Let's "recall" some basics about matrices and linear transformations.

A matrix is a rectangular array of numbers. Actually those "numbers" can come from any field you want, like $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, or more exotic ones like $\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Q}_p$ (if you know what those are).

Matrices multiply by a "mating" rule: to get the entry of AB in the ith row and jth column, you take the dot product of the ith row vector of A with the jth column vector of B. That is, mate the rows of A with the columns of B to get the entries of AB. Example:

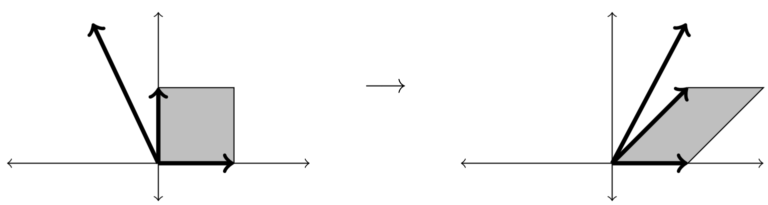
$$\begin{bmatrix} 5 & \pi \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 10+\pi \\ -1 & -2 \end{bmatrix}.$$

The *determinant* is defined for square matrices, and is particularly easy to define for 2×2 matrices, which are basically all we care about for complex analysis purposes. We define $\det \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = ad - bc$. Fact: determinant is multiplicative, meaning that $\det(AB) = \det(A)\det(B)$. (Easy to check for 2×2s.)

Note that $I = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ is a special matrix called "the identity" that has the property that $AI = IA = A$ for any 2×2 matrix A. We can see that $\det I = 1$.

The *inverse* of a matrix A is a matrix A^{-1} that cancels out A, meaning that $A \cdot A^{-1} = A^{-1} \cdot A = I$. Because $\det I = 1$, it follows that $\det(A^{-1}) = \frac{1}{\det(A)}$.

Multiplying vectors by a 2×2 matrix can be regarded geometrically as a *linear transformation* of $\mathbb{R}^2$. For instance, let's look at $A = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. We have $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} x+y \\ y \end{smallmatrix}\right]$, so that gives us a rule to send input vectors to output vectors.



The result is a *shear* of the plane. Notice that it sent the unit square (area 1) to a parallelogram that, you can check, also has area 1. In general, this is the geometric interpretation of determinants: they tell you by what factor the areas or volumes of regions are changed when you multiply by the matrix.

What makes such maps "linear" is that they take lines to lines. In fact, since these maps also always preserve the origin, they take lines through the origin to lines through the origin.

Abstractly speaking, we'll call a function between vector spaces a *linear transformation* if $T(a\mathbf{v} + b\mathbf{w}) = a \cdot T(\mathbf{v}) + b \cdot T(\mathbf{w})$, where $\mathbf{v}, \mathbf{w}$ are vectors and a, b are scalars. (Why does this take lines to lines? Because a line is of the form $\{\mathbf{v} + t\mathbf{w}\}$. This condition is precisely what is needed to make the output be in the same form!)

Here's a useful observation: you can read off the effects of a matrix on the basis vectors $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ by just taking its columns. This is easily verified.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}.$$

It's a fact from basic linear algebra that *all matrix multiplications are linear transformations, and all linear transformations can be written in the form of matrix multiplication.*

Exercises.

(1) Compute $\left[\begin{smallmatrix} 5 & 2 \\ 3 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} -1 & 0 \\ -1 & 2 \end{smallmatrix}\right]$. Find the determinant of each matrix, and the determinant of the product.

(2) Check the following fact: if $ad - bc \neq 0$, then the inverse of $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is $A^{-1} = \frac{1}{\det A} \left[\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right]$; if $ad - bc = 0$, then the matrix has no inverse.

(3) Show that $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ does CCW rotation by θ. Find the matrix M whose effect is CCW rotation by a right angle. For that M, what is M^2?

(4) For a given $r > 0$, what is the matrix that scales all vectors by a factor of r?

(5) Show that a matrix has the effect of *stretching and rotating all vectors by the same amount* if and only if it is of the form $\left[\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right]$ for some numbers a, b. For such a matrix, how much does it stretch by? How much does it rotate by?

Since the complex plane $\mathbb{C}$ can be identified with the real plane $\mathbb{R}^2$, it will be useful to recall some facts about functions $\mathbb{R} \to \mathbb{R}^2$ and $\mathbb{R}^2 \to \mathbb{R}^2$ that you learned in calculus.

Parametrized curves. You can parametrize a curve in the plane by writing

$$\gamma(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \qquad \text{or} \qquad \gamma(t) = u(t) + i \cdot v(t).$$

For instance, $\gamma(t) = e^{it}$ is the curve traced out by $\cos(t) + i \cdot \sin(t)$ as t varies over real numbers. This is the unit circle!

As you remember from calculus, you can get the *tangent vector* to a parametrized curve by taking $\gamma'(t)$, and its component functions are just the derivatives of the real functions u and v. Written in complex notation, this is just saying that $\gamma'(t) = u'(t) + i \cdot v'(t)$. You'll recall that this is also known as the *velocity* vector, and its magnitude is the *speed* at each point. So we can compute that for the unit circle example, $\gamma(t) = e^{it}$, the derivative is $-\sin(t) + i \cdot \cos(t) = i(\cos(t) + i \cdot \sin(t)) = ie^{it}$. This has norm 1, which tells us that this parametrization travels around the unit circle at unit speed.

Jacobians. The curves above can be thought of as functions $\gamma : \mathbb{R} \to \mathbb{C}$. Now let's think about functions $f : \mathbb{C} \to \mathbb{C}$, which can be reinterpreted as functions $f : \mathbb{R}^2 \to \mathbb{R}^2$, with which we are already familiar from calculus.

If we write x, y for the input variables and u, v for the output variables, as usual, then we have $f(x, y) = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}$, or of course $f(x + iy) = u(x, y) + i \cdot v(x, y)$.

We can form a useful matrix called the *Jacobian* of the transformation by just recording all four partial derivatives:

$$J_f = Df = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix},$$

where u_x stands for $\frac{\partial u}{\partial x}$, and likewise for the other partials.

Notice that this is a matrix of *functions*: for instance, if $f(x, y) = \begin{bmatrix} xy^2 \\ \sin x \end{bmatrix}$, then $Df = \begin{bmatrix} y^2 & 2xy \\ \cos x & 0 \end{bmatrix}$ is the Jacobian, and $Df(\pi, 2) = \begin{bmatrix} 4 & 4\pi \\ -1 & 0 \end{bmatrix}$ is the Jacobian evaluated at the point $(\pi, 2)$.

Or to do an example in complex notation, the exponential map $f(z) = e^z$ can be written out as $f(x + yi) = e^x \cos y + i(e^x \sin y)$, so its Jacobian is $Df = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}$.

What are Jacobians good for? Lots of things: (1) they explain how tangent vectors are affected by the transformation at each particular point; (2) the determinant tells you how much area/volume is transformed by at each particular point; (3) they allow a very neat rule for change of variables in integration; (4) they give the best possible approximation to f by a linear transformation.

The last fact can be written this way: f has derivative Df if and only if an excellent approximation to $f(z_0 + \Delta z)$ is given by $f(z_0) + \big[Df(z_0)\big] \cdot \Delta z$. (Here Δz is supposed to be a small vector, otherwise known as a complex number of small norm.) How excellent? We demand that as Δz shrinks, the difference between these terms shrinks even faster:

$$\lim_{\Delta z \to 0} \frac{\left| f(z_0 + \Delta z) - f(z_0) - \left[Df(z_0) \right] \cdot \Delta z \right|}{|\Delta z|} = 0.$$

Stare at this for a second and do the cancellation. This is just the usual definition of derivative, repackaged!

Exercises.

(1) What curve is parametrized by $\gamma(t) = t^2 + (3t^2 + 1)i$? Plot this curve and label the points $\gamma(t)$ when $t = 0, 1, -1, 2, 3$. What is the tangent vector $\gamma'(t)$ at each of those points?

(2) The polar coordinates transformation is $(r, \theta) \mapsto (r\cos\theta, r\sin\theta)$. What is its Jacobian and what is the Jacobian determinant? (Notice that since r and θ are the input variables, your partial derivatives should be with respect to r and θ, not x and y.)

(3) One of the examples above was $f(x, y) = \left[\begin{smallmatrix} xy^2 \\ \sin x \end{smallmatrix}\right]$, where we found that $Df(\pi, 2) = \left[\begin{smallmatrix} 4 & 4\pi \\ -1 & 0 \end{smallmatrix}\right]$. One example of a parametrized curve through the point $P = (\pi, 2)$ is the curve $\gamma(t) = (\pi + t) + (2 + t)i$, which has $\gamma(0) = P$ and $\gamma'(0) = 1 + i$. Define a new curve by $\alpha(t) = f(\gamma(t))$. Find $\alpha(0)$ and $\alpha'(0)$. The Jacobian is supposed to be telling us what f does to tangent vectors, so check that $\left[Df(P)\right] \cdot \left[\gamma'(0)\right] = \left[\alpha'(0)\right]$, when everything is interpreted as matrices and vectors. Choose a second point Q and a curve through Q and carry out the same calculations.

Topology worksheet

This is based on the definitions of limits, continuity, open, closed, compact, and connected that are given in the book in §1.4.

Important Note: if you are working in the plane with its standard topology, as we usually are in this class, then you can use easier definitions than the general ones. *Open* means that every point in the set has an open disk centered there that stays in the set. *Closed* means that the complement is open, and it also means that if the set contains a sequence that converges, it contains the limit too. *Compact* means closed and bounded. And *connected* means that between every two points in the set there is a path that stays in the set.

Exercises.

(1) Show that $\{z \in \mathbb{C} : \ |\operatorname{Re} z - a| < \epsilon_1\} \cap \{z \in \mathbb{C} : \ |\operatorname{Im} z - b| < \epsilon_2\}$ is an open rectangle in the complex plane. Using this, show that for every disk in the plane, there are real numbers $a, b, \epsilon_1, \epsilon_2$ such that the rectangle they describe lies inside the given disk.

(2) We defined arg as a multivalued function $\mathbb{C} \to \mathbb{C}$. We defined the principal argument Arg as choosing the value of the argument in the standard interval $(-\pi, \pi]$, so that it becomes single-valued. Show that Arg is not continuous on $\mathbb{C}$ but it is continuous on the slit plane $\mathbb{C} \setminus (-\infty, 0]$.
(Hint: at what points of $\mathbb{C}$ is Arg NOT continuous?)

(3) Give a geometric proof that the composition of two continuous functions is continuous. Turn this into an ϵ–δ proof.

(4) Show that $\{1, 1/2, 1/3, 1/4, \ldots, 1/n \ldots\}$ is not compact by constructing an open cover that has no finite subcover. Is it closed? Is it bounded?

(5) The general definition of a connected set is as follows: we say that Ω is *disconnected* if it has a nonempty proper subset $A \subsetneq \Omega$ such that $A = \Omega \cap B$ for some open set B and $A = \Omega \cap C$ for some closed set C. (The nickname for such a set A is "clopen" because it is both closed and open relative to Ω.) If a set is not disconnected, then it's *connected*.

For $\Omega \subseteq \mathbb{C}$, suppose there is some line $L \subset \mathbb{C}$ such that Ω contains some points on either side of L but no point on L. Prove Ω is not connected.