3^n(L_{i-1}), y_3^n(i)\big) \in A_\epsilon^n \quad (10)$$

The error analysis for this decoding rule leads to the following rate constraints:

$$R_{13} \leq I(X_1; Y_3 | X_3, X_2, U, V), \\ R_1 \leq I(X_1, X_2; Y_3 | X_3) \quad (11)$$

By combining (7), (9) and (11), we obtain Theorem 1.

C. Discussion

Remark 1. A second-order block Markov encoding is used in this scheme where the transmitted codeword in block i depends on the codewords transmitted in blocks $i-1$ and $i-2$. This encoding technique is important in that it allows the relay to decode the public message using the received signals from the source and the destination. As shown in (6), the relay employs joint decoding over two consecutive blocks $(i, i+1)$ to decode $w_{12,i}$.

Remark 2. The scheme includes the following existing schemes as special cases:

- The original PDF scheme for the relay channel proposed in [2]. This can be verified by setting $U = X_3 = \hat{Y}_3 = \emptyset$ and $R_3 = 0$
- The PDF scheme for relay channels with destination feedback (DFB scheme) proposed in [5]. This can be verified by setting $U = \emptyset$ and requiring the relay to sequentially decode the compression index first and then the public message instead of jointly decode both of them.

Remark 3. The noisy network coding scheme (NNC) proposed in [6] can be applied to the channel model in Figure 1. By applying Theorem 1 in [6] to the channel model in Figure 1, the achievable rate can be expressed as

$$R_1 \leq \max \min\{J_1, J_2\}, \text{ where} \quad (12)$$
$$J_1 = I(X_1; \hat{Y}_2, \hat{Y}_3, Y_3 | X_2, X_3),$$
$$J_2 = I(X_1, X_2; \hat{Y}_d, Y_d | X_d) - I(Y_2; \hat{Y}_2 | X_1, X_2, X_3, \hat{Y}_3, Y_3)$$

and the maximization is over $p(x_1)p(x_2)p(x_3)$ $p(\hat{y}_2|x_2, y_2)p(\hat{y}_3|x_3, y_3)$. This achievable rate cannot be directly compared with that in Theorem 1. However, we compare them numerically for the Gaussian channel in Section IV.

Remark 4. For the channel model in Section II, the simple cut-set bound is given as follows.

Corollary 1. *The capacity of the relay channel with RD cooperation is upper bounded as*

$$R_1 \leq I(X_1; Y_2, Y_3 | X_2, X_3), \\ R_1 \leq I(X_1, X_2; Y_3 | X_3), \quad (13)$$

for some joint distribution $p(x_1, x_2, x_3)$,

IV. Gaussian Channel

A. Signaling

The Gaussian model for the relay channel with RD cooperation is given in (2). Since superposition encoding is equivalent to addition in Gaussian channels [7], S, R and D construct their transmit signals X_1 and X_2 in block i as follows.

$$X_1 = \sqrt{\rho_{13}} T_1(w_{13,i}) + \sqrt{\rho_{12}} V(w_{12,i}) \\ + \sqrt{\rho_{11}} U(w_{12,i-1}) + \sqrt{\rho_1} S_2(w_{12,i-2}), \\ X_2 = \sqrt{P_2} T_2(w_{12,i-2}), \\ X_3 = \sqrt{P_3} T_3(l_{i-1}), \\ \hat{Y}_3 = Y_3 + \hat{Z}_3, \quad (14)$$

where $\hat{Z}_3 \sim \mathcal{CN}(0, \sigma^2)$ and T_1, T_2, T_3, V and U are all i.i.d random variables distributed according to $\sim \mathcal{CN}(0,1)$. The power constraint at S is given as

$$\rho_{13} + \rho_{12} + \rho_{11} + \rho_1 = P_1. \quad (15)$$

B. Achievable Rate

The achievable rate can be derived as follows.

Corollary 2. *For the Gaussian relay channel with RD cooperation, the achievable rate* (R_1) *under the proposed scheme is given as in Theorem 1 with*

$$I_1 = \mathcal{C}(g_{13}^2 \rho_{13}) + \mathcal{C}\left(\frac{g_{12}^2 \rho_{11}}{1 + g_{12}^2(\rho_{13} + \rho_{12})}\right) \\ + \mathcal{C}\left(\frac{g_{13}^2 \rho_{12} + (1+\sigma^2) g_{12}^2 \rho_{12}}{(1+\sigma^2)(1 + g_{12}^2 \rho_{13}) + g_{13}^2 \rho_{13}}\right) \\ I_2 = \mathcal{C}(g_{13}^2 \rho_{13}) + \mathcal{C}(\frac{g_{12}^2(\rho_{12} + \rho_{11}) + g_{32}^2 P_3}{1 + g_{12}^2 \rho_{13}}) \\ - \mathcal{C}\left(\frac{1 + (g_{12}^2 + g_{13}^2)\rho_{13}}{\sigma^2(1 + g_{12}^2 \rho_{13})}\right) \\ I_3 = \mathcal{C}(g_{13}^2 P_1 + g_{23}^2 P_2 + 2 g_{13} g_{23} \sqrt{\rho_1 P_2}). \quad (16)$$

These constraints are subject to the condition in (5) which is given in the Gaussian channel as

$$\sigma^2 \geq \sigma_c^2 \text{ where} \\ \sigma_c^2 = \frac{\left(1 + g_{12}^2(\rho_{13} + \rho_{12})\right)\left(1 + (g_{12}^2 + g_{13}^2)\rho_{13}\right)}{(1 + g_{12}^2 \rho_{13}) g_{32}^2 P_3}, \quad (17)$$

Proof: By direct application of Theorem 1 into the Gaussian channel in (2) with signaling in Section IV-A. ■

C. Optimal σ_o

Since I_1 is a decreasing function of σ while I_2 is a an increasing function, the achievable rate in Corollary 2 is maximized with the optimal σ_o obtained form the intersection between I_1 and I_2 in (16).

Corollary 3. *The optimal* σ_o *that maximizes the achievable rate in Corollary 2 is given as*

$$\sigma_o^2 = \frac{1 + (g_{13}^2 + g_{12}^2)(\rho_{13} + \rho_{12}) + g_{12}^2 \rho_{11}(1+\mu)}{g_{32}^2 P_3} \\ \mu = \frac{g_{13}^2(\rho_{13} + \rho_{12})}{1 + g_{12}^2(\rho_{13} + \rho_{12})} \quad (18)$$

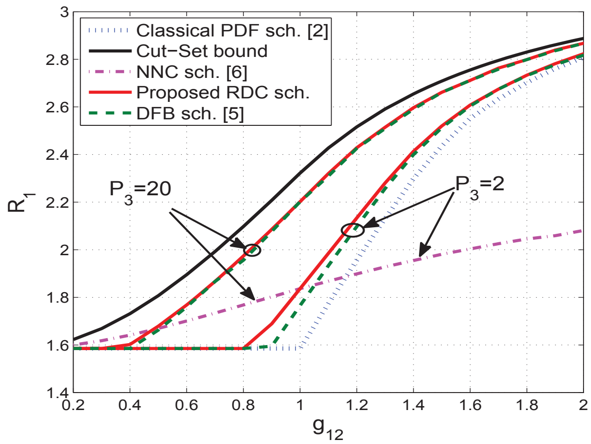

Fig. 2. Achievable rate versus g_{12} of the relay channel for the classical PDF scheme [2], NNC scheme [6], DFB scheme [5], proposed RDC scheme and the cut-set bound with $g_{13} = g_{23} = 1$ and $P_1 = P_2 = 2$ and different values of P_3.

Proof: (18) comes from the solution of $I_1 = I_2$ in (16). ∎

Remark 5. From the expressions in (17) and (18), it can be shown that $\sigma_o^2 > \sigma_c^2$.

Figure 2 compares between the achievability of the proposed RDC, DFB [5], classical PDF [2], NNC [6] schemes and the cut-set bound. Results shows that the proposed RDC scheme outperforms the classical PDF and the DFB schemes since they are special cases of the proposed scheme as mentioned in Remark 2. However, the difference between the proposed RDC and the DFB schemes decreases as P_3 and g_{12} increase. Moreover, NNC is insensitive to P_3 as it does not use the feedback link.

Remark 6. The relay in the NNC scheme performs quantize-forward relaying regardless of the strength of the signal it receives from the destination. However, in the proposed RDC and the DFB schemes, the relay uses the signal it receives from the destination and performs PDF relaying. Therefore, the relay can decode at a higher rate as P_3 increases.

Furthermore, NNC is the best scheme at low g_{12} and then becomes the worst at high g_{12}. For each value of P_3, there is a threshold value of g_{12} at which the proposed RDC scheme becomes better than NNC. Figure 3 illustrates this threshold g_{12} value versus P_3 and shows the regions where the proposed RDC scheme or the NNC scheme is preferred.

V. Capacity Achieving at $P_3 \to \infty$

In some applications such as the uplink in cellular networks, the destination (base station) has much more power than the mobiles such that the destination power is relatively infinite. Moreover, Figure 2 shows that gap between the achievable rate of the proposed scheme and the cut-set bound decreases as P_3 increases. Hence, it is also of interest to study the asymptotic achievable rate when $P_3 \to \infty$. In this section, we show that the proposed RDC scheme asymptotically achieves the cut-set bound as $P_3 \to \infty$.

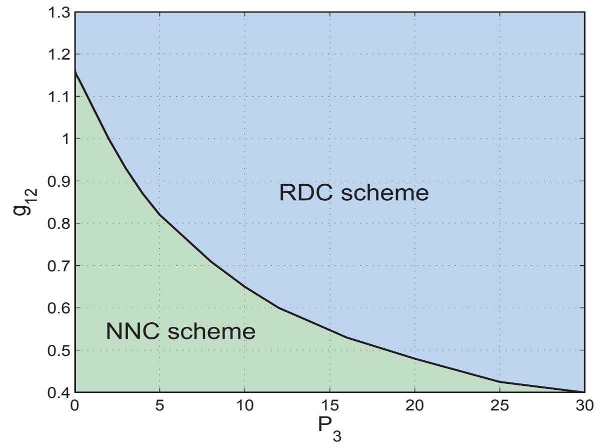

Fig. 3. Regions for the preferred coding scheme (the proposed RDC or NNC) versus P_3 and g_{12} where $g_{13} = g_{23} = 1$ and $P_1 = P_2 = 2$.

A. Achievability at $P_3 \to \infty$

The achievable rate at $P_3 \to \infty$ is given as follows.

Corollary 4. *The achievable rate of the proposed RDC scheme with $P_3 \to \infty$ is given as in Corollary 2 with*

$$
\begin{aligned}
I_1 &= \mathcal{C}(g_{13}^2\rho_{13}) + \mathcal{C}\left(\frac{g_{12}^2\rho_{11}}{1+g_{12}^2(\rho_{13}+\rho_{12})}\right) \\
&\quad + \mathcal{C}\left(\frac{(g_{13}^2+g_{12}^2)\rho_{12}}{1+(g_{13}^2+g_{12}^2)\rho_{13}}\right), \\
I_2 &= I_1, \\
I_3 &= \mathcal{C}(g_{13}^2P_1 + g_{23}^2P_2 + 2g_{13}g_{23}\sqrt{\rho_1P_2}). \qquad (19)
\end{aligned}
$$

Proof: By substituting $P_3 \to \infty$ and hence $\sigma_o^2 = \sigma_c^2 = 0$ in Corollary 2. ∎

B. Cut-Set Bound for Gaussian Channel

The cut-set bound for the Gaussian relay channel in (2) is given as follows.

Corollary 5. *The capacity of relay channel with RD cooperation is upper bounded by R_1 satisfying*

$$
\begin{aligned}
R_1 &\le \mathcal{C}\left((g_{12}^2+g_{13}^2)P_1\left(1-\frac{\delta_{12}^2+\delta_{13}^2-2\delta_{12}\delta_{13}\delta_{23}}{1-\delta_{23}^2}\right)\right), \\
R_1 &\le \mathcal{C}\big(g_{13}^2P_1(1-\delta_{13}^2)+g_{23}^2P_2(1-\delta_{23}^2) \\
&\quad + 2g_{13}g_{23}(\delta_{12}-\delta_{13}\delta_{23})\sqrt{P_1P_2}\big), \qquad (20)
\end{aligned}
$$

where $(\delta_{12}, \delta_{13}, \delta_{23}) \in [-1, +1]$ and

$$1-\delta_{12}^2-\delta_{13}^2-\delta_{23}^2+2\delta_{12}\delta_{13}\delta_{23} \ge 0. \qquad (21)$$

Proof: The cut-set bound in (2) is maximized when the input distribution (X_1, X_2, X_3) is jointly Gaussian. $\delta_{i,j}$ for i and $j \in \{1, 2, 3\}$ is the correlation factor between X_i and X_j. For the detailed proof, see Appendix A. ∎

From this cut-set bound and the achievability in Corollary 4, we obtain the following theorem:

Theorem 2. *The proposed scheme asymptotically achieves the capacity of the Gaussian relay channel with RD cooperation as $P_3 \to \infty$.*

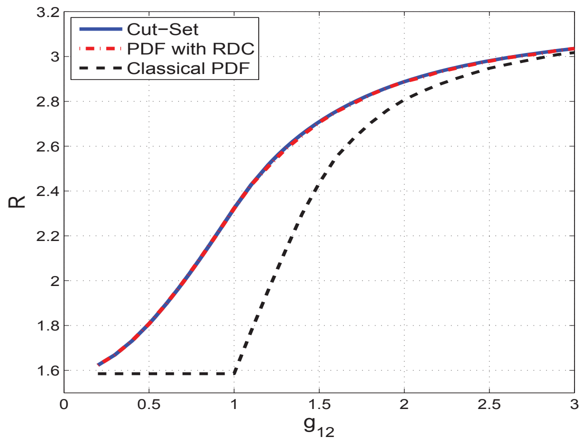

Fig. 4. Achievable rate versus g_{12} of the relay channel for the classical PDF scheme, proposed RDC scheme at $P_3 \rightarrow \infty$ and the cut-set bound with $g_{13} = g_{23} = 1$ and $P_1 = P_2 = 2$.

Proof: After some mathematical manipulations, we can show that formulas (19) with (15) and (20) with (21) are identical. For the detailed proof, see Appendix B. ■

The equivalency between the cut-set bound and the achievable rate with $P_3 \rightarrow \infty$ can be intuitively interpreted as follows.

Remark 7. When $P_3 \rightarrow \infty$, the destination has an infinite link to both relay and source. The destination virtually joins the relay in one entity. Hence, the first constraint in the cut-set bound in (13) is achieved. The second constraints in the cut-set bound is obtained easily from the coherent transmission of the source and the relay.

Figure 4 shows that the proposed RDC scheme is capacity achieving at $P_3 \rightarrow \infty$.

VI. Conclusion

In this paper, we showed how relay-destination cooperation helps improve the achievable rate of the relay channel until it asymptotically achieves the capacity of the Gaussian relay channel. We proposed a coding scheme based on rate splitting, second-order superposition block Markov encoding, PDF, QF relaying and backward decoding techniques. The proposed scheme includes as special cases the classical PDF and destination feedback (DFB) schemes. Furthermore, we applied the proposed scheme into the Gaussian channel and numerically compare between the proposed RDC, the PDF, DFB and NNC schemes. Finally, we proved that for Gaussian channels, the proposed scheme asymptotically achieves the capacity as the destination power approaches infinity. This result shows the importance of relay destination cooperation and encourages further studying of this cooperation in other multiuser channels.

Appendix A
Proof of Corollary 5

For the Gaussian channel in (2), we aim to identify the optimal joint density of (X_1, X_2, X_3) that maximizes the rate in Corollary 1. First, we start with $I(X_1; Y_2, Y_3|X_3, X_2)$

$$\begin{aligned} &I(X_1; Y_2, Y_3|X_3, X_2) \\ &= h(Y_2, Y_3|X_3, X_2) - h(Y_2, Y_3|X_3, X_2, X_1) \\ &= h(Y_2, Y_3|X_3, X_2) - h(Z_2, Z_3). \end{aligned} \tag{22}$$

The distribution of (Z_2, Z_3) is $\mathcal{CN}(0, I_{2\times 2})$. Therefore, by the entropy power inequality [7], (22) is maximized when $(Y_2, Y_3|X_3, X_2)$ is jointly Gaussian. From (2), $(Y_2, Y_3|X_3, X_2)$ is jointly Gaussian if $(X_1|X_2, X_3)$ is Gaussian.

Next, we move to $I(X_1, X_2; Y_3|X_3)$

$$\begin{aligned} &I(X_1, X_2; Y_3|X_3) \\ &= h(Y_3|X_3) - h(Y_3|X_3, X_2, X_1) \\ &= h(Y_3|X_3) - h(Z_3) \\ &= h(g_{13}X_1|X_3 + g_{23}X_2|X_3 + Z_3) - h(Z_3). \end{aligned} \tag{23}$$

Similar to $I(X_1; Y_2, Y_3|X_3, X_2)$, by the entropy power inequality, (23) is maximized when $(g_{13}X_1|X_3 + g_{23}X_2|X_3)$ is Gaussian because $Z_3 \sim \mathcal{CN}(0, 1)$.

Therefore, an optimal input distribution in Corollary 1 is jointly Gaussian because if (X_1, X_2, X_3) is jointly Gaussian, $(X_1|X_2, X_3)$, $(X_1|X_3)$, $(X_2|X_3)$ and $(X_1|X_3 + X_2|X_3)$ are Gaussian. Hence, $(X_1, X_2, X_3) \sim \mathcal{N}(0, \Sigma)$ where Σ is the covariance matrix given as

$$\begin{aligned} \Sigma &= \mathrm{cov}(X_1, X_2, X_3) \\ &= \begin{bmatrix} P_1 & \delta_{12}\sqrt{P_1P_2} & \delta_{13}\sqrt{P_1P_3} \\ \delta_{12}\sqrt{P_1P_2} & P_2 & \delta_{23}\sqrt{P_2P_3} \\ \delta_{13}\sqrt{P_1P_3} & \delta_{23}\sqrt{P_2P_3} & P_3 \end{bmatrix} \end{aligned} \tag{24}$$

where $(\delta_{12}, \delta_{13}, \delta_{23}) \in [-1, +1]$ and $\det(\Sigma) \geq 0$ such that Σ is positive semi-definite and a valid covariance matrix. This implies the condition in (21).

Appendix B
Proof of Theorem 2

For the signaling in Section IV-A, define the power allocations of the source as $\rho_k = \mu_k P_1$ for $k \in \{1, 11, 12, 13\}$. Then, the power constraint in (15) is equivalent to

$$\mu_{13} + \mu_{12} + \mu_{11} + \mu_1 = 1. \tag{25}$$

Then, after some mathematical manipulations, the achievable rate with $P_3 \rightarrow \infty$ in (19) can be expressed as

$$\begin{aligned} &R_1 \leq \mathcal{C}\left((g_{12}^2 + g_{13}^2)P_1(1 - \beta_1)\right), \\ &R_1 \leq \mathcal{C}\left(g_{13}^2 P_1 + g_{23}^2 P_2 + 2g_{13}g_{23}\sqrt{\mu_1 P_1 P_2}\right). \end{aligned} \tag{26}$$

where

$$\beta_1 = \frac{\zeta_1}{\zeta_2}, \text{ with} \tag{27}$$

$$\begin{aligned} \zeta_1 &= \left(1 - \mu_{12} - \frac{g_{12}^2\mu_{11} + g_{13}^2\mu_{13}}{g_{12}^2 + g_{13}^2}\right) \\ &\quad + \rho_{13}\left(g_{12}^2 + g_{13}^2 - g_{13}^2(\mu_{13} + \mu_{12}) - \frac{g_{13}^2 g_{12}^2 \mu_{11}}{g_{12}^2 + g_{13}^2}\right) \\ &\quad + g_{12}^2(\rho_{12} + \rho_{13})\left(1 + (g_{12}^2 + g_{13}^2)\rho_{13} - \eta\right) \\ \zeta_2 &= \left(1 + g_{12}^2(\rho_{12} + \rho_{13})\right)\left(1 + (g_{12}^2 + g_{13}^2)\rho_{13}\right), \\ \eta &= \mu_{12} + \mu_{11} + \frac{g_{13}^2\mu_{13}}{g_{12}^2 + g_{13}^2} + g_{13}^2\rho_{13}(\mu_{13} + \mu_{12} + \mu_{11}) \end{aligned}$$

Given the power constraint in (25), it can be easily verified that $0 \leq \beta_1 \leq 1$ because $0 \leq \zeta_1 \leq \zeta_2$.

Now, the cut-set bound in (20) can be expressed as

$$\begin{aligned} R_1 &\leq \mathcal{C}\left((g_{12}^2 + g_{13}^2)P_1(1-\beta_2)\right), \\ R_1 &\leq \mathcal{C}\big(g_{13}^2 P_1(1-\delta_{13}^2) + g_{23}^2 P_2(1-\delta_{23}^2) \\ &\quad + 2g_{13}g_{23}(\delta_{12} - \delta_{13}\delta_{23})\sqrt{P_1 P_2}\big), \end{aligned} \tag{28}$$

where

$$\beta_2 = \frac{\delta_{12}^2 + \delta_{13}^2 - 2\delta_{12}\delta_{13}\delta_{23}}{1-\delta_{23}^2} \tag{29}$$

With the condition in (21), it can be also shown that $0 \leq \beta_2 \leq 1$. Furthermore, setting $\delta_{13} = \delta_{23} = 0$ does not change the range of β_2 nor affect the cut-set bound as in (28). After this setting, the conditions in (26) and (28) are identical.

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