TUFTS UNIVERSITY DEPARTMENT OF MATHEMATICS
Fair Division Handout
Math 87, Duchin **Spring 2020**

1 Setup

Suppose M parties need to share a resource with N types, each assumed homogeneous and infinitely divisible. We can introduce an $M \times N$ taste matrix, where t_{ij} represents the preference coefficient for the Mth party and the Nth good. The rows are *taste vectors* $\vec{t_i}$ telling you how much the ith party prefers each of the goods, relative to each other. For instance, suppose a large park with ponds and trees must be split up for three groups: children, teens, and seniors. The children really like ponds and barely care about trees, the teens like trees, and the older folks like both. So the kids' taste vector is $[.92\ .08]$, the teens' is $[0\ 1]$, and the seniors' is $[.5\ .5]$, say. This is just telling you the weight of preference for one versus the other, scaled to add up to 1. This gives us

$$T = \begin{bmatrix} .92 & .08 \\ 0 & 1 \\ .5 & .5 \end{bmatrix}.$$

Again, because the rows are interpreted as proportions of each person's preference, they add to one. (A matrix with rows adding to one is called *stochastic*.)

I can record a way of allocating resources with a *division matrix* D. This is also $M \times N$ and you can read the columns as telling you how each good was divided. For instance, even division is D_0 and two other possible divisions D_1 and D_2 are as follows:

$$D_0 = \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & 1/3 \\ 1/3 & 1/3 \end{bmatrix} \qquad D_1 = \begin{bmatrix} .3 & .5 \\ .2 & .4 \\ .5 & .1 \end{bmatrix} \qquad D_2 = \begin{bmatrix} .5 & 0 \\ 0 & .5 \\ .5 & .5 \end{bmatrix}$$

In division D_1, 30% of the pond supply goes to kids, 20% to teens, and 50% to seniors. In D_2, the ponds are split evenly between kids and seniors. And so on. I can let $\mathcal{D}$ be the set of all valid division matrices:

$$\mathcal{D} = \left\{ (d_{ij}) \ : \ 1 \leqslant i \leqslant M,\ 1 \leqslant j \leqslant N,\ \ 0 \leqslant d_{ij} \leqslant 1 \quad \forall i,j, \quad \sum_{i=1}^{M} d_{ij} = 1 \quad \forall j \right\},$$

so that $D \in \mathcal{D}$ just means that D is $M \times N$, has positive entries, and has column sums 1.

We've seen that the columns of D are proportional allocations of each resource. As for the rows, $\vec{d_i}$ just records party i's share of each resource. Since $\vec{t_i}$ recorded their level of preference for each resource, I can combine those to get a total *value*—how much of each resource I received *weighted by* how much I prefer each type. Thus $v_{ii} = \vec{t_i} \cdot \vec{d_i} = \sum_{j=1}^{N} t_{ij} d_{ij}$ is the value to party i of their share. But by the same token any v_{ij} is the value to party i of the share allotted to party j.

Red: first person's preferences. ($\vec{t_1}$)
Blue: first person's share. ($\vec{d_1}$)
Green: second person's share. ($\vec{d_2}$)

$$\begin{bmatrix} .92 & .08 \\ 0 & 1 \\ .5 & .5 \end{bmatrix} \cdot \begin{bmatrix} .3 & .5 \\ .2 & .4 \\ .5 & .1 \end{bmatrix}^{\mathsf{T}}$$

$v_{11} = \vec{t_1} \cdot \vec{d_1}$ = how first person values own share
$v_{12} = \vec{t_1} \cdot \vec{d_2}$ =: how first person values 2nd share
etc

2 Finding fairness

Let's start with a simple observation.

Lemma 1. Under equal division D_0 (which has $1/M$ in every entry), each party values each share equally as $v_{ij} = 1/M$ for any i and j.

Proof. The value matrix has entries

$$v_{ij} = \vec{t_i} \cdot \vec{d_j} = \sum_k t_{ik} d_{jk} = \sum_k \left(t_{ik} \cdot \tfrac{1}{M}\right) = \frac{1}{M} \sum_k t_{ik} = \frac{1}{M},$$

because $\sum_k t_{ik}$ is a row-sum of T, which is 1. □

This helps us find some axiomatic notions that correspond to fairness ideals.

Definition 2.

- A division D is called **envy-free** if each person likes their own share at least as well as they like the other shares: $v_{ii} \geqslant v_{ij}$ for all i, j.
- D is called **equitable** if everyone likes their share just as much as everyone else likes their own: $v_{ii} = v_{jj}$ for all i, j. Let us use $\mathcal{D}_{\mathsf{eq}}$ to denote the subset of $\mathcal{D}$ that satisfies equitability.
- D is called **fair** if every party likes their share at least as much as they would have liked the equal division D_0: that is, $v_{ii} \geqslant {}^1\!/_M$ for all i.
- Given division matrices D and $\bar{D}$, suppose the corresponding value matrices are V and $\bar{V}$. We say that $\bar{D}$ **dominates** D (written $\bar{D} \gg D$) if $\bar{v}_{ii} \geqslant v_{ii}$ for all i.
- D is called **Pareto-optimal** if for all $\bar{D} \in \mathcal{D}$, defining $\bar{V} = T \cdot \bar{D}^T$ with entries $\bar{v}_{ij}$, we have

  $$\bar{D} \gg D \implies \bar{v}_{ii} = v_{ii} \quad \forall i.$$

  The subset of $\mathcal{D}$ satisfying Pareto optimality is called the **Pareto frontier**.

So we can rephrase Lemma 1 to conclude that equal division is envy-free, equitable, and fair. In fact, this lets us see that "fairness" (in this definition) is just saying that everyone is at least as happy as they would have been under equal division.

Lemma 3. If D is envy-free, then D is fair. If D is equitable and Pareto-optimal, then D is fair.

Proof. The value matrix V is $M \times M$. Envy-free means that each v_{ii} is (at least tied for) the biggest in its row. So no v_{ii} can be less than $1/M$ or else the row would fail to sum to one. (See Lemma 5.)

Equitable means $v_{ii} = v_{jj}$ for all i, j. That means the V matrix has all the same values on the diagonal. But if the division were not fair, then someone's v_{ii} is less than $1/M$, which means that they all are. But then the division is strictly dominated by D_0, so it's not Pareto-optimal. □

Theorem 4. If T has no zero entries, then the following are equivalent:

(a) D is equitable and Pareto-optimal. (So therefore fair, by Lemma 3.)

(b) D maximizes $\min\limits_{1 \leqslant i \leqslant M} v_{ii}$ over $\mathcal{D}$.

(c) D maximizes v_{11} over $\mathcal{D}_{\mathsf{eq}}$.

3 Fairness through linear programming

The equivalence of (a) and (c) gives us a great way to find a dominating, fair, and equitable division given any matrix of preferences. First we note that $\mathcal{D}$ and $\mathcal{D}_{\text{eq}}$ are defined by systems of equalities and inequalities in the d_{ij}. That's because $\mathcal{D}$ just requires column sums to be one and entries to be bounded above and below, and $\mathcal{D}_{\text{eq}}$ imposes the additional linear constraints that $v_{ii} = v_{jj}$ for all i, j, or equivalently, that all v_{ii} equal v_{11}.

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The t_ij are given.  The d_ij are decision variables.
Maximize z = v_11 = t_11 d_11 + t_12 d_12 + ··· + t_1N d_1N
subject to the linear constraints of D_eq:
  0 ⩽ d_ij ⩽ 1 for i = 1,...,M and j = 1,...,N (valid division)
  Σ_i d_ij = 1 for j = 1,...,N (valid division)
  Σ_j t_ij d_ij = Σ_j t_1j d_1j for i = 2,...,M (equitable).
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That is $2MN$ inequalities and $N + M - 1$ equalities in all.

A Some nice proofs for your reading pleasure

First let's see why the value matrix V is stochastic. We had that the rows of T sum to one and the columns of D sum to one. That means that $V = T \cdot D^\intercal$ is the product of two row-stochastic matrices.

Lemma 5. The product of (row-)stochastic matrices is stochastic.

Proof. Here's a really clever argument. Let $\mathbb{1}$ be the all-ones vector. Then a matrix A is stochastic precisely if $A \cdot \mathbb{1} = \mathbb{1}$. But then suppose A, B are stochastic. We then have $(AB)\mathbb{1} = A(B\mathbb{1}) = A\mathbb{1} = \mathbb{1}$, so AB is stochastic as well. □

Now let's go back and prove the biggest result above.

Theorem 4. If T has no zero entries, then the following are equivalent:

(a) D is equitable and Pareto-optimal. (So therefore fair, by Lemma 3.)

(b) D maximizes $\min_{1 \leqslant i \leqslant M} v_{ii}$ over $\mathcal{D}$.

(c) D maximizes v_{11} over $\mathcal{D}_{\mathsf{eq}}$.

Proof. $[(b) \Rightarrow (a)]$ Fix a D which realizes the maximum possible $\min_i v_{ii}$ over $\mathcal{D}$. I must show that it's equitable and Pareto-optimal. If it were not equitable, then consider the largest of the diagonal entries and compare it to the smallest, say $v_{jj} > v_{ii}$. But then I can take a small share $\epsilon > 0$ of each type of resource from party j and reallocate it evenly to all the parties with minimal value. By taking ϵ very close to zero, I can be sure that v_{jj} stays greater than v_{ii}, but the latter gets bigger, which contradicts the hypothesis by improving the lowest value.

I've shown that D is equitable. Now let's see that it's Pareto-optimal. If it were not, then there would be some dominating $\bar{D} \gg D$ where for some index k, we have $\bar{v}_{kk} > v_{kk}$. Since D is equitable, all its diagonal values are equal to v_{kk}. By domination, the previous paragraph applies to $\bar{D}$ as well, so all of its diagonal values are equal to $\bar{v}_{kk}$, which means D didn't maximize its worst diagonal after all.

$[(a) \Rightarrow (b)]$ This direction is easier! Suppose D is equitable and Pareto-optimal. Then all diagonals equal v_{11}. To contradict (b), we'd need there to exist some other division with the minimum (and therefore all) diagonal value entries greater than this. But that would clearly contradict Pareto-optimality.

$[(a) \Rightarrow (c)]$ We've seen (a) implies (b), and it's obvious that an equitable D, since it has all diagonal entries equal, optimizes v_{11} if it optimizes v_{ii}. This gives (a) implies (c).

$[(c) \Rightarrow (a)]$ Suppose D maximizes v_{11} over $\mathcal{D}_{\mathsf{eq}}$. This D is equitable, so for contradiction I will suppose it is not Pareto-optimal. Then there is some $\bar{D}$ with some largest diagonal value $\bar{v}_{ii} > v_{ii}$. As above, I can take a tiny slice ϵ of each resource type and redistribute it to the ones that are tied for lowest, with a ratio that keeps them equal. I can keep doing this until more values become equal, and iterate until they are all equal. This produces an improvement on D within $\mathcal{D}_{\mathsf{eq}}$, which is a contradiction, completing the proof. □